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Dynamics of superattracting skew products on the attracting basins: Böttcher coordinates and plurisubharmonic functions

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Abstract: We study the dynamics of a superattracting skew product f on the attracting basin. As the first strategy, we find out forward f -invariant wedge-shaped regions in the basin, on some of which f is conjugate to monomial maps, and consider whether the unions of all the preimages of the regions coincide with the basin. As the second strategy, we show the existence and properties of several kinds of plurisubharmonic functions of f , the main functions of which are induced from the Böttcher coordinates, and investigate the asymptotic behaviour of the functions toward the boundaries of the unions. Consequently, we obtain a plurisubharmonic function on the complement of specific fibres in the basin, which is continuous and pluriharmonic on open and dense subsets of the complement and describes an certain weighted vertical dynamics well.

Keywords: complex dynamics; skew products; superattracting fixed points; Böttcher coordinates; plurisubharmonic functions.

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Biographical notes: Kohei Ueno received his PhD from Kyoto University in 2008. He worked at Toba National College of Maritime Technology in 2010–2014, and works at Daido University since 2014.

1 Introduction

1.1 Background

The dynamics around superattracting fixed points of holomorphic functions is completely understood in dimension one. Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with a superattracting fixed point at the origin: $p(z) = az^\delta + O(z^{\delta+1})$, where $a \neq 0$ and $\delta \geq 2$. Then Böttcher's theorem Böttcher (1904) indicates that the limit

$$\varphi_p(z) = \lim_{n \rightarrow \infty} p_0^{-n} \circ p^n(z)$$

exists on a small neighbourhood of the origin, where p^n is the n^{th} iterate of p , $p_0(z) = az^\delta$ and the branch is taken such that $p_0^{-n} \circ p_0^n = \text{id}$. This limit is called the Böttcher coordinate for p at the origin, and it conjugates p to p_0 . We refer to Milnor (2006) for details. We remark that the order of p^n at the origin is clearly δ^n , which is also called the attraction rate of p^n . Let A_p be the attracting basin of p at the origin, and E_p the union of all the preimages of the origin. We can extend φ_p from the neighbourhood to A_p until it meets the other critical points than the origin. On the other hand, the limit

$$G_p(z) = \lim_{n \rightarrow \infty} \frac{1}{\delta^n} \log |p^n(z)|$$

exists on A_p . It is subharmonic and negative on A_p and harmonic on $A_p - E_p$. Note that $G_p(z) = \log |\varphi_p(z)| + (\delta - 1)^{-1} \log |a|$ on the neighbourhood of the origin. We remark that similar statements hold even if p is a holomorphic germ.

Several studies have been made toward the generalisation of Böttcher's theorem to higher dimensions. For example, Ushiki (1992), Ueda (1993) and Buff et al. (2012) studied the case in which holomorphic germs, with superattracting fixed points, have the Böttcher coordinates on neighbourhoods of the points. The germs in Ushiki (1992) are conjugate to monomial maps, whereas the germs in Ueda (1993) and Buff et al. (2012) are conjugate to homogeneous and quasihomogeneous maps, respectively.

However, Böttcher's theorem does not extend to higher dimensions entirely as pointed out by Hubbard and Papadopol (1994). Let f be a two dimensional holomorphic germ with a superattracting fixed point at the origin. Unlike the one dimensional case, in which a superattracting fixed point of a holomorphic germ is an isolated critical point and forward invariant under the germ, the origin is contained in the critical set of f , which may have a singularity at the origin and is not forward invariant under f in general. Hence it is difficult or impossible to take a nice normal form of f on a neighbourhood of the origin.

Rigidity is a keyword for the study of the local dynamics of superattracting germs. We say that f is rigid if the critical set is contained in a totally invariant set with normal crossing singularities. Favre (2000) introduced and classified attracting rigid germs. Favre and Jonsson (2007) proved that any two dimensional superattracting germ can be blown up to a rigid germ, and hence the original germ is rigid on an open set whose closure contains the fixed point. Using the normal form on the open set, they proved that the asymptotic attraction rate is a quadratic integer, gave an inequality on the attraction rates, and constructed a dynamically nice plurisubharmonic function defined on a neighbourhood of the fixed point. Ruggiero (2012) extended their results to more general cases.

We restrict our attention to superattracting skew products in this paper. A skew product is a map or germ of the form $f(z, w) = (p(z), q(z, w))$ defined on somewhere in $\mathbb{C}^2$. The fundamental properties of the dynamics of polynomial skew products is well studied and summarised in Jonsson (1999) and Favre and Guedj (2001), in which the main topics are the Green functions, currents and measures. Lilov (2004) studied the local and semi-global dynamics of holomorphic skew products near a superattracting invariant fibre. As natural extensions of the one dimensional results, he obtained nice normal forms on neighbourhoods of periodic points which are geometrically attracting, parabolic and Siegel on fibre direction, except the superattracting case. See also Peters and Smit (2018) and Peters and Raissy (2019) for the dynamics of skew products near an invariant fibre of different types. Whereas the dynamics of skew products is mild in the

sense that it can be regarded as the intermediate between the one and two dimensional dynamics, it has very complicated aspects and gives new phenomena of the dynamics in dimension two. For example, Astorg et al. (2016) found polynomial skew products with wandering domains, which is the first example of non-invertible polynomial maps with wandering domains. See also Astorg et al. (2023) and Astorg and Boc-Thaler (2024) for other types of polynomial skew products with wandering domains. Furthermore, polynomial skew products are used in Dujardin (2017) and Taflin (2021) to construct robust bifurcations, i.e., open sets contained in a bifurcation locus.

1.2 Previous results: Böttcher coordinates

We have studied the dynamics of superattracting skew product germs and succeeded in constructing Böttcher coordinates on invariant wedge-shaped regions in Ueno (2016b), Ueno (2019a) and Ueno (2019b); we also studied the semi-global dynamics in Ueno (2019a) in order to deal with indeterminacy points instead of superattracting fixed points. Let us briefly review main results in Ueno (2019b), which are a consequence of the studies in Ueno (2016b) and Ueno (2019a). Let $f(z, w) = (p(z), q(z, w))$ be a skew product germ with a superattracting fixed point at the origin. More precisely, we assume that the eigenvalues of $Df(0)$ are both zero; this assumption is called nilpotent in Ruggiero (2012), whereas the assumption $Df(0) = 0$ is imposed in Favre and Jonsson (2007). Hence we may write $p(z) = az^\delta + O(z^{\delta+1})$ and $q(z, w) = bz + \sum_{i,j \geq 0, i+j \geq 2} b_{ij}z^i w^j$, where $a \neq 0$ and $\delta \geq 2$. Let $bz = b_{10}z^1 w^0$ and $q(z, w) = \sum b_{ij}z^i w^j$ for short.

It is clear that the dominant term of p is az^δ . We found in Ueno (2019b) that the order δ of p and the Newton polygon of q assign a term $b_{\gamma d} z^\gamma w^d$ of q that is dominant on a forward f -invariant wedge-shaped region U , where $U = U_r = \{|z|^{l_1+l_2} < r^{l_2}|w|, |w| < r|z|^{l_1}\}$ for some rational numbers $0 \leq l_1 < \infty$ and $0 < l_2 \leq \infty$. The study depends on where δ falls in a partition of $\mathbb{R}_{\geq 0}$ given by a finite decreasing sequence $\{T_k\}$, defined from the Newton polygon of q . Let $f_0(z, w) = (az^\delta, b_{\gamma d} z^\gamma w^d)$.

Lemma 1.1 (Ueno, 2019b): If $d \geq 2$ or if $d = 1$ and $\delta \neq T_k$ for any k , then $f \sim f_0$ on U_r as $r \rightarrow 0$, and $f(U_r) \subset U_r$ for small $r > 0$.

The notation $f \sim f_0$ on U_r as $r \rightarrow 0$ means that the ratios of the first and second components of f and f_0 tends to 1 on U_r as $r \rightarrow 0$, and the definition of T_k is given below. Moreover, we proved that the limit

$$\phi(z, w) = \lim_{n \rightarrow \infty} f_0^{-n} \circ f^n(z, w)$$

exists on U , where the branch is taken such that $f_0^{-n} \circ f^n = id$.

Theorem 1.2 (Ueno, 2019b): If $d \geq 2$ or if $d = 1$ and $\delta \neq T_k$ for any k , then ϕ is biholomorphic on U_r for small $r > 0$, which conjugates f to f_0 . Moreover, $\phi \sim id$ on U_r as $r \rightarrow 0$.

We call ϕ the Böttcher coordinate for f on U , although the second component of ϕ looks like the Kœnigs coordinate if $d = 1$.

Let us give the precise definitions in the results. We define the Newton polygon $N(q)$ of q as the convex hull of the union of $D(i, j)$ with $b_{ij} \neq 0$, where $D(i, j) = \{(x, y) | x \geq i, y \geq j\}$. Let $(n_1, m_1), (n_2, m_2), \dots, (n_s, m_s)$ be the vertices of $N(q)$, where $n_1 < n_2 < \dots < n_s$ and $m_1 > m_2 > \dots > m_s$. Let T_k be the y -intercept of the line L_k passing through the vertices (n_k, m_k) and (n_{k+1}, m_{k+1}) for each $1 \leq k \leq s-1$.

Case 1 If $s = 1$, then $N(q)$ has the only one vertex, which is denoted by (γ, d) . We define $l_1 = l_2^{-1} = 0$ and so $U = \{|z| < r, |w| < r\}$.

The results for Case 1 are classical and the proofs are simpler than the other cases. We remark that f in Case 1 is rigid and belongs to class 6 or class 4 in Favre (2000) if $d \geq 2$ or $d = 1$, respectively, and a description on Böttcher coordinates is given in pp.498–499 in Favre (2000).

Therefore, we only deal with the case $s > 1$ hereafter, which is divided into the following three cases:

Case 2 If $s > 1$ and $\delta \leq T_{s-1}$, then we define

$$(\gamma, d) = (n_s, m_s), \quad l_1 = \frac{n_s - n_{s-1}}{m_{s-1} - m_s} \quad \text{and} \quad l_2^{-1} = 0.$$

$$\text{Hence } U = \{|z| < r, |w| < r|z|^{l_1}\}.$$

Case 3 If $s > 1$ and $T_1 \leq \delta$, then we define

$$(\gamma, d) = (n_1, m_1), \quad l_1 = 0 \quad \text{and} \quad l_2 = \frac{n_2 - n_1}{m_1 - m_2}.$$

$$\text{Hence } U = \{|z|^{l_2} < r^{l_2}|w|, |w| < r\} = \{r^{-l_2}|z|^{l_2} < |w| < r\}.$$

Case 4 If $s > 2$ and $T_k \leq \delta \leq T_{k-1}$ for some $2 \leq k \leq s-1$, then we define

$$(\gamma, d) = (n_k, m_k), \quad l_1 = \frac{n_k - n_{k-1}}{m_{k-1} - m_k} \quad \text{and} \quad l_1 + l_2 = \frac{n_{k+1} - n_k}{m_k - m_{k+1}}.$$

$$\text{Hence } U = \{|z|^{l_1+l_2} < r^{l_2}|w|, |w| < r|z|^{l_1}\} = \{r^{-l_2}|z|^{l_1+l_2} < |w| < r|z|^{l_1}\}.$$

We remark that if $\delta > T_1$ and $d = 1$, then f is rigid and belongs to class 4 in Favre (2000) and hence f is conjugate to f_0 not only on U but also on a neighbourhood of the origin; see pp.44–45 in Ueno (2019b) for a detail.

Note that if $\delta = T_k$ for some k , then there are two different dominant terms of q , which correspond to the vertices (n_k, m_k) and (n_{k+1}, m_{k+1}) , whereas there is the only one dominant term if $\delta \neq T_k$ for any k . Moreover, if $\delta = T_k$ for some k and $m_{k+1} \geq 2$, then we have two disjoint invariant regions on which f is conjugate to each of the two different monomial maps.

Roughly speaking, we dealt with Case 2 in Ueno (2016b) and Case 3 in Ueno (2019a), respectively, without using the Newton polygon. We finally succeeded in Ueno (2019b) to construct the Böttcher coordinate for Case 4 and to unify all the cases in terms of the Newton polygon. Using the same idea and arguments as in Ueno (2019b), we gave inequalities on the attraction rates in Ueno (2024), which imply that the asymptotic attraction rate is δ if $\gamma > 0$ or $\delta \leq d$, and is d if $\gamma = 0$ and $\delta > d$.

1.3 New results: weighted plurisubharmonic functions

We assume that f is defined on $\{|z| < R\} \times \mathbb{C}$ for large R and that A_p is relatively compact in $\{|z| < R\}$, and focus on the dynamics of f on $A_p \times \mathbb{C}$. We remark that similar statements hold even if f is a germ. However, we require f to be defined semi-globally in order to study the dynamics on the whole attracting basin, which may be unbounded, and the asymptotic behaviour of plurisubharmonic functions toward the boundary of the union of all the preimages of U . We also assume that $a = b_{\gamma d} = 1$ for the study on plurisubharmonic functions in this paper, whereas we do not assume it for the study on invariant wedges and the union of all the preimages of the wedges. This assumption is not essential if $d \geq 2$ because the map $(z, w) \rightarrow (az^\delta, b_{\gamma d} z^\gamma w^d)$ is affine conjugate to the map $(z, w) \rightarrow (z^\delta, z^\gamma w^d)$. On the other hand, when $d = 1$, it seems that the case $a = b_{\gamma d} = 1$ is most interesting.

We give a summary of main results for Cases 2, 3 and 4, respectively, and Tables 1, 2 and 3 as comparison charts of the results for Cases 2, 3 and 4, respectively, in the next section. Although those results are important, there are so many of them and it is required to introduce the intervals $\mathcal{I}_f^1$ and $\mathcal{I}_f^2$ instead of l_1 and l_2 , which are combined into the rectangle $\mathcal{I}_f$ in $[0, \infty]^2$. Therefore, we only give three consequences of the main results in this subsection, which are separated into the case $\delta < T_{s-1}$ and the case $\delta \geq T_{s-1}$.

Let us first state two consequences of the main results for the case $\delta < T_{s-1}$. Let A_f be the union of all the preimages of U and A_0 the attracting basin. If $\delta < T_{s-1}$, then A_f almost coincides with A_0 . Let E_z and E_w be the unions of all the preimages of $\{z = 0\}$ and $\{w = 0\}$, respectively, and $E = E_z \cup E_w$.

Theorem 1.3: If $\delta < T_{s-1}$, then $A_f = A_0 - E_z$.

To describe the vertical dynamics, we introduce the following limits in Case 2:

$$G_z^\alpha(w) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{z_n^\alpha} \right| \text{ if } \delta \neq d, \text{ and}$$

$$G_z^\infty(w) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{|z_n|^{\frac{\gamma}{d}n}} \right| \text{ if } \delta = d,$$

where $\alpha = \gamma/(\delta - d)$ and $(z_n, w_n) = f^n(z, w)$. Although z_n^α is not well-defined if α is rational, the absolute value is well-defined and so we use this notation. Note that

$$G_z^\alpha(w) = \log \left| \frac{\phi_2(z, w)}{\phi_1(z)^\alpha} \right| \text{ and } G_z^\infty(w) = \log |\phi_2(z, w)|$$

if the Böttcher coordinate ϕ exists, where $\phi = (\phi_1, \phi_2)$, and that $G^\alpha \circ f = dG^\alpha$ and $G^\infty \circ f^n = d^n G^\infty + n\gamma d^{n-1} G_p \circ \pi_z$, where $G^\alpha(z, w) = G_z^\alpha(w)$, $G^\infty(z, w) = G_z^\infty(w)$ and π_z is the projection to the z -coordinate. Hence the existence of ϕ on U induces the existence of the limits above on A_f , whose properties depend on the magnitude relation between δ and d .

