Rigid block models for masonry structures

Maurizio Angelillo* and Antonio Fortunato
Department of Civil Engineering, University of Salerno, via Giovanni Paolo II 132, Fisciano (Sa) 84084, Italy
Email: mangelillo@unisa.it
Email: a.fortunato@unisa.it
*Corresponding author

Antonio Gesualdo, Antonino Iannuzzo and Giulio Zuccaro
Department of Structures for Engineering and Architecture, University of Naples Federico II, via Claudio 21, Napoli 80125, Italy
Email: fabiana.deserio@unina.it
Email: gesualdo@unina.it
Email: antonino.iannuzzo@unina.it
Email: zuccaro@unina.it

Abstract: Masonry structural modelling needs of a completely different methodology from the ones adopted for ductile structures. In fact, the concepts of strength, stiffness and elastic stability, fundamental for the latter structures, play a marginal role in masonry mechanics. In this respect, Heyman’s theory, gives a modern turn to the old methods of masonry design, adopting a set of very simple and clear mechanical hypotheses. In these papers, the basic ingredients of a new method based on unilateral equilibrium and rigid block kinematics, which may allow the implementation of Heyman’s model for masonry on a computer, is introduced. In particular a simple method based on energy minimisation, with the possibility of combining the effects of loads and settlements on real masonry structures, is developed.

Keywords: masonry structures; unilateral constraints; unilateral materials; multi-body structures; contact and friction.


Biographical notes: Maurizio Angelillo is a Professor of Structural Mechanics at the University of Salerno. He is the author of more than 100 scientific papers, several of which are published in leading international journals of structural and solid mechanics. Masonry structures and unilateral materials behaviour, propagation of brittle fracture in elastic solids, brain and corneal biomechanics are his main research topics.
Antonio Fortunato obtained a PhD in Structural Engineering at the University of Naples and thereafter in the University of Salerno, he has been a Postdoctoral Researcher fellow. His research interests cover a wide range of topics such as structural masonry analysis, fracture mechanics, variational optimisation problems with no-convex energies and biomechanics of running.

Antonio Gesualdo is an Assistant Professor of Structural Engineering at the University of Naples Federico II where he received his PhD in Structural Engineering. His research fields are limit analysis, masonry mechanics and homogenisation.

Antonino Iannuzzo gained his PhD in Structural Engineering, Geotechnical and Seismic at the University of Naples Federico II. His research areas, besides the modelling of masonry structures and rocking of rigid blocks, include homogenisation of discrete structures and seismic vulnerability.

Giulio Zuccaro is an Associate Professor of Structural Engineering at the University of Naples Federico II. He is Scientific Director of several European Research Projects and is member of the Task Group 3 “Seismic Risk and Seismic Scenarios”. He is the Director of the Plinius research center at LUPT, Napoli. His main research activities concern masonry mechanics and seismic vulnerability of structures.

1 Introduction

Peculiarities of masonry behaviour

The renaissance of the old technical literature on masonry together with an accurate revisiting of the old building techniques and design rules in the last decades, has led to rediscover an almost lost code of conduct for masonry constructions. In this respect, starting from the seminal work of Kooharian (1952), the individuals which gave the more important contribution to this rediscovering were Jacques Heyman with a series of works originating from the seminal paper (Heyman, 1966) of 1966, whose title is ‘The stone skeleton’ (and later, with the monography (Heyman, 1995) by the same title), and Santiago Huerta, with a number of works originating from his doctoral thesis of 1992, and among which the cutting paper (Huerta, 2006) by the provoking title “Galileo was wrong”, stands out. The main message of Heyman’s theory is that masonry structures are essentially unilateral and that the theorems of limit analysis can be used to assess their stability, as remarked in Livesley (1978), Como (1992), Angelillo (2014, 2015); Brandonisio et al. (2013, 2015, 2017), Gesualdo et al. (2018), Angelillo et al. (2014), Fortunato et al. (2015, 2018) and Portioli et al. (2014).

Although in the more historical and theoretical restoration literature a constant growth of sensibility toward less invasive and more attentive techniques and procedures for masonry retrofitting can be acknowledged (Bergamasco et al., 2018), in most of practical applications, still an inertia in recognising and accepting the diversity of masonry behaviour still persists. Also building codes (such as the Eurocodes) often appear imprisoned by the schematic approach of the elastic model, proposing methods which appear as simply translated from framed structures of concrete and steel, to masonry.
The original structural approach here proposed, following in the steps of Heyman and Huerta, has been the object of study of a group of researchers of the University of Salerno and Napoli, among which the first author played a leading part (see for example Angelillo et al. (2010, 2012), Fortunato et al. (2016, 2018), Angelillo and Fortunato (2004) and De Serio et al. (2018)) and references therein). The main object of the research of this group is the implementation of numerical methods based on the unilateral model for masonry, through computer codes allowing to predict the effect of forces and settlements, and assess the safety of real structures (Iannuzzo et al., 2017; Angelillo et al., 2005, 2016; Fraldi et al., 2009; Cennamo et al., 2013; Chiozzi et al., 2017; Gesualdo et al., 2017; Monaco et al., 2014; Gesualdo and Monaco, 2015).