Theorem 1.4: Let $\delta < T_{s-1}$. If $\delta < d$, then G_z^α is plurisubharmonic and negative on A_0 and $G_z^\alpha(w) = \log |z^{-\alpha} w| + o(1)$ on U_r as $r \rightarrow 0$. If $\delta = d$, then G_z^∞ is plurisubharmonic on $A_0 - E_z$ and $G_z^\infty(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$. If

$\delta > d \geq 1$, then G_z^α is plurisubharmonic on $A_0 - E_z$ and $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on U_r as $r \rightarrow 0$. For all the cases, the functions are pluriharmonic on $A_0 - E$.

Detailed versions of the two theorems above are given as Theorems 2.3 and 2.4, respectively, in the next section. It follows from Theorems 1.2 and 1.3 that $A_0 - E$ is open. We remark that the notation α_0 is used instead of α in Ueno (2019b).

Let us next state a consequence of the main results for the case $\delta \geq T_{s-1}$. Note that $d \geq 1$ for this case. The weight α is not defined for the special case $\gamma = 0$ and $\delta = d$, whereas $\delta > d$ and so $\alpha \geq 0$ except the special case if $\delta \geq T_{s-1}$. By redefining α for the case $\delta \geq T_{s-1}$, which coincides with $\gamma/(\delta - d)$ except the special case, we can always define the limit G_z^α for the case $\delta \geq T_{s-1}$, which exists on $A_f - E_z$ if ϕ exists on U . Moreover, the limit

$$G_z^{\alpha,+}(w) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \left| \frac{w_n}{z_n^\alpha} \right|$$

exists and behaves well on the complement of specific fibres in the basin. In the definition of $G_z^{\alpha,+}$ and following theorem we set (γ, d) as (n_k, m_k) if f has two dominant terms (n_k, m_k) and (n_{k+1}, m_{k+1}) ; we remark that α depends only on f .

Theorem 1.5: Let $\delta \geq T_{s-1}$ and let $d \geq 2$ or $\delta \neq T_k$ for any k . If $\gamma > 0$ or $\delta = d$, then $\alpha > 0$, and $G_z^{\alpha,+}$ is plurisubharmonic and continuous on $A_0 - E_z$, pluriharmonic outside the boundary of $\{G_z^{\alpha,+} > 0\} \cap (A_0 - E_z)$ and $G_z^{\alpha,+}(w) = \log |w/z^\alpha| + o(1)$ on $U_r - \{z = 0\}$ as $r \rightarrow 0$. If $\gamma = 0$ and $\delta > d$, then $\alpha = 0$, and $G_z^{\alpha,+}$ is plurisubharmonic and continuous on A_0 , pluriharmonic outside the boundary of $\{G_z^{\alpha,+} > 0\} \cap A_0$ and $G_z^{\alpha,+}(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$.

If $\alpha > 0$ and $d \geq 2$, then the set $\{G_z^{\alpha,+} > 0\} \cap (A_0 - E_z)$ is expressed by the union of all the preimages of another invariant wedge-shaped region, which is included in U and whose slope relates to α . See Theorems 2.9 and 2.17 for details.

The limits G_z^α and $G_z^{\alpha,+}$ are good tools for the study of the dynamics of w_n/z_n^α on $A_f - E_z$ and $A_0 - E_z$, respectively, when α is an integer. In particular, the set where $G_z^{\alpha,+}$ is not pluriharmonic seems to be a Julia set of the dynamics of w_n/z_n^α and we expect that this Julia set is fractal when $\delta = T_k$ for some k , which is true at least for Examples 6.16 and 6.17. When α is rational, then we can consider the dynamics of $w_n^{\delta-d}/z_n^\gamma$ and expect that similar claims hold.

1.4 New results: classical plurisubharmonic functions

There are other limits such as

$$G_z(w) = \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \log |w_n| \quad \text{and} \quad G_f(z, w) = \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \log |(z_n, w_n)|,$$

where $\lambda = \max\{\delta, d\}$ and $|(z, w)| = \max\{|z|, |w|\}$. We remark that λ depends only on f . Although these limits are classical to study the dynamics of a holomorphic map f , they are less useful than the previous limits to discover the weighted vertical dynamics; roughly speaking, if $\delta \geq d$, then either these limits are constant on the intersection of each fibre and A_f , or $\gamma = 0$ and so f_0 is just a product. Tables 4 and 5 in the next section help us to understand the statements below.

First, the existence of the Böttcher coordinate induces the following.

Theorem 1.6: If $\delta < d$, then f is in Case 2, $G_z = G_z^\alpha$ on A_0 and $G_f = 0$ on $A_0 - E_z$. If $\gamma > 0$ and $\delta = d$, then f is in Case 2, $G_z = -\infty$ and $G_f = G_p \circ \pi_z$ on A_0 . If $\gamma = 0$ and $\delta = d$, then f is in Case 3, G_z is pluriharmonic on A_f and $G_z(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$. If $\gamma > 0$ and $\delta > d \geq 2$ or if $\gamma > 0$, $\delta > d = 1$ and $\delta \neq T_k$ for any k , then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$, $A_f - E_w$ or A_f . More precisely, $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ if $\delta < T_{s-1}$ and $d \geq 1$, on $A_f - E_w$ if $\delta = T_{s-1}$ and $d \geq 2$, and on A_f if $\delta > T_{s-1}$, and if $d \geq 2$ or if $d = 1$ and $\delta \neq T_k$. If $\gamma = 0$ and $\delta > d$, then f is in Case 3 and $G_z = 0$ on A_f .

Although G_z is useful for the special case $\gamma = 0$ and $\delta = d$, the monomial map f_0 is just a product and the dynamics is rather easy.

Even if $d = 0$, we can prove that $G_z = \alpha G_p \circ \pi_z$ on A_0 unless $\delta = T_{s-1}$, although the Böttcher coordinate does not exist on U .

Theorem 1.7: If $T_{s-1} > \delta > d = 0$, then f is in Case 2 and $G_z = \alpha G_p \circ \pi_z$ on A_0 .

If $\delta = T_{s-1}$, then the dominant term (m_{s-1}, n_{s-1}) belongs to Case 3 or Case 4, and the other dominant term (m_s, n_s) belongs to Case 2. Moreover, if $\delta > m_{s-1}$ and $m_s \geq 2$, then $G_z = \alpha G_p \circ \pi_z$ on the two disjoint invariant regions as stated in Theorem 1.6. Hence it follows from Corollary 2.15 that $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$.

Corollary 1.8: If $T_{s-1} = \delta > m_{s-1} > m_s \geq 2$, then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$.

The limit

$$G_f^\alpha(z, w) = \lim_{n \rightarrow \infty} \frac{1}{\lambda^n} \log |(z_n^\alpha, w_n)|$$

provides a plurisubharmonic function not only on A_f but also on A_0 for the case $\gamma = 0$ and $\delta = d$ and the case $\delta > d$.

Corollary 1.9: If $\gamma = 0$ and $\delta = d$, then G_f^α is plurisubharmonic on A_0 , $G_f^\alpha = G_z > \alpha G_p \circ \pi_z$ on A_f and $G_f^\alpha = \alpha G_p \circ \pi_z$ on $A_0 - A_f$. If $\gamma > 0$, $\delta > d$ and $(n, m_1) \neq (0, \delta)$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 unless $d \leq 1$ and $\delta = T_k$ for some k . If $\gamma = 0$ and $\delta > d$, then $G_f^\alpha = 0$ on A_0 .

The three theorems above are consequences of Theorems 2.5, 2.10 and 2.19. More strongly, we can show that G_f^α is plurisubharmonic on A_0 even if $\gamma > 0$, $\delta > d$, $d \leq 1$ and $\delta = T_k$ for some k , by using the other dominant term that corresponds to (n_k, m_k) or (n_{k+1}, m_{k+1}) . By setting (γ, d) as (n_k, m_k) if f has two dominant terms (n_k, m_k) and (n_{k+1}, m_{k+1}) , we can remove the exceptional case $d \leq 1$ and $\delta = T_k$ for some k from the theorem above.

Theorem 1.9: If $\gamma = 0$ and $\delta = d$, then G_f^α is plurisubharmonic on A_0 , $G_f^\alpha = G_z > \alpha G_p \circ \pi_z$ on A_f and $G_f^\alpha = \alpha G_p \circ \pi_z$ on $A_0 - A_f$. If $\gamma > 0$, $\delta > d$ and $(n, m_1) \neq (0, \delta)$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 . If $\gamma = 0$ and $\delta > d$, then $G_f^\alpha = 0$ on A_0 .

Corollary 1.10: At least one of the limits G_z , G_f and G_f^α is plurisubharmonic on A_0 for any case. More precisely, we can choose the limit as G_z for the case $\delta < d$, G_f for the case $\gamma > 0$ and $\delta = d$, and G_f^α for the other cases, which is non-constant and continuous on $A_0 - E$ except the case $\gamma = 0$ and $\delta > d$.

The plurisubharmonic function G_{FJ} constructed in Favre and Jonsson (2007) has to satisfy the equation $G_{FJ} \circ f = c_\infty G_{FJ}$, where c_∞ is the asymptotic attraction rate for f . If $\gamma > 0$ or $\delta \leq d$, then $G_p \circ \pi_z$ is a candidate of G_{FJ} since $c_\infty = \delta$. Moreover, G_f^α is another candidate when $\alpha > 0$, which is essentially different from $G_p \circ \pi_z$ only when $\gamma = 0$ and $\delta = d$. However, G_z^α , G_z^∞ and $G_z^{\alpha,+}$ do not satisfy the equation above. On the other hand, if $\gamma = 0$ and $\delta > d$, then G_z^α is a candidate of G_{FJ} since $c_\infty = d$, whereas $G_p \circ \pi_z$ is not. However, $\alpha = 0$ and f_0 is just a product for this case.

1.5 Comparison with our results on polynomial skew products

Similar studies have been made for polynomial skew products in Ueno (2011, 2014, 2016a, 2025) to understand the dynamics near infinity. Although the idea of assigning suitable weights is the same and we apply similar arguments in Ueno (2011, 2014) to investigate the asymptotic behaviour of plurisubharmonic functions, which derives similar statements, the dynamics around the superattracting fixed point of a superattracting skew product and the dynamics near infinity of a polynomial skew product are different. For example, whereas polynomial skew products preserve the line at infinity except indeterminacy points, superattracting skew products do not preserve the z -axis and w -axis in general. The definitions of Newton polygons and weights are also different. Moreover, we remark that we only dealt with the case when the specific term in q is dominant on a wedge-shaped region in Ueno (2011, 2014, 2016a) and with the specific weight in Ueno (2011, 2014); roughly speaking, a polynomial skew product dealt with in Ueno (2011, 2014, 2016a) belongs to Case 1 or Case 2 defined in Ueno (2025) in terms of a Newton polygon. Furthermore, we only studied Böttcher coordinates in Ueno (2025). In this paper we have succeeded in adapting arguments in Ueno (2011, 2014) for the specific class of polynomial skew products to a superattracting skew product f for all the cases, in order to clarify the asymptotic behaviour of plurisubharmonic functions of f toward the boundary of A_f . Besides this adaptation, we provide some statements on degenerate fibres. Furthermore, by combining this adaptation with the results on the relation between A_f and A_0 , which are described in Theorems 1.3 and 2.14, we have managed to obtain dynamically nice plurisubharmonic functions of f defined on the complement of fibres in A_0 and on A_0 , which are summarised in Theorems 1.4, 1.5 and Corollary 1.10. On the other hand, we gave further study of the dynamics near infinity in Ueno (2011) and Ueno (2014) such as the approximations of weighted Green functions in terms of the weighted homogeneous part of q and as the dynamics on weighted projective spaces, whose adaptations to superattracting skew products and to polynomial skew products in other cases are future works.

1.6 Organisation

We give main results for each case in Section 2. More generally, Lemma 1.1 and Theorem 1.2 hold when we let (l_1, l_2) vary on a suitable rectangle $\mathcal{I}_f$ determined by the combinatorial data. In this sense, there are infinitely many invariant wedges if $\delta \neq T_k$ for any k . For Cases 2 and 4, we find out new invariant wedges, on which $b_{\gamma d} z^\gamma w^d$ may not be the dominant term in q , and claim that the unions of all the preimages of the wedges coincide with $A_0 - E_z$ for almost all the cases. These results imply Theorem 1.3 and are important to induce Theorems 1.5 and 1.7 and Corollary 1.8. Tables are given in each subsection as comparison charts of the main results.

We review our previous study in Ueno (2019b) more precisely in Section 3. We give the definitions of $\mathcal{I}_f^1$, $\mathcal{I}_f^2$ and $\mathcal{I}_f$, calculate them, and illustrate detailed versions of Lemma 1.1 and Theorem 1.2 in terms of blow-ups.

The proofs of the results for Case 2 are given in Sections 4 and 5. With an illustration in terms of blow-ups, we first prove the results on invariant wedges and the unions of all the preimages of the wedges in Section 4. We next show the existence and properties of plurisubharmonic functions, and investigate the asymptotic behaviour of the functions toward the boundaries of the unions in Section 5, which makes clear the inclusion relation between the unions. The proofs of the results for Cases 3 and 4 are given in Sections 6 and 7, respectively. Moreover, we provide examples of polynomial skew products that are semiconjugate to polynomial products in Section 6, which demonstrate that G_z^α and $G_z^{\alpha,+}$ are good tools for the study of the weighted vertical dynamics.

2 Summary of main results for each case

Main results for each case are given in this section. We deal with Cases 2, 3 and 4 in Subsections 2.1, 2.2 and 2.3, respectively. Moreover, we provide tables for each case, which help us to recognise the differences of properties of plurisubharmonic functions depending on the magnitude relation between δ and d , between δ and T_j and between γ and 0. Furthermore, we provide tables of properties of the classical plurisubharmonic functions in Subsection 2.4.

Let $U^{l(1),l(2)} = U_r^{l(1),l(2)} = \{|z|^{l(1)+l(2)} < r^{l(2)}|w|, |w| < r|z|^{l(1)}\}$. Then $U = U^{l_1,\infty}$ for Case 2, $U = U^{0,l_2}$ for Case 3, and $U = U^{l_1,l_2}$ for Case 4. Let $A_f^{l(1),l(2)}$ be the union of all the preimages of $U^{l(1),l(2)}$. Moreover, let

$$U_{r_1,r_2}^{l(1),l(2)} = \{|z|^{l(1)+l(2)} < r_1^{l(2)}|w|, |w| < r_2|z|^{l(1)}\}.$$

In particular, note that $U_{r_1,r_2}^{l(1),\infty} = \{|z| < r_1, |w| < r_2|z|^{l(1)}\}$.

2.1 Main results for Case 2

Let $s > 1$, $\delta \leq T_{s-1}$ and $(\gamma, d) = (n_s, m_s)$ in this subsection. Then $\gamma > 0$. Table 1 is given as a comparison chart of main results for Case 2. We remark that the descriptions on the asymptotic behaviour of G_z^α and G_z^∞ in the table are not precise. Actually, we have to remove E_z from the boundary $\partial A_f^{l,\infty}$ of $A_f^{l,\infty}$; see Propositions 5.16 and 5.27 in Section 5 for complete descriptions.

We first exhibit a result on invariant wedges, and give an affirmative answer to the question whether the unions of all the preimages of the wedges coincide with the basin. The following is a detailed version of Lemma 1.1, which is stated also in the next section as Lemma 3.1.

Table 1 Comparison chart of main results for Case 2

	$T_{s-1} > d > \delta$	$T_{s-1} > \delta = d$	$T_{s-1} > \delta > d$	$T_{s-1} = \delta > d$
$\mathcal{I}_f^1$	$[l_1, \infty)$	$[l_1, \infty)$	$[l_1, \alpha]$	$\{l_1\} = \{\alpha\}$
α	$\alpha < 0$		$\alpha > 0$	$\alpha = l_1 > 0$
d	$d > 2$	$d \geq 2$	$d \geq 0$	$d \geq 0$
G_z^α	psh on A_0	psh on $A_0 - E_z$	psh on $A_0 - E_z$	psh on A_f
or	ph on $A_0 - E$	ph on $A_0 - E$	ph on $A_0 - E$	ph on $A_f - E$
G_z^∞	$\log z^{-\alpha} w + o(1)$ on U_r	$\log w + o(1)$ on U_r	$\log w/z^\alpha + o(1)$ on U_r if $d \geq 1$	$\log w/z^\alpha + o(1)$ on U_r if $d \geq 2$
lim	$G_z^\alpha \rightarrow 0$	$G_z^\infty \rightarrow \infty$	$G_z^\alpha \rightarrow \infty$ if $l < \alpha$	$G_z^\alpha \rightarrow 0$
sup	as $\rightarrow \partial A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1$	as $\rightarrow \partial A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1$	and $d \geq 1$ $G_z^\alpha \rightarrow 0$ if $l = \alpha$ and $d \geq 2$	as $\rightarrow \partial A_f^{\alpha, \infty}$ if $d \geq 2$
$A_f^{l, \infty}$	$A_f^{l, \infty} = A_f^{l_1, \infty}$ $= A_0 - E_z$ for any l in $\mathcal{I}_f^1$	$A_f^{l, \infty} = A_f^{l_1, \infty}$ $= A_0 - E_z$ for any l in $\mathcal{I}_f^1$	$A_f^{l, \infty} = A_f^{l_1, \infty}$ $= A_0 - E_z$ for any $l < \alpha$ $A_f^{\alpha, \infty} \subsetneq A_f^{l_1, \infty}$ if $d \geq 2$	
G_z	$G_z = G_z^\alpha$ on A_0 (psh on A_0)	$G_z = -\infty$ on A_0	$G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ if $d \geq 1$	$G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ or $A_f - E_w$ if $d \geq 2$
G_f	$G_f = 0$	$G_f = G_p \circ \pi_z$ on A_0 (psh on A_0)	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 (psh on A_0)	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 if $n_{s-1} > 0$ or $A_f^{\alpha, \infty}$ if $n_{s-1} = 0$
or	on $A_0 - E_z$			
G_f^α				

Lemma 2.1 (Ueno, 2019b): The following holds for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$ and $\delta < T_{s-1}$, where $\mathcal{I}_f^1 = [l_1, \infty)$ if $\delta \leq d$ and $\mathcal{I}_f^1 = [l_1, \alpha]$ if $\delta > d$: $f \sim f_0$ on $U_r^{l, \infty}$ as $r \rightarrow 0$, and $f(U_r^{l, \infty}) \subset U_r^{l, \infty}$ for small $r > 0$.

The interval $\mathcal{I}_f^1$ plays an important role to study the dynamics of f as explained in the next section. Note that $\min \mathcal{I}_f^1 = l_1$ and $U^{l_1, \infty}$ is the largest wedge among $U^{l, \infty}$ for any l in $\mathcal{I}_f^1$. We have succeeded in extending the second statement of this lemma to the following.

Theorem 2.2: The following holds for any $l > 0$ if $\delta \leq d$ and for any $0 < l < \alpha$ if $\delta > d$: $f(U_{r_1, r_2}^{l, \infty}) \subset U_{r_1, r_2}^{l, \infty}$ for suitable small r_1 and r_2 .

Whereas we used the property $f \sim f_0$ on $U_r^{l, \infty}$ as $r \rightarrow 0$ to show that $f(U_r^{l, \infty}) \subset U_r^{l, \infty}$ in the lemma above, we apply different arguments to prove this theorem; the property above does not hold since l does not belong to $\mathcal{I}_f^1$ if $0 < l < l_1$. Note that $U_{r_1, r_2}^{l, \infty}$ is

larger than $U_{r_1, r_2}^{l_1, \infty}$ for any $0 < l < l_1$ and that there is no condition on the degree d in the theorem.

We have also succeeded in giving the following affirmative answer to the question, which implies Theorem 1.3 since $l_1 < \alpha$ if $\delta < T_{s-1}$.

Theorem 2.3: It follows that $A_f^{l, \infty} = A_0 - E_z$ for any $l > 0$ if $\delta \leq d$ and for any $0 < l < \alpha$ if $\delta > d$ and $(n_1, m_1) \neq (0, \delta)$.

We next exhibit main results on plurisubharmonic functions, assuming that $a = b_{\gamma d} = 1$. If the Böttcher coordinate exists on U , then one can show that the limits G_z^α and G_z exist and are plurisubharmonic on $A_f - E_z$ and A_f , respectively. Moreover, thanks to Theorem 1.3, we obtain the following theorems.

Theorem 2.4: Let $\delta < T_{s-1}$. If $\delta < d$, then $G_z^\alpha = G_z$ on A_0 , which is plurisubharmonic and negative on A_0 and $G_z^\alpha(w) = \log |z^{-\alpha} w| + o(1)$ on U_r as $r \rightarrow 0$. If $\delta = d$, then G_z^∞ is plurisubharmonic on $A_0 - E_z$ and $G_z^\infty(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$. If $\delta > d \geq 1$, then G_z^α is plurisubharmonic on $A_0 - E_z$ and $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on U_r as $r \rightarrow 0$. For all the cases, the functions are pluriharmonic on $A_0 - E$.

On the other hand, if $T_{s-1} = \delta > d \geq 2$, then G_z^α is plurisubharmonic on A_f , pluriharmonic on $A_f - E_w$ and $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on U_r as $r \rightarrow 0$.

Theorem 2.5: Let $\delta < T_{s-1}$. If $\delta < d$, then $G_z = G_z^\alpha$ on A_0 . Hence $G_f = 0$ on $A_0 - E_z$ and $G_f = -\infty$ on E_z . If $\delta = d$, then $G_z = -\infty$ and $G_f = G_p \circ \pi_z$ on A_0 . If $\delta > d \geq 1$, then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ and $-\infty$ on E_w . If $d = 0$, then $G_z = \alpha G_p \circ \pi_z$ on A_0 . In particular, if $\delta > d$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

On the other hand, if $T_{s-1} = \delta > d \geq 2$, then $G_z = \alpha G_p \circ \pi_z$ on $A_f - E_w$ and $-\infty$ on E_w . Hence $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_f . Moreover, if $n_{s-1} > 0$, then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ and $-\infty$ on E_w . Hence $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

By investigating the asymptotic behaviour of G_z^α toward the boundary $\partial A_f^{l, \infty}$, we can show that $A_f^{\alpha, \infty}$ is smaller than $A_f^{l_1, \infty}$.

Proposition 2.6: If $T_{s-1} > \delta > d \geq 2$, then $A_f^{\alpha, \infty} \subsetneq A_f^{l_1, \infty}$.

Moreover, we give a sufficient condition for $A_f^{\alpha, \infty}$ and $A_f^{l_1, \infty}$ being unbounded.

Proposition 2.7: Let $T_{s-1} > \delta > d \geq 2$. If q is polynomial and $f^{-1}\{w = 0\} \not\supseteq \{w = 0\}$, then both $A_f^{\alpha, \infty}$ and $A_f^{l_1, \infty}$ are unbounded.

2.2 Main results for Case 3

Let $s > 1$, $\delta \geq T_1$ and $(\gamma, d) = (n_1, m_1)$ in this subsection. Then $\delta \geq T_1 \geq d \geq 1$, and $\delta > d$ if $\gamma > 0$. Table 2 is given as a comparison chart of main results for Case 3; see Proposition 6.4 for a complete description on the asymptotic behaviour of G_z^α toward $\partial A_f^{0, l} \cap A_0$.

Although we want to define α as $\gamma/(\delta - d)$, it is not defined if $\gamma = 0$ and $\delta = d$, which corresponds to the exceptional case $(n_1, m_1) = (0, \delta)$ in Theorem 2.3. To

overcome this problem, we redefine α as the minimum of $l \geq 0$ such that a line $x + ly - l\delta = 0$ intersects $N(q)$; this definition works not only when $\delta \geq T_1$ but also when $\delta > m_s$. Then the line that takes the minimum passes through $(0, \delta)$ and (γ, d) , and α coincides with $\gamma/(\delta - d)$ unless $\gamma = 0$ and $\delta = d$. Moreover, α is well-defined even if $\gamma = 0$ and $\delta = d$, which coincides with $l_2 = n_2/(\delta - m_2)$.

Table 2 Comparison chart of main results for Case 3

	$\gamma > 0$ and $\delta > T_1$	$\gamma > 0$ and $\delta = T_1$	$\gamma = 0$ and $\delta > T_1$	$\gamma = 0$ and $\delta = T_1$
$\mathcal{I}_f^2$	$[\alpha, l_2]$	$\{\alpha\} = \{l_2\}$	$(0, l_2]$	$(0, l_2]$
α	$\alpha > 0$	$\alpha > 0$	$\alpha = 0$	$\alpha = l_2 > 0$
d	$\delta > d \geq 1$	$\delta > d \geq 1$	$\delta > d \geq 2$	$\delta = d \geq 2$
f_0	$(z^\delta, z^\gamma w^d)$	$(z^\delta, z^\gamma w^d)$	(z^δ, w^d)	(z^δ, w^δ)
G_z^α	ph on $A_f - E_z$ $\log w/z^\alpha + o(1)$ on U_r	ph on $A_f - E_z$ $\log w/z^\alpha + o(1)$ on U_r if $d \geq 2$	ph on A_f $\log w + o(1)$ on U_r	ph on $A_f - E_z$ $\log w/z^\alpha + o(1)$ on U_r
lim	$G_z^\alpha \rightarrow 0$ as $\rightarrow \partial A_f^{0,l} \cap A_0$ if $l = \alpha$ and $d \geq 2$ $G_z^\alpha \rightarrow -\infty$ for any $l > \alpha$	$G_z^\alpha \rightarrow 0$ as $\rightarrow \partial A_f^{0,l} \cap A_0$ if $d \geq 2$	$G_z^\alpha \rightarrow -\infty$ as $\rightarrow \partial A_f^{0,l} \cap A_0$ for any l in $\mathcal{I}_f^2$	$G_z^\alpha \rightarrow (l - \alpha)G_p \circ \pi_z$ as $\rightarrow \partial A_f^{0,l} \cap A_0$ for any l in $\mathcal{I}_f^2$ In particular, $G_z^\alpha \rightarrow 0$ if $l = \alpha$
$G_z^{\alpha,+}$	psh and cont. on $A_0 - E_z$ $G_z^{\alpha,+} > 0$ on $A_f^{0,\alpha}$ $= 0$ on $A_0 - A_f^{0,\alpha}$ if $d \geq 2$	psh and cont. on $A_0 - E_z$ $G_z^{\alpha,+} > 0$ on $A_f^{0,\alpha}$ $= 0$ on $A_0 - A_f^{0,\alpha}$ if $d \geq 2$	psh and cont. on A_0	psh and cont. on $A_0 - E_z$ $G_z^{\alpha,+} > 0$ on $A_f^{0,\alpha}$ $= 0$ on $A_0 - A_f^{0,\alpha}$
$A_f^{0,l}$	$A_f^{0,l} = A_f^{0,l_2}$ for any $l > \alpha$ $A_f^{0,\alpha} \subsetneq A_f^{0,l_2}$ if $d \geq 2$		$A_f^{0,l} = A_f^{0,l_2}$ for any l in $\mathcal{I}_f^2$	$A_f^{0,l'} \subsetneq A_f^{0,l''}$ for any $l' < l''$
G_z	$G_z = \alpha G_p \circ \pi_z$ on A_f	$G_z = \alpha G_p \circ \pi_z$ on A_f if $d \geq 2$	$G_z = 0$ on A_f	$= \log w + o(1)$ on U_r ph on A_f
G_f^α	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 (psh on A_0)	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 (psh on A_0)	$G_f^\alpha = 0$ on A_0 (psh on A_0)	psh on A_0 $G_f^\alpha > \alpha G_p \circ \pi_z$ on $A_f^{0,\alpha}$ $= \alpha G_p \circ \pi_z$ on $A_0 - A_f^{0,\alpha}$

We exhibit results only on plurisubharmonic functions, assuming that $a = b_{\gamma d} = 1$; we have no results on invariant wedges and do not know whether the unions of all the preimages of the wedges coincide with the basin.

The existence of the Böttcher coordinate induces the following.

Proposition 2.8: Let $d \geq 2$ or $\delta > T_1$. If $\gamma > 0$ or $\delta = d$, then $\alpha > 0$ and G_z^α is pluriharmonic on $A_f - E_z$. Moreover, $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on $U_r - \{z = 0\}$ as $r \rightarrow 0$. If $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and G_z^α is pluriharmonic on A_f . Moreover, $G_z^\alpha(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$.

Although the interval $\mathcal{I}_f^2$ plays an important role as the same as Case 2, where $\mathcal{I}_f^2 = [\alpha, l_2]$ if $\gamma > 0$ and $\mathcal{I}_f^2 = (0, l_2]$ if $\gamma = 0$, we avoid to use this notation in this

subsection. By investigating the asymptotic behaviour of G_z^α and G_z toward the boundary $\partial A_f^{0,l}$, we obtain the theorems and proposition below.

Theorem 2.9: Let $d \geq 2$ or $\delta > T_1$. If $\gamma > 0$ or $\delta = d$, then $G_z^{\alpha,+}$ is plurisubharmonic and continuous on $A_0 - E_z$ and pluriharmonic outside the boundary of $\{G_z^{\alpha,+} > 0\} \cap (A_0 - E_z)$. More precisely, if $d \geq 2$, then $G_z^{\alpha,+} > 0$ on $A_f^{0,\alpha}$ and $G_z^{\alpha,+} = 0$ on $A_0 - A_f^{0,\alpha}$, which is pluriharmonic on $A_0 - \partial A_f^{0,\alpha} \cup E_z$. If $\gamma = 0$ and $\delta > d$, then $G_z^{\alpha,+}$ is plurisubharmonic and continuous on A_0 and pluriharmonic outside the boundary of $\{G_z^{\alpha,+} > 0\} \cap A_0$.

Theorem 2.10: If $\gamma > 0$ and $\delta > d \geq 2$ or if $\gamma > 0$, $\delta > d = 1$ and $\delta > T_1$, then $G_z = \alpha G_p \circ \pi_z$ on A_f and $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 . If $\gamma = 0$ and $\delta > d$, then $G_z = 0$ on A_f and $G_f^\alpha = 0$ on A_0 . If $\gamma = 0$ and $\delta = d$, then G_z is pluriharmonic on A_f , $G_z(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$, G_f^α is plurisubharmonic and continuous on A_0 , $G_f^\alpha > \alpha G_p \circ \pi_z$ on A_f and $G_f^\alpha = \alpha G_p \circ \pi_z$ on $A_0 - A_f$.

We remark that the equality $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 holds even if $\gamma > 0$, $d = 1$ and $\delta = T_1$ as stated in Remark 6.8.

Proposition 2.11: If $\delta > T_1$, then $A_f^{0,l} = A_f^{0,l_2}$ for any $\alpha < l \leq l_2$. Moreover, if $\gamma > 0$ and $d \geq 2$, then $A_f^{0,\alpha} \subsetneq A_f^{0,l_2}$. If $\gamma = 0$ and $\delta = d$, then $\delta = T_1$ and $A_f^{0,l'} \subsetneq A_f^{0,l''}$ for any $0 < l' < l'' \leq l_2$.

2.3 Main results for Case 4

Let $s > 2$, $T_k \leq \delta \leq T_{k-1}$ and $(\gamma, d) = (n_k, m_k)$ for some $2 \leq k \leq s - 1$ in this subsection. Then $\gamma > 0$ and $\delta > d \geq 1$. Table 3 is given as a comparison chart of main results for Case 4; see Propositions 7.13, 7.16 and 7.19 for complete descriptions on the asymptotic behaviour of G_z^α toward $\partial A_f^{l(1),l(2)}$. In the table the notations A_f^{k-1} and A_f^1 mean the attracting sets corresponding to the other dominant terms (n_{k-1}, m_{k-1}) and (n_1, m_1) , respectively.

We first exhibit a result on invariant wedges, and give a partial affirmative answer to the question whether the unions of all the preimages of the wedges coincide with the basin, which are similar to Case 2. A detailed version of Lemma 1.1 is given in the next section as Lemma 3.10, in which the wedge $U^{l(1),l(2)}$ and intervals $\mathcal{I}_f^1$ and $\mathcal{I}_f^2$ are used. Lemma 3.10 induces the following, in which we use $U^{l,\infty}$ for simplicity, although it is more precise to use $\{|z| < r^{1+1/l(2)}, |w| < r|z|^{l(1)}\}$ to adapt to $U^{l(1),l(2)}$.