Quality of masonry. It is worth pointing out that Heyman’s model applies to structures composed of masonry elements having the quality of masonry, that is built with the rules of art and whose building blocks have a sufficient integrity. Indeed, masonry is not a bunch of blocks arranged randomly, but rather a collection of well-organised elements (bricks, stones, voussoirs) disposed in such a way to avoid sliding. There are essentially two tricks to obtain this goal: friction and interlocking.

Friction precludes sliding on planar joints where large compressive forces are present (horizontal planes in solid walls).

Interlocking prevents sliding on interfaces where there are feeble or no compressive forces (vertical planes in solid walls).

Such tricks are those allowing for the construction of vertical walls and pillars and guarantee the applicability of the theorems of limit analysis.

If we make stupid masonry structures (say a wall with vertical joints or with scarce interlocking) they may collapse even if a compressive equilibrium state exists, that is, for them, the theorems of limit analysis do not apply.

Heyman’s model. Once the quality of the structure, namely the correct execution of the masonry apparatus, is granted, it is possible to proceed to the analysis of such masonry structure as composed of macro-elements, on adopting a simplified model based on Heyman’s assumptions: masonry has no tensile strength, it is infinitely resistant in compression and does not slide along fracture lines. This model can catch the essential aspects of masonry behaviour and, overcoming the difficulties connected with the mechanical description of brittleness and friction (on introducing the no-tension/no-sliding assumptions), allows for the application of the theorems of limit analysis, bringing back the study of masonry structures within a consolidated framework.

2 Statics and kinematics of rigid no-tension bodies

In this section, the Heyman’s model will be generalised to 2d continua adopting a set of unilateral restrictions on stress and strain tensors.

A 2d masonry structure \( S \), is modelled as a continuum occupying the region \( \Omega \) of the Euclidean space \( \mathbb{E}^2 \). The stress inside \( \Omega \) is denoted \( T \) and the displacement of material points \( x \) belonging to \( \Omega \) is denoted \( u \). Restricting to the case of small displacements and strains, the infinitesimal strain \( E \) is adopted as the strain measure.

The so-called Normal Rigid No-Tension (NRNT) material is defined by the following restrictions:
\( T \in \text{Sym}^-, \ E \in \text{Sym}^+, \ T \cdot E = 0, \) \hspace{1cm} (1)

\( \text{Sym}^- , \text{Sym}^+ \) being the mutually polar cones of negative semidefinite and positive semidefinite symmetric tensors.

Restrictions (1) are equivalent to the following conditions, called normality conditions:

\[ T \in \text{Sym}^-, \ \left( T - T^* \right) \cdot E \geq 0, \ \forall \ T^* \in \text{Sym}^- , \] \hspace{1cm} (2)

and, dually, to the conditions, called dual normality conditions, listed below

\[ E \in \text{Sym}^+, \ \left( E - E^* \right) \cdot T \geq 0, \ \forall \ E^* \in \text{Sym}^+. \] \hspace{1cm} (3)

The restrictions defining the NRNT material in the particular form (2), are the essential ingredients for the application of the theorems of Limit Analysis (see Angelillo et al. (2014), Drei et al. (2016), Vanderbei (2015), Dantzig (2016), Arcidiacono et al. (2016), Mele et al. (2012), Dantzig and Thapa (2006), Dorn and Greenberg (1957), Milani et al. (2012) and Bertolesi et al. (2016)).

The following boundary value problem (BVP) represents the mathematical continuum formulation of the equilibrium of a two-dimensional masonry construction considered as a NRNT material subjected to assigned loads and settlements:

“Find a displacement field \( u \), the associated deformation \( E \), and a stress field \( T \) such that

\[ E = \frac{1}{2} \left( \nabla u + \nabla u^T \right), \ E \in \text{Sym}^+, \ u = \bar{u} \text{ on } \partial \Omega_D , \] \hspace{1cm} (4)

\[ \text{div} T + b = 0, \ T \in \text{Sym}^-, \ Tn = \bar{s} \text{ on } \partial \Omega_N , \] \hspace{1cm} (5)

\[ T \cdot E = 0^*, \] \hspace{1cm} (6)

where \( n \) is the unit outward normal to the boundary and \( \partial \Omega \), and \( \partial \Omega_D , \partial \Omega_N \) is a fixed partition of the boundary into the constrained and loaded parts.

On introducing the set \( K \) of kinematically admissible displacements, and the set \( H \) of the statically admissible stresses, defined as follows:

\[ K = \left\{ u \in S / \ E = \frac{1}{2} \left( \nabla u + \nabla u^T \right) \in \text{Sym}^+ \& \ u = \bar{u} \text{ on } \partial \Omega_D \right\} , \] \hspace{1cm} (7)

\[ H = \left\{ T \in S' / \text{div} T + b = 0, \ T \in \text{Sym}^-, \ Tn = \bar{s} \text{ on } \partial \Omega_N \right\} , \] \hspace{1cm} (8)

in which \( S, S' \) are two suitable function spaces, a solution of the BVP for NRNT materials, is the triplet \( \left( u^o, E \left( u^o \right), T^o \right) \) such that \( u^o \in K , \ T^o \in H \), and \( T^o \cdot E \left( u^o \right) = 0 \).