Lemma 2.12: The following holds for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$ and $T_k < \delta < T_{k-1}$, where $\mathcal{I}_f^1 = [l_1, \alpha]$ if $T_k < \delta < T_{k-1}$, $\mathcal{I}_f^1 = [l_1, \alpha)$ if $\delta = T_k$, and $\mathcal{I}_f^1 = \{l_1\} = \{\alpha\}$ if $\delta = T_{k-1}$: $f(U_r^{l,\infty}) \subset U_r^{l,\infty}$ for small $r > 0$.

Note that $\min \mathcal{I}_f^1 = l_1$ and $U^{l_1,\infty}$ is the largest wedge among $U^{l,\infty}$ for any l in $\mathcal{I}_f^1$. We can extend this lemma to the following theorem by the same arguments as Case 2.

Theorem 2.13: The following holds for any $0 < l < \alpha$: $f(U_{r_1, r_2}^{l, \infty}) \subset U_{r_1, r_2}^{l, \infty}$ for suitable small r_1 and r_2 .

Note that $U_{r_1, r_2}^{l, \infty}$ is larger than $U_{r_1, r_2}^{l_1, \infty}$ for any $0 < l < l_1$ and that there is no condition on the degree d in this theorem.

Table 3 Comparison chart of main results for Case 4

	$T_k < \delta < T_{k-1}$	$T_k = \delta < T_{k-1}$	$T_k < \delta = T_{k-1}$ and $n_{k-1} > 0$	$T_k < \delta = T_{k-1}$ and $n_{k-1} = 0$
$\mathcal{I}_f$	$[l_1, \alpha] \times [\alpha, l_1 + l_2]$ $-\{(\alpha, \alpha)\}$	$[l_1, \alpha] \times \{\alpha\}$	$\{\alpha\} \times (\alpha, l_1 + l_2]$	$\{\alpha\} \times (\alpha, l_1 + l_2]$
d	$d \geq 1$	$d \geq 1$	$d \geq 1$	$d \geq 1$
$A_f^{l, \infty}$	$A_f^{l, \infty} = A_0 - E_z$ for any $0 < l < \alpha$	$A_f^{l, \infty} = A_0 - E_z$ for any $0 < l < \alpha$	$A_f^{l, \infty} = A_0 - E_z$ for any $0 < l < \alpha$	
G_z^α	ph on A_f $\log w/z^\alpha + o(1)$ on U_r	ph on A_f $\log w/z^\alpha + o(1)$ on U_r if $d \geq 2$	ph on A_f $\log w/z^\alpha + o(1)$ on U_r if $d \geq 2$	ph on A_f $\log w/z^\alpha + o(1)$ on U_r if $d \geq 2$
lim	$G_z^\alpha \rightarrow \infty$ if $l < \alpha$	$G_z^\alpha \rightarrow \infty$	$G_z^\alpha \rightarrow 0$	$G_z^\alpha \rightarrow 0$
sup	$G_z^\alpha \rightarrow 0$ if $l = \alpha$ and $d \geq 2$ as $\rightarrow \partial A_f^{l, \infty}$	as $\rightarrow \partial A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$	as $\rightarrow \partial A_f^{\alpha, \infty}$ if $d \geq 2$	as $\rightarrow \partial A_f^{\alpha, \infty}$ if $d \geq 2$
lim	$G_z^\alpha \rightarrow 0$ if $l = \alpha$ and $d \geq 2$ $G_z^\alpha \rightarrow -\infty$ if $l > \alpha$ as $\rightarrow$ $\partial A_f^{l_1, l-l_1} \cap A_0$	$G_z^\alpha \rightarrow 0$ as $\rightarrow$ $\partial A_f^{l_1, \alpha-l_1} \cap A_0$ if $d \geq 2$	$G_z^\alpha \rightarrow -\infty$ as $\rightarrow$ $\partial A_f^{\alpha, l-\alpha} - \partial A_f^{\alpha, \infty}$ for any l in $\mathcal{I}_f^2$ if $d \geq 2$	$G_z^\alpha \rightarrow -\infty$ as $\rightarrow$ $\partial A_f^{\alpha, l-\alpha} - \partial A_f^{\alpha, \infty}$ for any l in $\mathcal{I}_f^2$ if $d \geq 2$
$A_f^{l(1), l(2)}$	$A_f^{l(1), l(2)} = A_f$ for most $l(1)$ and $l(2)$ $A_f^{l_1, \alpha-l_1} \subsetneq A_f$ $A_f^{\alpha, l_1+l_2-\alpha} \subsetneq A_f$ if $d \geq 2$	$A_f^{l, \alpha-l} = A_f$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$	$A_f^{\alpha, l-\alpha} = A_f$ for any l in $\mathcal{I}_f^2$ if $d \geq 2$	$A_f^{\alpha, l-\alpha} = A_f$ for any l in $\mathcal{I}_f^2$ if $d \geq 2$
other	$\nexists$	(n_{k+1}, m_{k+1})	$(n_{k-1}, m_{k-1}) \Rightarrow A_f^{k-1}$	$(n_1, m_1) \Rightarrow A_f^1$
term		Case 2 or Case 4	Case 3 or Case 4	Case 3
$G_z^{\alpha, +}$	psh and cont. on $A_0 - E_z$ > 0 on $A_f^{l_1, \alpha-l_1}$ $= 0$ on $A_0 - A_f^{l_1, \alpha-l_1}$ if $d \geq 2$	psh and cont. on $A_0 - E_z$ > 0 on $A_f^{l_1, \alpha-l_1}$ $= 0$ on $A_0 - A_f^{l_1, \alpha-l_1}$ if $d \geq 2$	psh and cont. on $A_0 - E_z$ > 0 on A_f^{k-1} $= 0$ on $A_0 - A_f^{k-1}$	psh and cont. on $A_0 - E_z$ > 0 on A_f^1 $= 0$ on $A_0 - A_f^1$
G_z	$G_z = \alpha G_p \circ \pi_z$ on A_f	$G_z = \alpha G_p \circ \pi_z$ on A_f if $d \geq 2$	$G_z = \alpha G_p \circ \pi_z$ on A_f if $d \geq 2$	$G_z = \alpha G_p \circ \pi_z$ on A_f if $d \geq 2$
G_f^α	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0	psh on A_0 $G_f^\alpha > \alpha G_p \circ \pi_z$ on $A = \alpha G_p \circ \pi_z$ on $A_0 - A$

We obtain the following partial affirmative answer to the question as the same as Case 2.

Theorem 2.14: If $(n_1, m_1) \neq (0, \delta)$, then $A_f^{l, \infty} = A_0 - E_z$ for any $0 < l < \alpha$.

This theorem implies the following since $l_1 < \alpha$ if $\delta < T_{k-1}$.

Corollary 2.15: If $\delta < T_{k-1}$, then $A_f^{l_1, \infty} = A_0 - E_z$.

We next exhibit main results on plurisubharmonic functions, assuming that $a = b_{\gamma d} = 1$. The existence of the Böttcher coordinate induces the following.

Proposition 2.16: If $d \geq 2$ or $T_k < \delta < T_{k-1}$, then G_z^α is pluriharmonic on A_f and $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on U_r as $r \rightarrow 0$.

Thanks to Corollary 2.15, we obtain the following theorem by investigating the asymptotic behaviour of G_z^α toward the boundary $\partial A_f^{l_{(1)}, l_{(2)}}$.

Theorem 2.17: Let $\delta < T_{k-1}$. If $d \geq 2$ or $\delta > T_k$, then $G_z^{\alpha, +}$ is plurisubharmonic and continuous on $A_0 - E_z$ and pluriharmonic outside the boundary of $\{G_z^{\alpha, +} > 0\} \cap (A_0 - E_z)$. More precisely, if $d \geq 2$, then $G_z^{\alpha, +} > 0$ on $A_f^{l_1, \alpha - l_1}$ and $G_z^{\alpha, +} = 0$ on $A_0 - A_f^{l_1, \alpha - l_1}$, which is pluriharmonic on $A_0 - \partial A_f^{l_1, \alpha - l_1} \cup E_z$.

The investigation of the asymptotic behaviour of G_z^α toward $\partial A_f^{l_{(1)}, l_{(2)}}$ also shows us the inclusion relation between $A_f^{l_{(1)}, l_{(2)}}$ for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$.

Proposition 2.18: If $T_k < \delta < T_{k-1}$, then $A_f^{l_{(1)}, l_{(2)}} = A_f^{l_1, l_2}$ for any $l_1 \leq l_{(1)} < \alpha$ and $\alpha < l_{(1)} + l_{(2)} \leq l_1 + l_2$. Moreover, $A_f^{l_1, \alpha - l_1} \subsetneq A_f^{l_1, l_2}$, $A_f^{\alpha, l_1 + l_2 - \alpha} \subsetneq A_f^{l_1, l_2}$ and $A_f^{l_1, \alpha - l_1} \cap A_f^{\alpha, l_1 + l_2 - \alpha} = \emptyset$ if $d \geq 2$. If $\delta = T_k$ and $d \geq 2$, then $A_f^{l, \alpha - l} = A_f^{l_1, \alpha - l_1}$ for any $l_1 \leq l < \alpha$. If $\delta = T_{k-1}$ and $d \geq 2$, then $A_f^{\alpha, l - \alpha} = A_f^{\alpha, l_2 - \alpha}$ for any $0 < l \leq l_2$.

The existence of the Böttcher coordinate and Theorems 2.13 and 2.14 induce the following.

Theorem 2.19: If $d \geq 2$ or $T_k < \delta < T_{k-1}$, then $G_z = \alpha G_p \circ \pi_z$ on A_f . If $(n_1, m_1) \neq (0, \delta)$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

2.4 Tables for classical plurisubharmonic functions

We end this section with Tables 4 and 5 for the classical plurisubharmonic functions.

3 Intervals of weights and blow-ups

More precise review of the study in the previous paper (Ueno, 2019b) is given in this section. We deal with Cases 2, 3 and 4 in Subsections 3.1, 3.2 and 3.3, respectively. For each case, we give the definitions of the intervals $\mathcal{I}_f^1$ and $\mathcal{I}_f^2$ and the rectangle $\mathcal{I}_f$, calculate them, restate Lemma 1.1 and Theorem 1.2 by using $\mathcal{I}_f^1$ and/or $\mathcal{I}_f^2$, and illustrate

them in terms of blow-ups. In addition, explanations of weights as a generalisation of the degree are given at the end of each subsection.

Table 4 Comparison chart of classical psh functions

	G_z	G_f	G_f^α
$\delta < d$	$G_z = G_z^\alpha$ on A_0 (psh on A_0)	$G_f = 0$ on $A_0 - E_z$ $= -\infty$ on E_z	$G_f^\alpha = 0$ on $A_0 - E_z$ $= \infty$ on E_z
$\delta = d$ and $\gamma > 0$	$G_z = -\infty$ on A_0	$G_f = G_p \circ \pi_z$ on A_0 (psh on A_0)	
$\delta = d$ and $\gamma = 0$	ph on A_f $G_z = \log w + o(1)$ on U_r	ph on A_f	psh on A_0 $G_f^\alpha = \log w + o(1)$ on U_r
$\delta > d$, $\gamma > 0$ and $d \neq 1$ or $\delta \neq T_k$	$G_z = \alpha G_p \circ \pi_z$ on $\cdots$ (see Table 5)	psh on $\cdots$ (see Table 5)	$G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 (psh on A_0)
$\delta > d$ and $\gamma = 0$	$G_z = 0$ on A_f	$G_f = 0$ on A_f	$G_f^\alpha = 0$ on A_0

Table 5 Comparison chart of G_z and G_f when $\delta > d$ and $\gamma > 0$

on $\cdots$	if $\cdots$
$A_0 - E_w$	if $\delta < T_{s-1}$ and $d \geq 1$
A_0	if $\delta < T_{s-1}$ and $d = 0$
$A_f - E_w$	if $\delta = T_{s-1}$ and $d \geq 2$
A_f	if $\delta > T_{s-1}$, and if $d \geq 2$ or if $d = 1$ and $\delta \neq T_k$

3.1 Intervals of weights and blow-ups for Case 2

Let $s > 1$,

$$\delta \leq T_{s-1}, (\gamma, d) = (n_s, m_s), l_1 = \frac{n_s - n_{s-1}}{m_{s-1} - m_s} \text{ and } l_2 = \infty.$$

Note that $\gamma > 0$ and $d \geq 0$ by the setting.

3.1.1 Intervals and Böttcher coordinates

We define the intervals $\mathcal{I}_f^1, \mathcal{I}_f^2$ and the rectangle $\mathcal{I}_f$ as

$$\mathcal{I}_f^1 = \{l > 0 \mid l\delta \leq \gamma + ld \leq i + lj \text{ for any } i \text{ and } j \text{ s.t. } b_{ij} \neq 0\},$$

$\mathcal{I}_f^2 = \{\infty\}$ and $\mathcal{I}_f = \mathcal{I}_f^1 \times \{\infty\}$. It follows that $\min \mathcal{I}_f^1 = l_1$. In fact, if $\delta > d$, then

$$\mathcal{I}_f^1 = \left[\max \left\{ \frac{\gamma - i}{j - d} \right\}, \frac{\gamma}{\delta - d} \right] = \left[\frac{\gamma - n_{s-1}}{m_{s-1} - d}, \frac{\gamma}{\delta - d} \right] = [l_1, \alpha].$$

If $\delta \leq d$, then the inequality $l\delta \leq \gamma + ld$ is trivial and so $\mathcal{I}_f^1 = [l_1, \infty)$. We remark that

$$\mathcal{I}_f^1 = \{l > 0 \mid l\delta \leq \gamma + ld \leq n_j + lm_j \text{ for any } 1 \leq j \leq s-1\},$$

which expression is used in Case 4.

Recall that $U = U^{l, \infty}$, where $U^{l, \infty} = U_r^{l, \infty} = \{|z| < r, |w| < r|z|^l\}$.

Lemma 3.1 (Ueno, 2019b): The following hold for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$ and $\delta < T_{s-1}$: $f \sim f_0$ on $U_r^{l, \infty}$ as $r \rightarrow 0$, and $f(U_r^{l, \infty}) \subset U_r^{l, \infty}$ for small $r > 0$.

This lemma induces the Böttcher coordinate for f on $U^{l, \infty}$.

Theorem 3.2 (Ueno, 2019b): The following holds for any l in the previous lemma: ϕ is biholomorphic on $U_r^{l, \infty}$ and conjugates f to f_0 for small $r > 0$. Moreover, $\phi \sim id$ on $U_r^{l, \infty}$ as $r \rightarrow 0$.

Note that $U^{l_1, \infty}$ is the largest open set among $U^{l, \infty}$ for any l in $\mathcal{I}_f^1$.

3.1.2 Blow-ups

Assuming that l in $\mathcal{I}_f^1$ is an integer, we illustrate our previous results in terms of blow-ups. We also assume that $p(z) = z^\delta$ and $b_{\gamma d} = 1$ for simplicity. Let $\pi_1(z, c) = (z, z^l c)$ and $\tilde{f} = \pi_1^{-1} \circ f \circ \pi_1$. Note that π_1 is the l^{th} compositions of the blow-up $(z, c) \rightarrow (z, zc)$. Then

$$\begin{aligned} \tilde{f}(z, c) &= (\tilde{p}(z), \tilde{q}(z, c)) = \left(p(z), \frac{q(z, z^l c)}{p(z)^l} \right) \\ &= \left(z^\delta, z^{\gamma + ld - l\delta} c^d + \sum b_{ij} z^{i+l j - l\delta} c^j \right) \\ &= \left(z^\delta, z^{\gamma + ld - l\delta} c^d \left\{ 1 + \sum b_{ij} z^{(i+l j) - (\gamma + ld)} c^{j-d} \right\} \right). \end{aligned}$$

Let $\tilde{\gamma} = \gamma + ld - l\delta$ and $\tilde{i} = i + l j - l\delta$. Then $\tilde{q}(z, c) = z^{\tilde{\gamma}} c^d + \sum b_{ij} z^{\tilde{i}} c^j$ and $0 \leq \tilde{\gamma} \leq \tilde{i}$ for any (i, j) such that $b_{ij} \neq 0$. Therefore, the Newton polygon of $\tilde{q}$ has just one vertex $(\tilde{\gamma}, d)$, $N(\tilde{q}) = D(\tilde{\gamma}, d)$, and we have the following.