Concentrated strain and stress. For NRNT materials, it is possible to admit that strain and stress are bounded measures. Bounded measures can be decomposed into the sum of two parts (see Angelillo (2014)):

\[ E = E' + E^*, \ T = T' + T^* , \] \hspace{1cm} (9)
where $(\cdot)'$ is the part that is absolutely continuous with respect to the area measure (that is $(\cdot)'$ is a density per unit area) and $(\cdot)\delta$ is the singular part.

On admitting singular strains and stresses, it is possible to admit that both the displacement $u$ and the stress vector $S$ be discontinuous. The stress vector is the contact force transmitted across a surface of unit normal $n$, and, in Cauchy’s sense, is related to the regular part of the stress through the relation $s = T' n$.

**Displacement jumps.** If the displacement vector exhibits a jump discontinuity across a regular curve $\Gamma$, on such a curve the strain is concentrated, namely is a line Dirac delta whose intensity coincides with the value of the jump of $u$ across $\Gamma$ (see Figure 1).

**Figure 1** A representation of a generic displacement jump across $\Gamma$. The unit tangent and the unit normal to $\Gamma$ are denoted with $t$ and $n$ whilst $\Omega^-$, $\Omega^+$ are the two parts of the domain $\Omega$ having $\Gamma$ as common boundary (see online version for colours)

Denoting $t$, $n$ the unit tangent and the unit normal to $\Gamma$, and calling $\Omega^-$, $\Omega^+$ the two parts on the two sides of $\Gamma$, $\Omega^+$ being the part toward which $n$ points, the jump of $u$ on $\Gamma$ can be denoted as follows:

$$[u] = u^+ - u^-,$$

and decomposed into tangential and normal components:

$$[u] = w t + v n, \quad w = [u] \cdot t, \quad v = [u] \cdot n.$$

Denoting $\delta(\Gamma)$ the unit line Dirac delta with support on $\Gamma$, the concentrated strain on $\Gamma$, taking into account the relation defining the infinitesimal strain in terms of the displacement: $E = \frac{1}{2} (\nabla u + \nabla u^T)$, and the material restrictions on strains for NRNT materials, takes the form

$$E = v \delta(\Gamma) n \otimes n, \quad v \geq 0,$$
since, taking into account the restriction \( E \in \text{Sym}^+ \), it must be
\[ w = 0 \]
That is, the two parts \( \Omega^- , \Omega^+ \) may separate but cannot penetrate each other, and the sliding \( w \) along \( \Gamma \) must be zero.

**Stress vector jumps.** If the stress vector exhibits a jump discontinuity across a regular curve \( \Gamma \), on such a curve the stress is concentrated, namely is a line Dirac delta whose intensity \( P \) is related to the jump of \( s \) across \( \Gamma \). Recalling the definition introduced above for \( t, n \), on adopting the previous notation the jump of \( s \) across \( \Gamma \) can be denoted as follows
\[ [s] = s^+ - s^- , \] (14)
and decomposed into normal and tangential components
\[ [s] = \begin{bmatrix} p \ t + q \ n \end{bmatrix}, \quad p = [s] \cdot t, \quad q = [s] \cdot n . \] (15)

Denoting \( \delta(\Gamma) \) the unit line Dirac delta with support on \( \Gamma \), the stress concentrated on \( \Gamma \), taking into account the balance equations \( \text{div} \ T + b = 0 \), and the material restrictions for NRNT materials, takes the form (see Figure 2):
\[ T = P \ \delta(\Gamma) \ t \otimes t, \quad P + p = 0, \quad P \rho + q = 0, \quad P \leq 0 , \] (16)
where \( \rho \) is the curvature of the line \( \Gamma \) and \( P' \) is the derivative of \( P \) with respect to its argument, namely the arc length along \( \Gamma \).

**Figure 2** A representation of a singular stress field along a regular curve \( \Gamma \) (see online version for colours)

The intensity \( P \) of the concentrated stress represents a concentrated axial contact force acting along the 1d substructure \( \Gamma \). The last relation in equation (16) says that such a force must be compressive.

### 3 Energy based approach of the BVP for NRNT materials

The BVP for NRNT materials can be decomposed into two parts: the search of a displacement field belonging to \( \mathcal{K} \), and the search of a stress field belonging to \( \mathcal{H} \). The first problem is named kinematical problem (KP) and the second problem is called
equilibrium problem (EP). The two problems are coupled only through condition (6), and can be taken up independently.

If the solution of the BVP is attacked considering first the KP and taking as primal variable the displacement, then we say that a displacement approach is adopted. If, instead, the EP is considered first, by taking the stress as the primal variable, then we say that a force type approach is followed.

Compatibility of force and displacement data. First of all, it is to be pointed out that both the KP and the EP can be incompatible, in the sense that the displacement or the load data could be given in such a way that the set \( \mathcal{K} \), or the set \( \mathcal{H} \), are empty. In particular, the compatibility of the EP is an issue involved in the theorems of Limit Analysis. Such theorems, dealing with the possibility or the impossibility of collapse, can be viewed as follows: the safe theorem as a definition of compatible loads, and the kinematical theorem as an indirect way to assess the incompatibility of the loads.