Proposition 3.3 (Ueno, 2019b): If l in $\mathcal{I}_f^1$ is an integer, then $\tilde{f}$ is holomorphic and skew product on a neighbourhood of the origin. More precisely,

$$\tilde{f}(z, c) = (z^\delta, z^{\tilde{\gamma}} c^d \{1 + \eta(z, c)\}),$$

where $\eta \rightarrow 0$ as z and $c \rightarrow 0$. Moreover, it has a superattracting fixed point at the origin if $d \geq 2$ or if $d = 1$ and $l\delta < \gamma + ld$.

Remark 3.4: Even if l is rational, we can lift f to a holomorphic skew product similar to $\tilde{f}$ as stated in Subsection 5.1 in Ueno (2019b). In particular, l_1 is rational.

Because $\tilde{f}$ is a holomorphic skew product in Case 1, it is easy to construct the Böttcher coordinate for $\tilde{f}$ on a neighbourhood of the origin, which induces the Böttcher coordinate for f on $U^{l, \infty}$.

Remark 3.5: The affine transformation

$$A_1 \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i + lj - l\delta \\ j \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} l\delta \\ 0 \end{pmatrix}$$

maps the basis $\{(1, 0), (-l, 1)\}$ to $\{(1, 0), (0, 1)\}$. In other words, it maps a horizontal line and a line with slope $-l^{-1}$ to the same horizontal line and a vertical line. More precisely, it maps the line with slope $-l^{-1}$ that passes through (i, δ) to the vertical line $\{x = i\}$. In particular, if $l = l_1$, then it maps the line L_{s-1} with slope $-l_1^{-1}$ and the horizontal line that intersect at (γ, d) , to the vertical and horizontal lines that intersect at $(\tilde{\gamma}, d)$.

3.1.3 Another explanation of weights

Although $z^\gamma w^d$ is a term in q of lowest degree with respect to w , it is not a term in q of lowest degree with respect to z . We define the weight of a monomial $z^n w^m$ as $n + lm$. Then $z^\gamma w^d$ is a term in q of lowest weight if l belongs to $\mathcal{I}_f^1$. More precisely, it is the unique term in q of lowest weight if $l > l_1$, and there are other terms in q of lowest weight if $l = l_1$. Furthermore, if $d \geq 1$, then the second component Q^n of f^n contains the term $z^{\gamma^n} w^{d^n}$ and it is the unique term in Q^n of lowest degree with respect to w and of lowest weight for any $n \geq 1$. See Ueno (2024) for details.

3.2 Intervals of weights and blow-ups for Case 3

Let $s > 1$,

$$T_1 \leq \delta, \quad (\gamma, d) = (n_1, m_1), \quad l_1 = 0 \text{ and } l_2 = \frac{n_2 - n_1}{m_1 - m_2}.$$

Note that $\gamma \geq 0$ and $\delta \geq T_1 \geq d \geq 1$ by the setting.

3.2.1 Intervals and Böttcher coordinates

We define the intervals $\mathcal{I}_f^1, \mathcal{I}_f^2$ and the rectangle $\mathcal{I}_f$ as $\mathcal{I}_f^1 = \{0\}$,

$$\mathcal{I}_f^2 = \{l > 0 \mid \gamma + ld \leq l\delta \text{ and } \gamma + ld \leq i + lj \text{ for any } i \text{ and } j \text{ s.t. } b_{ij} \neq 0\}$$

and $\mathcal{I}_f = \{0\} \times \mathcal{I}_f^2$. It follows that $\max \mathcal{I}_f^2 = l_2$. In fact, if $\gamma > 0$, then $\delta > d$ and

$$\mathcal{I}_f^2 = \left[\frac{\gamma}{\delta - d}, \min \left\{ \frac{i - \gamma}{d - j} \right\} \right] = \left[\frac{\gamma}{\delta - d}, \frac{n_2 - \gamma}{d - m_2} \right] = [\alpha, l_2].$$

If $\gamma = 0$, then the inequality $\gamma + ld \leq l\delta$ is trivial since $d \leq \delta$, and so $\mathcal{I}_f^2 = (0, l_2]$.

Recall that $U = U^{0, l_2}$, where $U^{0, l} = U_r^{0, l} = \{|z|^l < r^l |w|, |w| < r\}$.

Lemma 3.6 (Ueno, 2019b): The following hold for any l in $\mathcal{I}_f^2$ if $d \geq 2$ and for any l in $\mathcal{I}_f^2 - \{\alpha\}$ if $d = 1$ and $\delta > T_1$: $f \sim f_0$ on $U_r^{0, l}$ as $r \rightarrow 0$, and $f(U_r^{0, l}) \subset U_r^{0, l}$ for small $r > 0$.

Theorem 3.7 (Ueno, 2019b): The following holds for any l in the previous lemma: ϕ is biholomorphic on $U_r^{0,l}$ and conjugates f to f_0 for small $r > 0$. Moreover, $\phi \sim id$ on $U_r^{0,l}$ as $r \rightarrow 0$.

Note that U^{0,l_2} is the largest wedge among $U^{0,l}$ for any l in $\mathcal{I}_f^2$.

3.2.2 Blow-ups

Assuming that l^{-1} is an integer for l in $\mathcal{I}_f^2$, we illustrate our previous results in terms of blow-ups. Let $\pi_2(t, w) = (tw^{l^{-1}}, w)$ and $\tilde{f} = \pi_2^{-1} \circ f \circ \pi_2$. Note that π_2 is the l^{-1} th compositions of the blow-up $(t, w) \rightarrow (tw, w)$. Let $b_{\gamma d} = 1$ for simplicity. Then

$$\begin{aligned} \tilde{f}(t, w) &= (\tilde{p}(t, w), \tilde{q}(t, w)) = \left(\frac{p(tw^{l^{-1}})}{q(tw^{l^{-1}}, w)^{l^{-1}}}, q(tw^{l^{-1}}, w) \right) \text{ and} \\ \tilde{q}(t, w) &= q(tw^{l^{-1}}, w) = t^\gamma w^{l^{-1}\gamma+d} + \sum b_{ij} t^i w^{l^{-1}i+j} \\ &= t^\gamma w^{l^{-1}\gamma+d} \left\{ 1 + \sum b_{ij} t^{i-\gamma} w^{(l^{-1}i+j)-(l^{-1}\gamma+d)} \right\}. \end{aligned}$$

Let $\tilde{d} = l^{-1}\gamma + d$ and $\tilde{j} = l^{-1}i + j$. Then $\tilde{q}(t, w) = t^\gamma w^{\tilde{d}} + \sum b_{ij} t^i w^{\tilde{j}}$, $d \leq \tilde{d} \leq \delta$ and $\tilde{d} \leq \tilde{j}$ for any (i, j) such that $b_{ij} \neq 0$. Therefore, the Newton polygon of $\tilde{q}$ has just one vertex $(\gamma, \tilde{d})$, $N(\tilde{q}) = D(\gamma, \tilde{d})$, and we have the following.

Proposition 3.8 (Ueno, 2019b): If l^{-1} is an integer for l in $\mathcal{I}_f^2$, then $\tilde{f}$ is holomorphic on a neighbourhood of the origin. More precisely,

$$\tilde{f}(t, w) = \left(t^{\delta-l^{-1}\gamma} w^{l^{-1}(\delta-\tilde{d})} \{1 + \zeta(t, w)\}, t^\gamma w^{\tilde{d}} \{1 + \eta(t, w)\} \right),$$

where ζ and $\eta \rightarrow 0$ as t and $w \rightarrow 0$. Moreover, it has a superattracting fixed point at the origin if $d \geq 2$ or if $d = 1$ and $\gamma + ld < l\delta$.

Although $\tilde{f}$ is not skew product, it is a perturbation of a monomial map near the origin. Hence we can construct the Böttcher coordinate for $\tilde{f}$ on a neighbourhood of the origin by similar arguments as in Section 6 in Ueno (2019b), or one may refer to pp.498–499 in Favre (2000), which induces the Böttcher coordinate for f on $U^{0,l}$.

Remark 3.9: The linear transformation

$$A_2 \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ l^{-1}i + j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l^{-1} & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

maps the basis $\{(1, -l^{-1}), (0, 1)\}$ to $\{(1, 0), (0, 1)\}$. In other words, it maps a line with slope $-l^{-1}$ and a vertical line to a horizontal line and the same vertical line. In particular, if $l = l_2$, then it maps the line L_1 with slope $-l_2^{-1}$ and the vertical line that intersect at (γ, d) , to the horizontal and vertical lines that intersect at $(\gamma, \tilde{d})$.

3.2.3 Another explanation of weights

Note that $z^\gamma w^d$ is a term in q of lowest degree with respect to z . Although it is not a term in q of lowest degree with respect to w , it is a term in q of lowest weight if

l belongs to $\mathcal{I}_f^2$. More precisely, it is the unique term in q of lowest weight if $l < l_2$, and there are other terms in q of lowest weight if $l = l_2$. Furthermore, Q^n contains the term $z^{\gamma_n} w^{d^n}$ and it is the unique term in Q^n of lowest degree with respect to z and of lowest weight for any $n \geq 1$.

3.3 Intervals, rectangles and blow-ups for Case 4

Let $s > 2$,

$$T_k \leq \delta \leq T_{k-1} \text{ for some } 2 \leq k \leq s-1,$$

$$(\gamma, d) = (n_k, m_k), \quad l_1 = \frac{n_k - n_{k-1}}{m_{k-1} - m_k} \text{ and } l_1 + l_2 = \frac{n_{k+1} - n_k}{m_k - m_{k+1}}.$$

Note that $\gamma > 0$ and $\delta > d \geq 1$ by the setting.

3.3.1 Intervals, rectangles and Böttcher coordinates

We define the interval $\mathcal{I}_f^1$ as

$$\mathcal{I}_f^1 = \left\{ l_{(1)} > 0 \left| \begin{array}{l} l_{(1)}\delta \leq \gamma + l_{(1)}d \\ \gamma + l_{(1)}d \leq n_j + l_{(1)}m_j \text{ for } 1 \leq j \leq k-1 \\ \gamma + l_{(1)}d < n_j + l_{(1)}m_j \text{ for } k+1 \leq j \leq s \end{array} \right. \right\},$$

the interval $\mathcal{I}_f^2$ associated with $l_{(1)}$ in $\mathcal{I}_f^1$ as

$$\mathcal{I}_f^2 = \mathcal{I}_f^2(l_{(1)}) = \left\{ l_{(2)} > 0 \left| \begin{array}{l} \tilde{\gamma} + l_{(2)}d \leq l_{(2)}\delta \text{ and } \tilde{\gamma} + l_{(2)}d \leq \tilde{i} + l_{(2)}j \\ \text{for any } i \text{ and } j \text{ s.t. } b_{ij} \neq 0 \end{array} \right. \right\},$$

where $\tilde{\gamma} = \gamma + l_{(1)}d - l_{(1)}\delta$ and $\tilde{i} = i + l_{(1)}j - l_{(1)}\delta$, and the rectangle $\mathcal{I}_f$ as

$$\mathcal{I}_f = \{(l_{(1)}, l_{(1)} + l_{(2)}) | l_{(1)} \in \mathcal{I}_f^1, l_{(2)} \in \mathcal{I}_f^2\}.$$

Let us calculate the intervals and rectangle more practically. Note that $\alpha > 0$ since $\delta > d$ and $\gamma > 0$. Since $n_j < \gamma$ and $m_j > d$ for any $1 \leq j \leq k-1$, and $n_j > \gamma$ and $m_j < d$ for any $k+1 \leq j \leq s$,

$$\begin{aligned} \mathcal{I}_f^1 &= \left[\max_{1 \leq j \leq k-1} \left\{ \frac{\gamma - n_j}{m_j - d} \right\}, \min_{k+1 \leq j \leq s} \left\{ \frac{n_j - \gamma}{d - m_j} \right\} \right) \cap \left(0, \frac{\gamma}{\delta - d} \right] \\ &= \left[\frac{\gamma - n_{k-1}}{m_{k-1} - d}, \frac{n_{k+1} - \gamma}{d - m_{k+1}} \right) \cap \left(0, \frac{\gamma}{\delta - d} \right] = [l_1, l_1 + l_2) \cap (0, \alpha]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{I}_f^2 &= \left[\frac{\tilde{\gamma}}{\delta - d}, \frac{\tilde{n}_{k+1} - \tilde{\gamma}}{d - m_{k+1}} \right] \cap \mathbb{R}_{>0} = \left[\frac{\gamma}{\delta - d} - l_{(1)}, \frac{n_{k+1} - \gamma}{d - m_{k+1}} - l_{(1)} \right] \cap \mathbb{R}_{>0} \\ &= [\alpha - l_{(1)}, l_1 + l_2 - l_{(1)}] \cap \mathbb{R}_{>0}. \end{aligned}$$

If $T_k < \delta < T_{k-1}$, then it follows from the inequality $l_1 < \alpha < l_1 + l_2$ that

$$\mathcal{I}_f^1 = [l_1, \alpha], \quad \mathcal{I}_f^2 = [\alpha - l_{(1)}, l_1 + l_2 - l_{(1)}] \cap \mathbb{R}_{>0}$$

and so $\mathcal{I}_f = [l_1, \alpha] \times [\alpha, l_1 + l_2] - \{(\alpha, \alpha)\}$.

If $T_k = \delta < T_{k-1}$, then it follows from the inequality $l_1 < \alpha = l_1 + l_2$ that

$$\mathcal{I}_f^1 = [l_1, \alpha), \quad \mathcal{I}_f^2 = \{\alpha - l_{(1)}\} \text{ and so } \mathcal{I}_f = [l_1, \alpha) \times \{\alpha\}.$$

If $T_k < \delta = T_{k-1}$, then it follows from the inequality $l_1 = \alpha < l_1 + l_2$ that

$$\mathcal{I}_f^1 = \{\alpha\}, \quad \mathcal{I}_f^2 = (0, l_2] \text{ and so } \mathcal{I}_f = \{\alpha\} \times (\alpha, l_1 + l_2].$$

In particular, $\min \mathcal{I}_f^1 = l_1$ and $\max\{l_{(1)} + l_{(2)} | l_{(1)} \in \mathcal{I}_f^1, l_{(2)} \in \mathcal{I}_f^2\} = l_1 + l_2$.

Recall that $U = U^{l_1, l_2}$, where $U^{l_{(1)}, l_{(2)}} = \{|z|^{l_{(1)}+l_{(2)}} < r^{l_{(2)}} |w|, |w| < r |z|^{l_{(1)}}\}$.

Lemma 3.10 (Ueno, 2019b): The following hold for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$ if $d \geq 2$, and for any $l_{(1)}$ in $\mathcal{I}_f^1 - \{\alpha\}$ and $l_{(2)}$ in $\mathcal{I}_f^2 - \{\alpha - l_{(1)}\}$ if $d = 1$ and $T_k < \delta < T_{k-1}$: $f \sim f_0$ on $U_r^{l_{(1)}, l_{(2)}}$ as $r \rightarrow 0$, and $f(U_r^{l_{(1)}, l_{(2)}}) \subset U_r^{l_{(1)}, l_{(2)}}$ for small $r > 0$.

Theorem 3.11 (Ueno, 2019b): The following holds for any $l_{(1)}$ and $l_{(2)}$ in the previous lemma: ϕ is biholomorphic on $U_r^{l_{(1)}, l_{(2)}}$ and conjugates f to f_0 for small $r > 0$. Moreover, $\phi \sim id$ on $U_r^{l_{(1)}, l_{(2)}}$ as $r \rightarrow 0$.