Dealing with the solution of a BVP for the unilateral material that we consider with a displacement approach, under the preliminary assumption that both the KP and the EP are compatible and non-homogeneous, the problem arises of selecting, among the infinitely many admissible displacement fields, that (or those) to which a statically admissible stress field, such to satisfy the zero-dissipation condition (6), can be associated.

The idea is to seek, among all the kinematically admissible displacement fields, a possible solution of the BVP, by minimising the total potential energy of the system. For NRNT materials the total potential energy reduces solely to the potential energy \( \mathcal{P} \) of the given contact and body loads. Then the minimum problem can be formulated as follows:

"Find the displacement field \( \mathbf{u}^* \in \mathcal{K} \) such that

\[
\min_{\mathbf{u} \in \mathcal{K}} \mathcal{P}(\mathbf{u}) = \mathcal{P}(\mathbf{u}^*), \tag{17}
\]

Where

\[
\mathcal{P}(\mathbf{u}) = -\int_{\Omega_1} \mathbf{\bar{b}} \cdot \mathbf{u} \, ds - \int_{\Omega_2} \mathbf{b} \cdot \mathbf{u} \, da, \tag{18}
\]

is the potential energy of the given external loads".

Minimum of \( \mathcal{P} \) and equilibrium. The proof of existence of the minimiser \( \mathbf{u}^* \) of \( \mathcal{P}(\mathbf{u}) \) for \( \mathbf{u} \in \mathcal{K} \), is a complex mathematical question. Due to the poor regularity of the admissible functions, this proof requires sophisticated tools of mathematical analysis (see Giaquinta and Giusti (1985) and Anzellotti (1985)), and is well beyond the scopes of the present paper, devoted to more mechanical aspects of masonry equilibrium as in Cennamo et al. (2013) and Vanderbei (2015).

What it is possible to show, by using only elementary tools of calculus, by making the preliminary assumption that the settlements are compatible (\( \mathcal{K} \neq \emptyset \)), is that:

1. If the load is compatible (that is if \( \mathcal{H} \neq \emptyset \)), the functional \( \mathcal{P}(\mathbf{u}) \) is bounded from below.
2. If the triplet \( (\mathbf{u}^*, E(\mathbf{u}^*), T^*) \) is a solution of the BVP, it corresponds to a weak minimum of the functional \( \mathcal{P}(\mathbf{u}) \).
Proof.

- If the load is compatible, then there exists a stress field $T \in \mathcal{H}$, through which the functional $\wp(u)$, defined on $\mathcal{K}$, for each $u \in \mathcal{K}$, can be rewritten as follows

$$
\wp(u) = -\int_{\partial \Omega} \bar{\sigma} \cdot u \, ds - \int_\Omega b \cdot u \, da - \int_{\partial \Omega} s(T) \cdot \bar{u} ds - \int_\Omega T \cdot E(u) \, da,
$$

in which $s(T)$ denotes the trace of $T$ at the boundary. On assuming that the displacement assigned on the constrained part of the boundary are sufficiently regular (say continuous), since $s(T)$ is a bounded measure, the integral $\int_{\partial \Omega} s(T) \, ds$ is finite; then, since $T \in \text{Sym}^-$ and $E \in \text{Sym}^+$, the volume integral term in the right hand side of (19) is non-negative and $\wp(u)$ is bounded from below.

- If $(u^e, E(u^e), T^e)$ is a solution of the BVP, then, for any $u \in \mathcal{K}$, one can write:

$$
\wp(u) - \wp(u^e) = -\int_{\partial \Omega} \bar{\sigma} \cdot (u - u^e) \, ds - \int_\Omega b \cdot (u - u^e) \, da = \int_\Omega T^e \cdot (E(u) - E(u^e)) \, da.
$$

The result: $\wp(u) - \wp(u^e) \geq 0$, $\forall u \in \mathcal{K}$, follows from the dual normality condition (3).

The physical interpretation of the previous result is the following. Since the displacement solving the BVP corresponds to a state of weak minimum for the potential energy, then it is, at least, a state of neutral equilibrium (that is not unstable), in the sense that the transition to a different state requires a non-negative supply of energy.

**Remark 1:** On the basis of the minimum principle, if the EP is compatible and the KP is homogeneous, then the constant displacement field: $u = 0$ is a possible minimiser. Indeed, in this case, one can write:

$$
\wp(u) = -\int_{\partial \Omega} \bar{\sigma} \cdot u \, ds - \int_\Omega b \cdot u \, da = -\int_\Omega T \cdot E(u) \, da,
$$

$T$ being a generic element of $\mathcal{H}$, certainly existing since, by assumption, $\mathcal{H}$ is not empty. Since the last member of (21) is always non-negative, then $\wp(0) = 0$ is the infimum of $\wp$, and the displacement field $u = 0$ is a minimiser of the potential energy.

In this case, any $T \in \mathcal{H}$ is a legitimate solution in terms of stress, since $T \cdot E(0)$ for any $T$.

Dealing with the solution of a BVP for the unilateral material that we consider, with a force approach, again under the preliminary assumption that both the KP and the EP are compatible and non-homogeneous, the problem arises of selecting, among the infinitely many admissible stress fields, that (or those) to which a kinematically admissible displacement field, such to satisfy the zero-dissipation condition (6), can be associated.
The idea is to seek, among all the statically admissible stress fields, a possible solution of the BVP, by minimising the complementary energy of the system. For NRNT materials the form of complementary energy to be minimised is the sole complementary energy $\mathcal{P}_c$ associated to the given settlements.

Then the minimum problem can be formulated as follows:

"Find the stress field $T^\circ \in \mathcal{H}$ such that

$$\mathcal{T}(T^\circ) = \min_{T \in \mathcal{H}} \mathcal{T}(T)^\circ,$$

(22)

in which

$$\mathcal{T}(u) = -\int_{\partial \Omega_0} \mathbf{s}(T) \cdot \mathbf{a} \, ds,$$

(23)

is the complementary energy associated to the given settlements.

Minimum of $\mathcal{P}_c$ and compatibility. Leaving to more mathematical works the proof of existence of the minimum $T^\circ$ di $\mathcal{T}(T)$ per $T \in \mathcal{H}$, what can be easily shown, on assuming preliminarily that the EP is compatible (i.e., $\mathcal{H} \neq \emptyset$), is that:

- If the settlements are compatible (that is if $\mathcal{K} \neq \emptyset$), the functional $\mathcal{T}(T)$ is bounded from below.
- If the triplet $(u^\circ, E(u^\circ), T^\circ)$ is a solution of the BVP, then it corresponds to a weak minimum of the functional $\mathcal{T}(T)$.

Proof

- If the given settlements are compatible, then there exists a displacement $u \in \mathcal{K}$, through which the functional $\mathcal{T}(T)$, defined on $\mathcal{H}$, for any $T \in \mathcal{H}$, can be rewritten as follows

$$\mathcal{T}(T) = -\int_{\partial \Omega_0} \mathbf{s}(T) \cdot \mathbf{a} \, ds = \int_{\partial \Omega} \mathbf{a} \cdot \mathbf{u} \, ds + \int_{\Omega} \mathbf{b} \cdot \mathbf{u} \, da - \int_{\Omega} T \cdot E(u) \, da,$$

(24)

In which $\mathbf{s}(T)$ denotes the trace of $T$ along the boundary. Taking into account that $u$ is a function of Bounded Variation, on assuming that the given surface and body loads have enough regularity (say continuity) for the first two products in the last member of (24) to be summable, the integral $\int_{\partial \Omega} \mathbf{a} \cdot \mathbf{u} \, ds + \int_{\Omega} \mathbf{b} \cdot \mathbf{u} \, da$ is finite; therefore, since $T \in \text{Sym}^-$ and $E \in \text{Sym}^+$, the volume term in the last member of (24) is non-negative, then $\mathcal{T}(T)$ is bounded from below.

- If $(u^\circ, E(u^\circ), T^\circ)$ is a solution of the BVP, then, for any $T \in \mathcal{H}$, one can write:

$$\mathcal{T}(T) - \mathcal{T}(T^\circ) = -\int_{\partial \Omega_0} (\mathbf{s}(T) - \mathbf{s}(T^\circ)) \cdot \mathbf{a} \, ds = \int_{\Omega} (T - T^\circ) \cdot E(u^\circ) \, da.$$

(25)

The result $\mathcal{T}(T) - \mathcal{T}(T^\circ) \geq 0$, $\forall T \in \mathcal{H}$, follows from the normality condition (2).
4 Example 1: a simple benchmark case

The two energy criteria introduced above provide tools for predicting the response of a structure, subject to non-vanishing loads and settlements. To illustrate the proposed method, we consider the solution of the trivial benchmark problem described in Figure 3(a). The problem is concerned with the non-homogeneous equilibrium and kinematical problem for a lintel, an element that can be considered as made up by many, conveniently arranged, small blocks (stones or bricks) or by a single monolithic piece. As shown in Figure 3(a), the lintel is loaded along the upper side by uniformly distributed forces and suffers a symmetrical horizontal settlement of the two lateral constraints.

An example of a similar structural element is shown in Figure 3(b). This kind of lintels are often visible in the front façade of old Churches and Palaces, and is not rare to see that they are cracked in correspondence of their middle section.

The cracks which are visible in such real lintels are invariably produced by relative movements of the supporting structures. For such real structures, the load is different from that considered in Figure 3(a), being actually the self-load of the lintel and a surface contact load transmitted by the hanging masonry. The last mentioned load is undoubtedly different from a uniform load, whilst, if the lintel does not crack horizontally, the self-load can be transferred, by means of feeble tensile normal stresses, to the upper side of the lintel, becoming a uniformly distributed load.