Note that U^{l_1, l_2} is the largest wedge among $U^{l_{(1)}, l_{(2)}}$ for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$.

3.3.2 Blow-ups

Assuming that $l_{(1)}$ and $l_{(2)}^{-1}$ are integers for $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$, we illustrate our previous results in terms of blow-ups. The strategy is to combine the blow-ups in Cases 2 and 3. We first blow-up f to $\tilde{f}_1$ by π_1 as in Case 2. It then turns out that $\tilde{f}_1$ is a holomorphic skew product in Case 3. We next blow-up $\tilde{f}_1$ to $\tilde{f}_2$ by π_2 as in Case 3. The map $\tilde{f}_2$ is a perturbation of a monomial map near the origin, and we obtain the Böttcher coordinate.

Let us restate the first blow-up. Let $\pi_1(z, c) = (z, z^{l_{(1)}}c)$ and $\tilde{f}_1 = \pi_1^{-1} \circ f \circ \pi_1$ as in Case 2, and let $\tilde{f}_1(z, c) = (\tilde{p}_1(z), \tilde{q}_1(z, c))$. Let $p(z) = z^\delta$ and $b_{\gamma d} = 1$ for simplicity. Then

$$\tilde{f}_1(z, c) = \left(z^\delta, z^{\gamma + l_{(1)}d - l_{(1)}\delta} c^d + \sum b_{ij} z^{i+l_{(1)}j - l_{(1)}\delta} c^j \right).$$

Let $\tilde{\gamma} = \gamma + l_{(1)}d - l_{(1)}\delta$ and $\tilde{i} = i + l_{(1)}j - l_{(1)}\delta$ as in Case 2. Then $0 \leq \tilde{\gamma} \leq \tilde{i}$ for any (i, j) such that $b_{ij} \neq 0$, and we have the following.

Proposition 3.12 (Ueno, 2019b): If $l_{(1)}$ in $\mathcal{I}_f^1$ is an integer, then $\tilde{f}_1$ is holomorphic and skew product on a neighbourhood of the origin. More precisely,

$$\tilde{f}_1(z, c) = \left(z^\delta, z^{\tilde{\gamma}} c^d + \sum b_{ij} z^{\tilde{i}} c^j \right).$$

Moreover, it has a superattracting fixed point at the origin if $d \geq 2$ or if $d = 1$ and $l_{(1)}\delta < \gamma + l_{(1)}d$.

Note that $(\tilde{\gamma}, d)$ is the vertex of the Newton polygon $N(\tilde{q}_1)$ whose x -coordinate is minimum, and that $N(\tilde{q}_1)$ has other vertices such as $(\tilde{n}_{k+1}, m_{k+1})$. Hence the situation resembles that of Case 3.

We illustrate that $\tilde{f}_1$ is actually in Case 3. Recall that L_k is the line passing through the vertices (γ, d) and (n_{k+1}, m_{k+1}) , and T_k is the y -intercept of L_k . The slope of L_k is $-(l_1 + l_2)^{-1}$ and so $T_k = (l_1 + l_2)^{-1}\gamma + d$. Let $\tilde{L}_k$ be the line passing through the vertices $(\tilde{\gamma}, d)$ and $(\tilde{n}_{k+1}, m_{k+1})$, and $\tilde{T}_k$ the y -intercept of $\tilde{L}_k$, where $\tilde{n}_{k+1} = n_{k+1} + l_{(1)}m_{k+1} - l_{(1)}\delta$. Then the slope of $\tilde{L}_k$ is $-(l_1 + l_2 - l_{(1)})^{-1}$ and so $\tilde{T}_k = (l_1 + l_2 - l_{(1)})^{-1}\tilde{\gamma} + d$. Therefore, the inequality $T_k \leq \delta$ implies the inequality $\tilde{T}_k \leq \delta$. More precisely, $\tilde{T}_k < \delta$ if $T_k < \delta$, and $\tilde{T}_k = \delta$ if $T_k = \delta$. In particular,

Proposition 3.13 (Ueno, 2019b): If $l_{(1)}$ in $\mathcal{I}_f^1$ is an integer, then $\tilde{f}_1$ is a holomorphic skew product in Case 3.

Let us restate the second blow-up. Let $\pi_2(t, c) = (tc^{l_{(2)}^{-1}}, c)$ and $\tilde{f}_2 = \pi_2^{-1} \circ \tilde{f}_1 \circ \pi_2$ as in Case 3, and let $\tilde{f}_2(t, c) = (\tilde{p}_2(t, c), \tilde{q}_2(t, c))$. Let $\tilde{d} = l_{(2)}^{-1}\tilde{\gamma} + d$ and $\tilde{j} = l_{(2)}^{-1}\tilde{i} + j$ as in Case 3. Then $\tilde{d} \leq \tilde{j}$ and $\tilde{d} \leq \delta$ for any (i, j) such that $b_{ij} \neq 0$. Note that $(\tilde{\gamma}, \tilde{d})$ is the minimum in the sense that $0 \leq \tilde{\gamma} \leq \tilde{i}$ and $d \leq \tilde{d} \leq \tilde{j}$ for any (i, j) such that $b_{ij} \neq 0$. Therefore, the Newton polygon of $\tilde{q}_2$ has just one vertex $(\tilde{\gamma}, \tilde{d})$, $N(\tilde{q}_2) = D(\tilde{\gamma}, \tilde{d})$, and we have the following.

Proposition 3.14 (Ueno, 2019b): If $l_{(1)}$ and $l_{(2)}^{-1}$ are integers for $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$, then $\tilde{f}_2$ is holomorphic on a neighbourhood of the origin. More precisely,

$$\tilde{f}_2(t, c) = \left(t^{\delta - l_{(2)}^{-1}\tilde{\gamma}} c^{l_{(2)}^{-1}(\delta - \tilde{d})} \{1 + \zeta_2(t, c)\}, t^{\tilde{\gamma}} c^{\tilde{d}} \{1 + \eta_2(t, c)\} \right),$$

where ζ_2 and $\eta_2 \rightarrow 0$ as t and $c \rightarrow 0$. Moreover, it has a superattracting fixed point at the origin if $d \geq 2$ or if $d = 1$, $l_{(1)}\delta < \gamma + l_{(1)}d$ and $\tilde{\gamma} + l_{(2)}d < l_{(2)}\delta$.

Consequently, we can construct the Böttcher coordinate for $\tilde{f}_2$ on $\{|t| < r, |c| < r\}$, which induces that for $\tilde{f}_1$ on $\{|z| < r|c|^{l_{(2)}^{-1}}, |c| < r\}$ and that for f on $U^{l_{(1)}, l_{(2)}}$.

Remark 3.15: The affine transformation

$$A \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_2^{-1} & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} l_1\delta \\ 0 \end{pmatrix} \right\}$$

is the composition of the two affine transformations

$$A_1 \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i + l_1j - l_1\delta \\ j \end{pmatrix} \text{ and } A_2 \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ l_2^{-1}i + j \end{pmatrix}.$$

Although statements similar to below hold for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$, instead of l_1 and l_2 , we illustrate only the case $l_{(1)} = l_1$ and $l_{(2)} = l_2$ for simplicity. The transformation A_1 maps the basis $\{(1, -(l_1 + l_2)^{-1}), (-l_1, 1)\}$ to $\{(1, -l_2^{-1}), (0, 1)\}$.

In other words, it maps the line L_k with slope $-(l_1 + l_2)^{-1}$ and the line L_{k-1} with slope $-l_1^{-1}$, which intersect at (γ, d) , to the line $\tilde{L}_k$ with slope $-l_2^{-1}$ and the vertical line that intersect at $(\tilde{\gamma}, d)$. The transformation A_2 maps the basis $\{(1, -l_2^{-1}), (0, 1)\}$ to $\{(1, 0), (0, 1)\}$. In other words, it maps the line $\tilde{L}_k$ and the vertical line that intersect at $(\tilde{\gamma}, d)$, to the horizontal and vertical lines that intersect at $(\tilde{\gamma}, \tilde{d})$.

3.3.3 Another explanation of weights

Note that $z^\gamma w^d$ is not a term in q of lowest degree with respect to z nor of lowest degree with respect to w . However, it is the unique term in q of lowest weight with $l = l_{(1)}$ and of lowest weight with $l = l_{(1)} + l_{(2)}$ if $(l_{(1)}, l_{(1)} + l_{(2)})$ belongs to $\mathcal{I}_f$. Furthermore, Q^n contains the term $z^{\gamma n} w^{d n}$ and it is the unique term in Q^n of lowest weight with $l = l_{(1)}$ and of lowest weight with $l = l_{(1)} + l_{(2)}$ for any $n \geq 1$.

4 Case 2: invariant wedges and attracting sets

We deal with Case 2 in this and the next sections: we prove Theorems 2.2 and 2.3, the main results on invariant wedges and the unions of all the preimages of the wedges, in this section and prove Theorems 2.4 and 2.5 and Propositions 2.6 and 2.7, the main results on plurisubharmonic functions, in the next section. Let $s > 1$,

$$\delta \leq T_{s-1} \text{ and } (\gamma, d) = (n_s, m_s).$$

Note that $\gamma > 0$ and $d \geq 0$ by the setting.

Before we give proofs of Theorems 2.2 and 2.3 in Subsection 4.2, we illustrate the theorems in terms of blow-ups in Subsection 4.1. Actually, we check the following.

Proposition 4.1: Let $\delta \leq d$. Then $\tilde{f}$ is superattracting at the origin and degenerates the c -axis for any $l > 0$. Let $\delta > d$. Then $\tilde{f}$ is superattracting at the origin for any $0 < l < \alpha$, even for $l = \alpha$ if $d \geq 2$, and degenerates the c -axis for any $0 < l < \alpha$ unless $(n_1, m_1) = (0, \delta)$.

Roughly speaking, if $\tilde{f}$ is superattracting at the origin, then the open set $U_{r_1, r_2}^{l, \infty}$ should be invariant under f , where $U_{r_1, r_2}^{l, \infty} = \{|z| < r_1, |w| < r_2 |z|^l\}$, and if $\tilde{f}$ degenerates the c -axis, then one may expect that $A_f^{l, \infty} = A_0 - E_z$.

Remark 4.2 (another illustration of Theorem 2.3): One may expect that the vertical dynamics eventually converges to the dynamics on the w -axis since $p^n(z) \rightarrow 0$ as $n \rightarrow \infty$. If $n_1 > 0$, then f degenerates the w -axis to the origin and so one may expect that Theorem 2.3 holds. If $n_1 = 0$, then $q(0, w) = b_{n_1 m_1} w^{m_1} (1 + o(1))$ and so one may expect that Theorem 2.3 holds if $\delta < m_1$ since $p(z) \sim a z^\delta$.

4.1 Illustration of Theorems 2.2 and 2.3 in terms of blow-ups

Recall that $\tilde{f} = \pi_1^{-1} \circ f \circ \pi_1$, where $\pi_1(z, c) = (z, z^l c)$. Let $\tilde{f}(z, c) = (\tilde{p}(z), \tilde{q}(z, c))$ and we assume that $p(z) = z^\delta$ and $b_{\gamma d} = 1$ for simplicity. Then $\tilde{p}(z) = z^\delta$ and

$\tilde{q}(z, c) = \sum b_{ij} z^{i+l_j-l\delta} c^j$. Let $\tilde{\gamma} = \gamma + ld - l\delta$ and $\tilde{n}_j = n_j + lm_j - l\delta$. The shape of the Newton polygon $N(\tilde{q})$ of $\tilde{q}$ is useful to check whether $\tilde{f}$ is superattracting at the origin, and whether $\tilde{f}$ degenerates the c -axis. Recall that A_1 maps the line with slope $-l^{-1}$ that passes through the point (i, δ) to the vertical line $\{x = i\}$, whereas it preserves all the horizontal lines.

Let us describe the shape of the Newton polygon of $\tilde{q}$ and properties of $\tilde{f}$ for the cases $\delta \leq d$, $\delta > d$ and $\delta < T_{s-1}$, and $\delta > d$ and $\delta = T_{s-1}$ in Subsections 4.1.1, 4.1.2 and 4.1.3, respectively, which implies Proposition 4.1.

4.1.1 Case 2: $\delta \leq d$

Recall that $\mathcal{I}_f^1 = [l_1, \infty)$ if $\delta \leq d$. See Table 6 for a summary of the shape of $N(\tilde{q})$ and properties of $\tilde{f}$, where the notation SA means that $\tilde{f}$ is superattracting at the origin, and the notation Deg means that $\tilde{f}$ degenerates the c -axis.

Table 6 Shape of $N(\tilde{q})$ when $\delta \leq d$

l	$0 < l < l_1$	l_1	$l_1 < l$
$\tilde{\gamma}$	$0 < \tilde{n}_{s-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 < \tilde{\gamma} = \tilde{n}_{s-1} < \tilde{n}_j$ for $j \leq s-2$	$0 < \tilde{\gamma} < \tilde{n}_j$ for $j \leq s-1$
$N(\tilde{q})$	has other vertices than $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d)$
$\tilde{f}$	SA and Deg	SA and Deg	SA and Deg

For any l in $\mathcal{I}_f^1$, $N(\tilde{q}) = (\tilde{\gamma}, d)$ and $\tilde{\gamma} > 0$. Hence

$$\tilde{q}(z, c) = z^{\tilde{\gamma}} c^d \{1 + \eta(z, c)\},$$

where $\eta \rightarrow 0$ as z and $c \rightarrow 0$, and $\tilde{f}$ is superattracting at the origin and degenerates the c -axis. More precisely, $\tilde{\gamma} = \tilde{n}_{s-1} < \tilde{n}_j$ for any $1 \leq j \leq s-2$ if $l = l_1$, and $\tilde{\gamma} < \tilde{n}_j$ for any $1 \leq j \leq s-1$ if $l > l_1$.

On the other hand, for any $0 < l < l_1$, $N(\tilde{q})$ has other vertices than $(\tilde{\gamma}, d)$, whereas $(\tilde{\gamma}, d)$ is the vertex of $N(\tilde{q})$ whose y -coordinate is the minimum and $\tilde{\gamma} > 0$. More precisely, the candidates of vertices of $N(\tilde{q})$ are $(\tilde{n}_j, m_j)$'s, and $(\tilde{n}_{s-1}, m_{s-1})$ is actually one of the vertices of $N(\tilde{q})$. However, since $\tilde{n}_j > 0$ for any j , $\tilde{f}$ is superattracting at the origin and degenerates the c -axis.

4.1.2 Case 2: $\delta > d$ and $\delta < T_{s-1}$

Recall that $\mathcal{I}_f^1 = [l_1, \alpha]$ if $\delta > d$ and $\delta < T_{s-1}$. See Table 7 for a summary of the shape of $N(\tilde{q})$ and properties of $\tilde{f}$.

For any l in $\mathcal{I}_f^1$, $N(\tilde{q}) = (\tilde{\gamma}, d)$. Moreover, if $l < \alpha$, then $\tilde{\gamma} > 0$ and so $\tilde{f}$ is superattracting at the origin and degenerates the c -axis. On the other hand, if $l = \alpha$, then $\tilde{\gamma} = 0$,

$$\tilde{q}(z, c) = c^d \{1 + \eta(z, c)\},$$

and so $\tilde{f}$ is superattracting at the origin only if $d \geq 2$, and $\tilde{f}$ does not degenerate the c -axis. For any $0 < l < l_1$, the same things hold as the previous case. We remark that if $l > \alpha$, then $\tilde{\gamma} < 0$ and so $\tilde{f}$ is not holomorphic.