Figure 3 Masonry panel (lintel/wall beam) under given loads and settlements (a) and stone lintel, cracked near the middle section due to a relative displacement of the abutments (b) (see online version for colours)

A possible solution. In Figure 4, a possible solution of the BVP described in Figure 3(a), is reported graphically. In Figure 4(a), a compatible displacement field and in Figure 4(b) a balanced stress field are reported. The displacement considered in Figure 4(a) is a piecewise rigid displacement with support on two rectangles, symmetrical with respect to the vertical axis \( x_2 \), and consists of a rigid block mechanism articulated on the three hinges indicated with dots in Figure 4(b). The strain associated to such a compatible displacement is concentrated on three vertical lines of fracture, and is represented by uniaxial deformations of pure detachment, linearly variable along such lines. The stress field represented in Figure 4(b) consists of singular and regular parts. The regular part of stress, that is a uniaxial compressive stress directed vertically, produces a jump discontinuity of the stress vector across the curve represented with a solid black line in
Figure 4(b). Such a discontinuity is balanced by the singular part of the stress, a concentrated uniaxial stress having the form (16)^1, tangent to such a curve, and satisfying the balance equations (16)^2, (16)^3: a concentrated axial stress of intensity \( P \), acts along the curve, forming a so-called line of thrust. Such a line of thrust transmits concentrated forces to the hinges, producing a thrust force at the boundary and a horizontal force at the key hinge. The internal work of the statically admissible stress field depicted in Figure 4(b), for the strain associated to the displacement field represented in Figure 4(b), is zero, therefore these two fields represent a possible solution of the BVP.

**Solution derived through the energy formulations.** The solution of the trivial problem shown in Figure 3, is here reconsidered through the energy approach, as a way to show, on a simple example, how the energy criteria introduced in Section 3 can be used to generate approximate strategies to solve equilibrium problems for masonry-like structures. The analysis of this simple problems allows also to enlighten, within an easy context, some peculiar characteristics of the two approaches.

Figure 4  A possible kinematically admissible displacement field (a) and a possible statically admissible stress field (b), for the BVP depicted in Figure 3(a) (see online version for colours)

By attacking the problem with a displacement approach, an elementary approximation of the KP is obtained by considering a piecewise rigid displacement with support on two rectangular blocks. Restricting to the case in which the two blocks are fixed and coincide with the two rectangles obtained by dividing the domain along the vertical symmetry axis, the unknowns of the problem reduce to the six rigid displacement parameters of the two blocks. Choosing as poles of the rigid displacement the lower external vertices (A, C) of the two blocks, and restricting to symmetrical displacements, the kinematical conditions at the internal and boundary interfaces, accounting for the material kinematical restrictions, can be written, in terms of the components of horizontal translation \( u \), vertical translation \( v \) and rotation \( \phi \) about the pole \( A \) (say \( C \) for the assumed symmetry), as follows:

\[
\begin{align*}
v &= 0 , \ y \geq 0 , \ u \leq \eta , \ u - \phi L \geq 0 , u - \phi L &\leq \eta.
\end{align*}
\]

The potential energy (18), taking into account equation (26)^1, reads

\[
\phi(u, \phi) = -\frac{PL^2}{2} \phi. \tag{27}
\]
The minimisation problem (17) reduces, in this approximated context, to a trivial minimisation problem for the linear function (22) under the side linear constraints (26). Such a problem can be represented graphically as shown in Figure 5. From Figure 5(a), it is deduced that the values \( u = \eta \) and \( \phi = \eta / L \) correspond to the minimum of the potential energy. Such values correspond to the mechanism depicted in Figure 4(a).

For what concerns the force approach, restricting to stress fields of the type represented in Figure 4(b), namely fields composed of a regular uniaxial part and of a singular part with support on a curve passing through three points \( A, B, C \), the complementary energy can be written in terms of the position of these three points. Restricting to symmetric curves, denoting \( y_A, y_B \) the variable y-coordinates of the points \( A (A') \) and \( B \) having fixed abscissae \( L (–L) \) and 0 respectively, taking into account equations (16)\(^2\), (16)\(^3\), one obtains:

\[
\rho C (y_A, y_B) = p L^2 / 2 (y_B - y_A)
\]

(28)

**Figure 5** In (a): contour plot of the potential energy \( \rho (u, \phi) = -(pL^2 / 2) \phi \) normalised with respect to \( pL\eta / 2 \) as a function of the normalised variables \( u^o = u / \eta \) (horizontal axis), \( \phi = \phi L / \eta \) (vertical axis). From the graph it is deduced that the minimiser is \( u = \eta \) and \( \phi = \eta / L \), corresponding to the value of energy \( -pL\eta / 2 \). In (b): plot of the complementary energy as a function of \( y_B \) for various values of \( y_A \) (from left to right: \( y_A = \{0.0, 0.1, \ldots, 0.9\} \)). The positions \( y_A, y_B \) are normalised with respect to \( L \) and the value of energy is normalised with respect to \( pL \). From the graph it is deduced that the minimiser is \( y_A = 0.0 \), \( y_B = L \), corresponding to a value of complementary energy of \( pL / 2 \) (see online version for colours).

Such a function has to be minimised under the condition that the y-coordinates belong to the interval \([0, L]\) and with the constraint \( y_B > y_A \), ensuring that the axial force is finite and compressive. From the graph of Figure 5(b) it is deduced that the minimiser is \( y_A = 0 \), \( y_B = L \), corresponding to a value of the complementary energy of \( pL / 2 \). In conclusion, both the displacement and force approaches indicate as the minimal solution the same state, that is the one depicted in Figure 4(a) and (b).
5 Rigid block model: a numerical approximation of the displacements

The approximate solution of the minimum problem (17) generated by restricting the search for the minimum to the restricted class $K_{pr}$ of piecewise rigid displacements, is considered. This infinite dimensional space is discretised on considering the partition

$$(\Omega_i)_{i=1}^{\omega}$$

of the domain $\Omega$ into a finite number $M$ of rigid pieces, such that

$$\sum_{i=1}^{M} P(\Omega_i) < \infty,$$

$P(\Omega_i)$ denoting the perimeter of the piece $\Omega_i$. In particular, on restricting to polygonal elements, the boundary $\partial \Omega_i$ of $\Omega_i$, is composed by segments $\Gamma$, of length $\ell$, whose extremities are denoted 0,1 (see also Figure 1).

The segments $\Gamma$, which are either the common boundary between elements or between elements and the constrained part of the boundary, are called “interfaces”.

On denoting $K_{pr}^M$ the finite dimensional approximation of $K_{pr}$ generated by the partition (29), the following minimum problem is considered

$$\min_{u \in K_{pr}^M} \varphi(u).$$

To represent a generic displacement $u \in K_{pr}^M$, one can consider the vector $U$ of $3M$ components represented by the $3M$ parameters of rigid displacement of translation and rotation of the $M$ elements. These parameters are restricted by the assumption that the strain must be positive semidefinite. For piecewise rigid displacements the deformation is concentrated on the interfaces (that is on the segments $\Gamma$), and, recalling (12), assumes the form:

$$E = \nu \delta(\Gamma) \cdot n \otimes n,$$

where

$$\nu = [a] \cdot n \geq 0.$$

Therefore, on the segment $\Gamma$, besides the unilateral restriction (33), also the following condition

$$w = [a] \cdot t = 0,$$

must be enforced.

Notice that conditions (33), (34), descending from the normality assumption (2), represent unilateral conditions of contact without sliding on the interfaces $\Gamma$.

The static counterpart of (33), (34) concerns the stress vector $s$ acting on the interfaces (both the internal and the external). Such a stress vector represents the reaction associated to the constraints (33) and (34), transmitted among the blocks and among the
blocks and the soil. The stress vector coincides with the given tractions \( \mathbf{\tau} \), where the boundary of the blocks represents the loaded boundary. On denoting

\[
\sigma = [\mathbf{s}] \cdot \mathbf{n} , \quad \mathbf{\tau} = [\mathbf{s}] \cdot \mathbf{t} ,
\]

(35)

The normal and tangential components of the stress vector along \( \Gamma \), the condition to be imposed on \( \mathbf{s} \) is

\[
\sigma \leq 0.
\]

(36)

Notice that the tangential component \( \mathbf{\tau} \) of \( \mathbf{s} \) is not restricted and can be non-zero also if on \( \Gamma \) one has: \( \sigma = 0 \).

Denoting \( N \) the total number of interfaces \( \Gamma \), and \( v(0), v(1), w(0), w(1) \) the normal and tangential components of the relative displacement across \( \Gamma \), of the ends 0, 1 of the segment itself, the restrictions (33), (34) are equivalent to the 2\( N \) inequalities

\[
v(0) \geq 0, \quad v(1) \geq 0,
\]

(37)

And to the 2\( N \) equalities

\[
w(0) = 0, \quad w(1) = 0.
\]

(38)

The restrictions (37), (38) can be expressed in terms of the components of \( \mathbf{U} \), and rewritten in the matrix forms

\[
\mathbf{A} \mathbf{U} \geq 0,
\]

(39)

\[
\mathbf{B} \mathbf{U} = 0.
\]

(40)

Finally, the minimum problem (26), approximating the minimum problem (17), takes the form

\[
\rho(\mathbf{U}) = \min_{\mathbf{U} \in \mathbb{R}^M} \rho(\mathbf{U}),
\]

(41)

in which \( \mathbb{R}^M \) is the set

\[
\mathbb{R}^M = \{ \mathbf{U} \in \mathbb{R}^{3M} / \mathbf{A} \mathbf{U} \geq 0, \quad \mathbf{B} \mathbf{U} = 0 \}.
\]

(42)

**Remark 2:** The minimisation problem (41) here proposed to approximate the minimum problem (17), transforms the minimum problem (17), formulated for a continuum, into a minimisation problem for a structure composed by a finite number of rigid elements in mutual unilateral contact among each other.

Problem (41) is a standard problem of Linear Programming, since the function \( \rho(\mathbf{U}) \) depends linearly on the 3M-vector \( \mathbf{U} \) and the side constraints are linear. The existence of the solution of the approximate problem is trivially assured, if the exact functional is bounded from below (see (19)). For a limited number of variables (say less than \( 10^3 \)) the problem can be solved effectively with the simplex method, see Dantzig et al. (1955) and Mehrotra (1992), for larger size problems (up to \( 10^6/10^7 \) unknowns and sparse matrices.
A and B there exist a number of efficient approximation alternatives (Dorn and Greenberg, 1957).