Table 7 Shape of $N(\tilde{q})$ when $\delta > d$ and $\delta < T_{s-1}$

l	$0 < l < l_1$	l_1	$l_1 < l < \alpha$	α
$\tilde{\gamma}$	$0 < \tilde{n}_{s-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 < \tilde{\gamma} = \tilde{n}_{s-1} < \tilde{n}_j$ for $j \leq s-2$	$0 < \tilde{\gamma} < \tilde{n}_j$ for $j \leq s-1$	$0 = \tilde{\gamma} < \tilde{n}_j$ for $j \leq s-1$
$N(\tilde{q})$	has other vertices than $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d) = (0, d)$
$\tilde{f}$	SA and Deg	SA and Deg	SA and Deg	SA if $d \geq 2$

4.1.3 Case 2: $\delta > d$ and $\delta = T_{s-1}$

If $\delta = T_{s-1}$, then $\delta > d$ and $\mathcal{I}_f^1 = \{l_1\} = \{\alpha\}$. The shape of $N(\tilde{q})$ and properties of $\tilde{f}$ differ whether $(n_1, m_1) = (0, \delta)$ or not.

If $(n_1, m_1) \neq (0, \delta)$, then the situation is somehow similar to the previous case. If $l = \alpha$, then $\tilde{\gamma} = \tilde{n}_{s-1} = 0 < \tilde{n}_j$ for any $1 \leq j \leq s-2$ and so $\tilde{f}$ is superattracting at the origin only if $d \geq 2$, and it does not degenerate the c -axis. For any $0 < l < l_1$, the same things hold as the previous case. See Table 8 for a summary.

Table 8 Shape of $N(\tilde{q})$ when $\delta = T_{s-1}$ and $(n_1, m_1) \neq (0, \delta)$

l	$0 < l < l_1 = \alpha$	$l_1 = \alpha$
$\tilde{\gamma}$	$0 < \tilde{n}_{s-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 = \tilde{\gamma} = \tilde{n}_{s-1} < \tilde{n}_j$ for $j \leq s-2$
$N(\tilde{q})$	has other vertices than $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d) = (0, d)$
$\tilde{f}$	SA and Deg	SA if $d \geq 2$

Table 9 Shape of $N(\tilde{q})$ when $(n_1, m_1) = (0, \delta)$

l	$0 < l < l_1 = \alpha$	$l_1 = \alpha$
$\tilde{\gamma}$	$0 = \tilde{n}_1 < \tilde{\gamma}$	$0 = \tilde{n}_1 = \tilde{\gamma}$
$N(\tilde{q})$	has other vertices than $(\tilde{\gamma}, d)$	has the only one vertex $(\tilde{\gamma}, d) = (0, d)$
$\tilde{f}$	SA	SA if $d \geq 2$

On the other hand, if $(n_1, m_1) = (0, \delta)$, then the situation differs from the previous case. Note that $s = 2$ for this case. If $l = \alpha$, then $\tilde{\gamma} = \tilde{n}_1 = 0$,

$$\tilde{q}(z, c) = b_{0\delta}c^\delta + \cdots + b_{\gamma d}c^d + \sum_{i+l_j > l\delta} b_{ij}z^{i+l_j-l\delta}c^j = c^d \{1 + \eta(z, c)\}$$

and so $\tilde{f}$ is superattracting at the origin only if $d \geq 2$, and $\tilde{f}$ does not degenerate the c -axis. For any $0 < l < l_1$, $\tilde{\gamma} > \tilde{n}_1 = 0$,

$$\tilde{q}(z, c) = b_{0\delta}c^\delta + \sum_{i+l_j>l\delta} b_{ij}z^{i+l_j-l\delta}c^j$$

and so $\tilde{f}$ is superattracting at the origin, but it does not degenerate the c -axis. See Table 9 for a summary.

4.2 Proof of Theorems 2.2 and 2.3

The following quantity plays an important role in the proofs:

$$D = D_l = \min \{l^{-1}i + j | b_{ij} \neq 0\}.$$

By definition, $D > d$ for any $l > 0$ since $D \geq l^{-1}\gamma + d$ and $\gamma > 0$. Note that $l^{-1}i + j$ is the y -intercept of the line with slope $-l^{-1}$ that passes through (i, j) . Hence, for any $l > 0$, the minimum is realised by a vertex (n_j, m_j) . Let (N, M) be such a vertex: $D = l^{-1}M + N$. Then we have the following.

Lemma 4.3: It follows that $(N, M) = (\gamma, d)$ and $T_{s-1} \geq D = l^{-1}\gamma + d \geq \delta$ for any l in $\mathcal{I}_f^1$. Moreover, $D = T_{s-1}$ for $l = l_1$, and $D = \delta$ for $l = \alpha$ if $\delta > d$. On the other hand, $(N, M) \neq (\gamma, d)$ and $D = l^{-1}M + N \geq T_{s-1} \geq \delta$ for any $0 < l < l_1$. More precisely, $(M, N) = (n_1, m_1)$ or $(M, N) = (n_k, m_k)$ for some $2 \leq k \leq s-1$

$$\text{if } 0 < l \leq \frac{n_2 - n_1}{m_1 - m_2} \text{ or } \frac{n_k - n_{k-1}}{m_{k-1} - m_k} \leq l \leq \frac{n_{k+1} - n_k}{m_k - m_{k+1}}.$$

Consequently, the inequality $D \geq \delta$ holds for any $l > 0$ if $\delta \leq d$, and for any $0 < l \leq \alpha$ if $\delta > d$. Let us make clear when the inequality $D > \delta$ or the equality $D = \delta$ holds. If $\delta \leq d$, then $D > d \geq \delta$ for any $l > 0$. If $\delta > d$, then $D \geq \delta$ for any $0 < l < \alpha$, and $D = \delta$ for $l = \alpha$. More precisely, if $\delta > d$, then the inequality $D > \delta$ holds for any $0 < l < \alpha$ except the special case; if $(n_1, m_1) = (0, \delta)$, then $s = 2$ and $D = T_1 = T_{s-1} = \delta$ for any $0 < l < \alpha$.

Lemma 4.4: The inequality $D > \delta$ holds for any $l > 0$ if $\delta \leq d$ and for any $0 < l < \alpha$ if $\delta > d$ and $(n_1, m_1) \neq (0, \delta)$. On the other hand, the equality $D = \delta$ holds for $l = \alpha$ if $\delta > d$ and for any $0 < l \leq \alpha$ if $\delta > d$ and $(n_1, m_1) = (0, \delta)$.

4.2.1 Proof of Theorem 2.2

Theorem 2.2 follows from the inequality $D \geq \delta$. We use the following for the case $D > \delta$.

Lemma 4.5: For any $l > 0$ and for any small r_1 and r_2 , there exists a constant $C > 0$ such that $|q(z, w)| \leq C|z|^{lD}$ on $U_{r_1, r_2}^{l, \infty}$.

Proof: Since $|w| < r_2|z|^l$ on $U_{r_1, r_2}^{l, \infty}$ and $i + lj \geq lD$ for any (i, j) such that $b_{ij} \neq 0$,

$$\begin{aligned} |q(z, w)| &\leq \sum |b_{ij}| |z|^i |w|^j < \sum |b_{ij}| r_2^j |z|^{i+l_j} \\ &= \sum_{i+l_j=lD} |b_{ij}| r_2^j |z|^{lD} + \sum_{i+l_j>lD} |b_{ij}| r_2^j |z|^{i+l_j} \end{aligned}$$

$$\begin{aligned}
 &= |z|^{lD} \left\{ \sum_{i+l_j=lD} |b_{ij}|r_2^j + \sum_{i+l_j>lD} |b_{ij}|r_2^j |z|^{i+l_j-lD} \right\} \\
 &\leq |z|^{lD} \left\{ \sum_{i+l_j=lD} |b_{ij}|r_2^j + \sum_{i+l_j>lD} |b_{ij}|r_2^j r_1^{i+l_j-lD} \right\}
 \end{aligned}$$

on $U_{r_1, r_2}^{l, \infty}$. The first sum in the last expression is bounded from above since the set of the pairs (i, j) such that $i + l_j = lD$ and $b_{ij} \neq 0$ is finite, and the second sum tends to 0 as $r_1 \rightarrow 0$ since $i + l_j - lD > 0$. $\square$

The following is a precise version of Theorem 2.2.

Theorem 4.6: The following holds for any $l > 0$ if $\delta \leq d$ and for any $0 < l < \alpha$ if $\delta > d$: $f(U_{r_1, r_2}^{l, \infty}) \subset U_{r_1, r_2}^{l, \infty}$ for suitable small r_1 and r_2 . More precisely, r_1 and r_2 have to satisfy the inequality $Cr_1^{l(D-\delta)} < r_2$ or $2Cr_1^{l(D^*-\delta)} < r_2$ for some constants C and D^* if $D > \delta$ or $D = \delta$, where $D^* > \delta$.

Proof: Let $U = U_{r_1, r_2}^{l, \infty}$ in the proof. Since the inequality $|p(z)| < r_1$ is trivial, it is enough to show the existence of r_1 and r_2 such that $|q(z, w)| < r_2 |p(z)|^l$ for any (z, w) in U . We have the inequality $D \geq \delta$ by Lemma 4.4. The proof differs whether $D > \delta$ or $D = \delta$.

We first consider the case $D > \delta$. It follows from Lemma 4.5 that

$$|p(z)| > C_1 |z|^\delta \text{ and } |q(z, w)| \leq C_2 |z|^{lD} \text{ on } U$$

for some constants C_1 and C_2 . Hence

$$\left| \frac{q(z, w)}{p(z)^l} \right| < \frac{C_2 |z|^{lD}}{(C_1 |z|^\delta)^l} = \frac{C_2}{C_1^l} |z|^{l(D-\delta)} < Cr_1^{l(D-\delta)} \text{ on } U,$$

where $C = C_2 \cdot C_1^{-l}$. Since $D > \delta$, the map f preserves U if $Cr_1^{l(D-\delta)} < r_2$.

We next consider the case $D = \delta$; that is, the special case $(n_1, m_1) = (0, \delta)$. Recall that $\delta > d$ and $(N, M) = (n_1, m_1)$ for any $0 < l < \alpha$ if $(n_1, m_1) = (0, \delta)$. Let us assume that $p(z) = z^\delta$ for simplicity. Then

$$\begin{aligned}
 \left| \frac{q(z, w)}{p(z)^l} \right| &\leq \frac{\sum |b_{ij}| |z|^i |w|^j}{|z|^{l\delta}} \leq \sum |b_{ij}| r_2^j |z|^{i+l_j-l\delta} \\
 &= \sum_{i+l_j=l\delta} |b_{ij}| r_2^j + \sum_{i+l_j>l\delta} |b_{ij}| r_2^j |z|^{i+l_j-l\delta} \\
 &= |b_{NM}| r_2^M + \sum_{i+l_j>l\delta} |b_{ij}| r_2^j |z|^{i+l_j-l\delta} \text{ on } U.
 \end{aligned}$$

Since $M = m_1 = \delta \geq 2$, $|b_{NM}| r_2^M \leq |b_{NM}| r_2^2 < r_2/2$ if $r_2 < (2|b_{NM}|)^{-1}$. Let

$$D^* = \min \{ l^{-1}i + j \mid (i, j) \neq (0, \delta) \text{ and } b_{ij} \neq 0 \}.$$

Then $D^* > D = \delta$ and the following inequality holds on U for some constant C :

$$\begin{aligned}
 \sum_{i+l_j>l\delta} |b_{ij}| r_2^j |z|^{i+l_j-l\delta} &= \left\{ \sum_{i+l_j \geq lD^*} |b_{ij}| r_2^j |z|^{i+l_j-lD^*} \right\} |z|^{l(D^*-\delta)} \\
 &< \left\{ \sum_{i+l_j \geq lD^*} |b_{ij}| r_2^j r_1^{i+l_j-lD^*} \right\} r_1^{l(D^*-\delta)} \leq Cr_1^{l(D^*-\delta)}.
 \end{aligned}$$

Therefore, if $2Cr_1^{l(D^*-\delta)} < r_2 < (2|b_{NM}|)^{-1}$, then

$$\left| \frac{q(z, w)}{p(z)^l} \right| < \frac{r_2}{2} + \frac{r_2}{2} = r_2.$$

□

Remark 4.7: As the same strategy, we can show that the statement in the theorem above holds even for $l = \alpha$ if $\delta > d \geq 2$, without using the fact $f \sim f_0$ on U_r as in Lemmas 2.1 and 3.1. Let us give a short proof. As in the proof above,

$$\left| \frac{q(z, w)}{p(z)^l} \right| \leq \sum_{i+\alpha j=\alpha\delta} |b_{ij}|r_2^j + \sum_{i+\alpha j>\alpha\delta} |b_{ij}|r_2^j|z|^{i+\alpha j-\alpha\delta} \text{ on } U_{r_1, r_2}^{\alpha, \infty}.$$

Note that the set of the pairs (i, j) such that $i + \alpha j = \alpha\delta$ and $b_{ij} \neq 0$ is finite, which includes (γ, d) , and the minimum j such that $i + \alpha j = \alpha\delta$ is d . Since $d \geq 2$, the first sum is less than $r_2/2$ for small r_2 . Let

$$\begin{aligned} D^* &= \min \{ \alpha^{-1}i + j | (i, j) \neq (\gamma, d) \text{ and } b_{ij} \neq 0 \} \text{ if } \delta < T_{s-1}, \text{ and} \\ D^* &= \min \{ \alpha^{-1}i + j | i + \alpha j > \alpha\delta \text{ and } b_{ij} \neq 0 \} \text{ if } \delta = T_{s-1}. \end{aligned}$$

Then $D^* > D = \delta$ and the second sum is less than or equal to $Cr_1^{l(D^*-\delta)}$ for some constant C , which is less than $r_2/2$ for small r_1 .

Remark 4.8: As in the proof above, if $D > \delta$, then r_1 has to satisfy the inequality

$$r_1 \leq \left(\frac{r_2}{C} \right)^{\frac{1}{l(D-\delta)}}.$$

Note that $l(D - \delta) = N + lM - l\delta \rightarrow N$ as $l \rightarrow 0$, and $N = n_1$ for small enough l by Lemma 4.3. Hence, for small enough l , it is enough to satisfy the inequality

$$r_1 \leq \left(\frac{r_2}{C} \right)^{\frac{1}{n_1}}.$$

Therefore, r_1 need not converge to 0 as $l \rightarrow 0$.

4.2.2 Proof of Theorem 2.3

Let $V^l = \{0 < |z| < r, |z|^l \leq |w| < r_3\}$. Then the union $U^{l, \infty} \cup V^l \cup \{z = 0\}$ is a neighbourhood of the origin.

Lemma 4.9: For any $l > 0$ and for any small r and r_3 , there exists a constant $C > 0$ such that $|q(z, w)| \leq C|w|^D$ on V^l .

Proof: Since $|z| \leq (r^{-1}|w|)^{l^{-1}}$ on V^l ,

$$\begin{aligned} |q(z, w)| &\leq \sum |b_{ij}z^i w^j| \leq \sum |b_{ij}|r^{-l^{-1}i}|w|^{l^{-1}i+j} \\ &= \sum_{i+l j=lD} |b_{ij}|r^{-l^{-1}i}|w|^D + \sum_{i+l j>lD} |b_{ij}|r^{-l^{-1}i}|w|^{l^{-1}i+j} \\ &\leq |w|^D \left\{ \sum_{i+l j=lD} |b_{ij}|r^{-l^{-1}i} + \sum_{i+l j>lD} |b_{ij}|r^{-l^{-1}i}r_3^{(l^{-1}i+j)-D} \right\} \end{aligned}$$

on V^l . The first sum in the last expression is bounded from above since the set of the pairs (i, j) such that $i + lj = lD$ and $b_{ij} \neq 0$ is finite, and the second sum tends to 0 as $r_3 \rightarrow 0$ since $l^{-1}i + j - D > 0$. $\square$

The proposition below follows from the inequality $D > \delta$ and the lemma above, which implies Theorem 2.3.

Proposition 4.10: The following holds for any $l > 0$ if $\delta \leq d$ and for any $0 < l < \alpha$ if $\delta > d$ and $(n_1, m_1) \neq (0, \delta)$: for any (z, w) in V^l with suitable small r and r_3 , there exists an integer n such that $f^n(z, w)$ belongs to $U^{l, \infty}$.