6 Example 2: The displacement piecewise-rigid method in a real case

The numerical method based on the displacement approach proposed to approximate the BVP for NRNT masonry materials, implemented with the program of symbolic calculus Mathematica (Wolfram, 2003), is here applied to a simple example in order to simulate the fracture pattern in the real old building shown in Figure 6(a). This masonry construction, due to an evident foundation settlement, exhibits a widespread cracking on its main façade (see Figure 6(a)).

In the simulation, the main façade, loaded by its self-weight, is discretised into 7364 triangular elements (Figure 6(b)). The effect of a differential settlement of the right part of the foundation is considered. The analysis is restricted to the right part of the structure (on the right side of the vertical line depicted in Figure 6(b)), represented in Figure 7(a). The piecewise rigid displacement field (Chambolle et al., 2007), with support on the triangular elements, produced by the settlement is obtained by minimising the potential energy \( \rho(U) \) with respect to the generalised displacement \( \hat{U} \), as described in Section 5, see Iannuzzo et al., 2017; Cundall, 1971; Sarhosis et al., 2016; Sarhosis et al., 2009; Simon and Bagi, 2016; Drei et al., 2016).

Figure 6 Façade of a XVII century building in Bergamo, presenting an extensive cracking due to an evident differential settlement of the foundation. In (a) front view and crack pattern; in (b) discretisation of the wall into triangular elements

Source: Redrawn from the site of the Fireworkers of Bergamo. Courtesy of Paolo Faccio
Due to the high number of elements and conditions, to solve numerically problem (41), the approximate Linear Programming method known as ‘Interior-Point’ method, has been adopted.

The optimal profile of the given settlement, controlled by three parameters, was obtained by executing a parametric analysis for various runs of the program on the grid of possible values of the parameters. In Figure 7, the solution corresponding to such optimal profile is shown for comparison, side by side, with the drawing of the real crack pattern. To appreciate the accuracy of the numerical solution, the deformed shape obtained numerically and shown in Figure 7, and the real deformed configuration, are compared in Figure 8. By looking at the results of the numerical analysis reported in Figure 7, it is evident that the solution that we obtain predicts the subdivision of the structure into a small number of macro-blocks. A clever way to identify such blocks is to make a colour map of the rotation, a piecewise constant function with support on the individual elements of the discretisation. Regions of uniform colour correspond to zones of constant rotation and, on the deformed configuration where fractures are clearly visible, identify rigid macro-blocks. For comparison, such a colour map is superimposed in transparency over the image of the real crack pattern in Figure 8. A good agreement between real and simulated macro-blocks and fractures is detected.

**Figure 7** Mechanism produced by a foundation settlement in the discretised structure of Figure 6(b). The real crack pattern and the deformed shape obtained as a result of the analysis are shown on above the other
Figure 8 Mechanism produced by a foundation settlement in the discretised structure of Figure 6(b). In the first image the movement of the individual blocks composing the discretised structure is depicted. In the second image, the real and the simulated deformed configurations are shown superimposed one on top of the other. From this second picture we can easily identify the rigid macro-blocks which form as a result of the numerical analysis. The colour map measures the rotation, then a uniform colour means constant rotation, a useful element to identify rigid macro-blocks (see online version for colours)

7 Conclusions

In the present work masonry structures are modelled as continua composed of Normal Rigid No-Tension (NRNT) material. The NRNT material represents an extension of the model material of Heyman to 2d/3d continua. The material is rigid in compression, but extensional deformations (representing fracture), allowed at zero energy price, can be either regular or singular; then extensional deformation can appear as either diffuse (smeared cracks) or concentrated (macroscopic cracks), and there is not any reason to prefer one upon another, on an energy ground. The fact that rigid block deformation seems to be the preferred failure mode for real masonry structures stems from mechanical
characteristics, such as toughness, interlocking, finite friction and cohesion, that are not inherent to the simplified NRNT continuum model.

There exists an extensive literature on discrete element approximation of real masonry structures (see e.g., Forgács et al., 2017 or the recent book (Sarhosis et al., 2016) may be consulted for reference) and many studies proposing macro-block analyses (see for example Angelillo et al. (2014) and references therein).

What is interesting to see here, is that rigid macro-block mechanisms arise naturally in solving the equilibrium problem for NRNT materials. The equilibrium problem is formulated as an energy minimum search, and two numerical methods for approximating the solution could be used (see Iannuzzo, 2018)). With the first method, the one which is exploited in the present paper, the energy is minimised in the set of piecewise-rigid (PR) displacements. With the second method (not considered here) the possibility to restrict the search of the minimum to continuous (C°) displacement fields, by adopting some classical Finite Element (FE) approximation, is considered.

The benchmark problems here analysed to illustrate the numerical performances of the first approach, show that the subdivision into macro-blocks can arise naturally in solving the minimisation problem and that the subdivisions into macro-blocks predicted by our analysis is in good.

References


Kooharian, A. (1952) Limit Analysis of Voussoir (Segmental) and Concrete Archs, Journal of the American Concrete Institute, Vol. 24, No. 4, pp.317–328.