Proof: It follows from Lemma 4.9 that

$$|p(z)| > (C_1|z|)^\delta \text{ and } |q(z, w)| < (C_2|w|)^D \text{ on } V^l$$

for some constants C_1 and C_2 . We derive a contradiction by assuming that $f^n(z, w)$ belongs to V^l for all $n > 0$. Let $f^n(z, w) = (z_n, w_n)$. Then $r|z_n|^l \leq |w_n|$ by the assumption. Since

$$|z_n| > C_1^{\frac{\delta^n - 1}{\delta - 1}} |z|^{\delta^n} > C_1^{\frac{\delta^n}{\delta - 1}} |z|^{\delta^n} \text{ and } |w_n| < C_2^{\frac{D^n - 1}{D - 1}} |w|^{D^n} < C_2^{\frac{D^n}{D - 1}} |w|^{D^n},$$

we have the inequality

$$r(C_1^{\frac{1}{\delta - 1}} |z|)^{l\delta^n} < r|z_n|^l \leq |w_n| < (C_2^{\frac{1}{D - 1}} |w|)^{D^n}.$$

Recall that $D > \delta$ by Lemma 4.4. By taking r and r_3 small enough, we may assume that $C_1^{\frac{1}{\delta - 1}} r < 1$ and $C_2^{\frac{1}{D - 1}} r_3 < 1$. Therefore, the convergence speed of $(C_2^{\frac{1}{D - 1}} |w|)^{D^n}$ is faster than that of $\{(C_1^{\frac{1}{\delta - 1}} |z|)^l\}^{\delta^n}$, which contradict the inequality above. $\square$

5 Case 2: plurisubharmonic functions

We continue the study of the dynamics for Case 2. We show the existence and properties of plurisubharmonic functions and prove Theorems 2.4 and 2.5 and Propositions 2.6 and 2.7 in this section, assuming that $a = b_{\gamma d} = 1$.

We first deal with the monomial skew products in Subsection 5.1. Although they are not exactly in Case 2, the investigation in this subsection is helpful to understand the other cases. We then deal with the case $\delta < T_{s-1}$ and $d \geq 1$ in Subsection 5.2. The Böttcher coordinate exists for this case, and the dynamics depends on the magnitude relation between δ and d . The case $d = 0$ is dealt with in Subsection 5.3. We finally deal with the case $\delta = T_{s-1}$ in Subsection 5.4, which is separated into the subcases $n_{s-1} > 0$ and $n_{s-1} = 0$. One may refer Table 1 in Subsection 2.1 for a comparison chart.

Theorem 2.4 follows from Propositions 5.5, 5.7 and 5.26. Theorem 2.5 follows from Proposition 5.8, Corollary 5.9, Propositions 5.25, 5.28 and 5.29. Propositions 2.6 and 2.7 follow from Corollaries 5.17 and 5.22, respectively.

5.1 Monomial maps

Let $f_0(z, w) = (z^\delta, z^\gamma w^d)$, $\delta \geq 2$, $\gamma \geq 0$, $d \geq 1$ and $\gamma + d \geq 2$. Then f_0 has a superattracting fixed point at the origin. Let

$$\alpha = \frac{\gamma}{\delta - d}$$

if $\delta \neq d$. Note that $\alpha < 0$ if $\delta < d$, and $\alpha > 0$ if $\delta > d$.

Let $f_0^n(z, w) = (z_n, w_n)$. Then $z_n = z^{\delta^n}$ and $w_n = z^{\gamma_n} w^{d^n}$, where

$$\gamma_n = \alpha(\delta^n - d^n) \text{ if } \delta \neq d, \text{ and } \gamma_n = n\gamma d^{n-1} \text{ if } \delta = d.$$

It is clear that $G_p(z) = \log |z|$.

We first consider the case $\delta \neq d$. If $\delta \neq d$ and $z \neq 0$, then

$$|w_n| = \left| z^{\alpha(\delta^n - d^n)} w^{d^n} \right| = \left| z^{\alpha \delta^n} \left(\frac{w}{z^\alpha} \right)^{d^n} \right|.$$

Hence

$$\left| \frac{w_n}{z_n^\alpha} \right| = \left| \frac{w}{z^\alpha} \right|^{d^n} \text{ and so } G_z^\alpha(w) = \log \left| \frac{w}{z^\alpha} \right|.$$

Lemma 5.1: If $\delta < d$, then $G_z^\alpha(w) = \log |z^{-\alpha} w|$ on $\mathbb{C}^2$. If $\delta > d$ and $\gamma > 0$, then $G_z^\alpha(w) = \log |w/z^\alpha|$ on $\{z \neq 0\}$. If $\delta \neq d$ and $\gamma = 0$, then $G_z^\alpha(w) = \log |w|$ on $\mathbb{C}^2$.

We also have the following.

Lemma 5.2: If $\delta < d$, then $G_z(w) = \log |z^{-\alpha} w|$ on $\mathbb{C}^2$. If $\delta > d$ and $\gamma > 0$, then $G_z(w) = \alpha \log |z|$ on $\{w \neq 0\}$ and $G_z(w) = -\infty$ on $\{w = 0\}$. If $\delta > d$ and $\gamma = 0$, then $G_z(w) = 0$ on $\{w \neq 0\}$ and $G_z(w) = -\infty$ on $\{w = 0\}$.

Proof: If $\delta < d$, then

$$\begin{aligned} G_z(w) &= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| z^{\alpha \delta^n} \left(\frac{w}{z^\alpha} \right)^{d^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{(z^{-\alpha} w)^{d^n}}{(z^{-\alpha})^{\delta^n}} \right| \\ &= \log |z^{-\alpha} w| - \lim_{n \rightarrow \infty} (-\alpha) \left(\frac{\delta}{d} \right)^n \log |z| = \log |z^{-\alpha} w| \end{aligned}$$

on $\{z \neq 0\}$. It is clear that $G_z(w) = -\infty$ on $\{z = 0\}$.

If $\delta > d$ and $\gamma > 0$, then

$$\begin{aligned} G_z(w) &= \lim_{n \rightarrow \infty} \frac{1}{\delta^n} \log \left| z^{\alpha \delta^n} \left(\frac{w}{z^\alpha} \right)^{d^n} \right| \\ &= \alpha \log |z| + \lim_{n \rightarrow \infty} \left(\frac{d}{\delta} \right)^n \log \left| \frac{w}{z^\alpha} \right| = \alpha \log |z| \end{aligned}$$

on $\{zw \neq 0\}$. It is clear that $G_z(w) = -\infty$ on $\{zw = 0\}$. □

Remark 5.3: If $\delta \neq d$, then f_0 is semiconjugate to the product $(z, c) \rightarrow (z^\delta, c^d)$ by $\pi(z, c) = (z^{\delta-d}, z^\gamma c)$. We can replace π by $\pi(z, c) = (z, z^\alpha c)$ if α is an integer.

We next consider the case $\delta = d$. If $\delta = d$ and $z \neq 0$, then

$$w_n = z^{n\gamma d^{n-1}} w^{d^n} \text{ and so } G_z^\infty(w) = \log |w|.$$

If $\delta = d$, $\gamma > 0$ and $|z| < 1$, then

$$\begin{aligned} G_z(w) &= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| z^{n\gamma d^{n-1}} w^{d^n} \right| \\ &= \lim_{n \rightarrow \infty} \left\{ n \frac{\gamma}{d} \log |z| + \log |w| \right\} = -\infty. \end{aligned}$$

Lemma 5.4: If $\delta = d$ and $\gamma > 0$, then $G_z^\infty(w) = \log |w|$ on $\{z \neq 0\}$ and $G_z(w) = -\infty$ on $\{|z| < 1\}$. If $\delta = d$ and $\gamma = 0$, then $G_z^\infty(w) = G_z(w) = \log |w|$ on $\mathbb{C}^2$.

These lemmas can be summarised into the following two tables.

Table 10 Psh functions for monomial maps when $\gamma > 0$

	G_z^α or G_z^∞	G_z	A_0
$\delta < d$	$\log z^{-\alpha} w $ on $\mathbb{C}^2$	$\log z^{-\alpha} w $ on $\mathbb{C}^2$	$\{ z < 1, z^{-\alpha} w < 1\}$
$\delta > d$	$\log w/z^\alpha $ on $\{z \neq 0\}$	$\alpha \log z $ on $\{w \neq 0\}$	$\{ z < 1\}$
$\delta = d$	$\log w $ on $\{z \neq 0\}$	$-\infty$ on $\{ z < 1\}$	$\{ z < 1\}$

Table 11 Psh functions for monomial maps when $\gamma = 0$

	G_z^α or G_z^∞	G_z	A_0
$\delta < d$	$\log w $ on $\mathbb{C}^2$	$\log w $ on $\mathbb{C}^2$	$\{ z < 1, w < 1\}$
$\delta > d$	$\log w $ on $\mathbb{C}^2$	0 on $\{w \neq 0\}$	$\{ z < 1, w < 1\}$
$\delta = d$	$\log w $ on $\mathbb{C}^2$	$\log w $ on $\mathbb{C}^2$	$\{ z < 1, w < 1\}$

5.2 Case 2: $\delta < T_{s-1}$ and $d \geq 1$

Let $\delta < T_{s-1}$ and $d \geq 1$ in this subsection. Then the Böttcher coordinate exists on U . The dynamics differs whether $\delta < d$, $\delta = d$ or $\delta > d$. Recall that $\alpha < 0$ and $\mathcal{I}_f^1 = [l_1, \infty)$ if $\delta < d$, that $\mathcal{I}_f^1 = [l_1, \infty)$ if $\delta = d$, and that $\alpha > 0$ and $\mathcal{I}_f^1 = [l_1, \alpha]$ if $T_{s-1} > \delta > d$.

We first show the existence and properties of plurisubharmonic functions on $A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1$ or $\mathcal{I}_f^1 - \{\alpha\}$ in Subsection 5.2.1, and then evaluate the limit supremum and inferior of G_z^α toward the boundary $\partial A_f^{l, \infty}$ in Subsections 5.2.2 and 5.2.3, respectively.

5.2.1 Existence and basic properties

Because f is conjugate to f_0 on $U^{l, \infty}$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, the limit G_z^α or G_z^∞ exists on $U^{l, \infty}$, where $U^{l, \infty} = \{|z| < r, |w| < r|z|^l\}$. More precisely, as shown below,

$$\begin{aligned}
G_z^\alpha(w) &= \log |\phi_1(z)^{-\alpha} \phi_2(z, w)| \\
&= \log |z^{-\alpha} w| + o(1) \text{ on } U_r^{l, \infty} \text{ as } r \rightarrow 0 \text{ if } \delta < d, \\
G_z^\infty(w) &= \log |\phi_2(z, w)| \\
&= \log |w| + o(1) \text{ on } U_r^{l, \infty} \text{ as } r \rightarrow 0 \quad \text{if } \delta = d, \\
G_z^\alpha(w) &= \log \left| \frac{\phi_2(z, w)}{\phi_1(z)^\alpha} \right| \\
&= \log \left| \frac{w}{z^\alpha} \right| + o(1) \text{ on } U_r^{l, \infty} \text{ as } r \rightarrow 0 \quad \text{if } \delta > d,
\end{aligned}$$

where $\phi = (\phi_1, \phi_2)$ is the Böttcher coordinate for f on $U^{l, \infty}$. It extends to a plurisubharmonic function on $A_f^{l, \infty}$ via the equations $G^\alpha \circ f = dG^\alpha$ and $G^\infty \circ f^n = d^n G^\infty + n\gamma d^{n-1} G_p \circ \pi_z$, where $G^\alpha(z, w) = G_z^\alpha(w)$, $G^\infty(z, w) = G_z^\infty(w)$ and π_z is the projection to the z -coordinate. We remark that $|\phi_1|$ and $|\phi_2|$ extend to continuous functions from $A_f^{l, \infty}$ to $[0, 1)$. Theorem 2.3 indicates that $A_f^{l, \infty} = A_0 - E_z$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$. Consequently, we obtain the following.

Proposition 5.5: Let $\delta < T_{s-1}$. If $\delta < d$, then $G_z^\alpha = G_z$ on A_0 . It is plurisubharmonic and negative on $A_0 - E_z$, pluriharmonic on $A_0 - E$ and $-\infty$ on $E \cap A_0$. Moreover, $G_z^\alpha(w) = \log |z^{-\alpha} w| + o(1)$ on $U_r^{l_1, \infty}$ as $r \rightarrow 0$. If $\delta = d$ or $\delta > d \geq 1$, then G_z^∞ or G_z^α is plurisubharmonic on $A_0 - E_z$. It is pluriharmonic on $A_0 - E$, $-\infty$ on $E_w - E_z$, ∞ on $E_z - E_w$ and not defined on $E_z \cap E_w$. Moreover, $G_z^\infty(w) = \log |w| + o(1)$ or $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on $U_r^{l_1, \infty}$ as $r \rightarrow 0$.

Proof: We only give a proof of the equality $G_z^\alpha = G_z$ on A_0 for the case $\delta < d$; the other parts follow from the lemma below. Let $\delta < d$ and $(z_n, w_n) = f^n(z, w)$. Then

$$\begin{aligned}
G_z^\alpha(w) &= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{z_n^\alpha} \right| = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log |w_n| - \lim_{n \rightarrow \infty} \frac{\alpha}{d^n} \log |z_n| \\
&= G_z(w) - 0 = G_z(w) \text{ on } U^{l_1, \infty},
\end{aligned}$$

which extends to $A_0 - E_z$. The equality $G_z^\alpha = -\infty$ on E_z follows from the definition. On the other hand, the equality $G_z = -\infty$ on $E_z \cap A_0$ follows from the dynamics of f on the w -axis. In fact, $q(0, w)$ degenerates the w -axis to the origin if $n_1 > 0$, or it has a superattracting fixed point at $w = 0$ with the order m_1 if $n_1 = 0$, where $m_1 > d$. $\square$

Lemma 5.6: Let $\delta < T_{s-1}$. If $\delta < d$, then $G_p = \log |\phi_1|$ on $\{|z| < r\}$ and $G_z^\alpha = \log |\phi_1^{-\alpha} \phi_2|$ on $U^{l_1, \infty}$. If $\delta = d$, then $G_z^\infty = \log |\phi_2|$ on $U^{l_1, \infty}$. If $\delta > d \geq 1$, then $G_z^\alpha = \log |\phi_2/\phi_1^\alpha|$ on $U^{l_1, \infty}$.

Proof: We treat explicitly only the case $\delta < d$ and the case $\delta = d$, the case $\delta > d$ being completely analogous to the case $\delta < d$ and left to the reader.

Let $\delta < d$. Since $f_0^n(z, w) = (z^{\delta^n}, z^{\gamma_n} w^{d^n})$, formally,

$$f_0^{-n}(Z, W) = \left(Z^{1/\delta^n}, \left(\frac{W}{Z^{\gamma_n/\delta^n}} \right)^{1/d^n} \right).$$

Let $(z_n, w_n) = f^n(z, w)$. Since $\phi(z, w) = \lim_{n \rightarrow \infty} f_0^{-n} \circ f^n(z, w)$,

$$\phi(z, w) = \lim_{n \rightarrow \infty} \left(z_n^{1/\delta^n}, \left(\frac{w_n}{z_n^{\gamma_n/\delta^n}} \right)^{1/d^n} \right).$$

We remark that the composition $f_0^{-n} \circ f^n$ is well-defined and converges uniformly to ϕ on $U^{l, \infty}$ as stated in Ueno (2019b). Hence $\log |\phi_1(z)| = \lim_{n \rightarrow \infty} \delta^{-n} \log |z_n| = G_p(z)$ and, since $\gamma_n = \alpha(\delta^n - d^n)$,

$$\begin{aligned} \log |\phi_2(z, w)| &= \lim_{n \rightarrow \infty} \log \left| \left(\frac{w_n}{z_n^{\gamma_n/\delta^n}} \right)^{1/d^n} \right| = \lim_{n \rightarrow \infty} \log \left| z_n^{\alpha/\delta^n} \left(\frac{w_n}{z_n^\alpha} \right)^{1/d^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{\alpha}{\delta^n} \log |z_n| + \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{z_n^\alpha} \right| = \alpha G_p(z) + G_z^\alpha(w). \end{aligned}$$

Therefore, $G_z^\alpha(w) = \log |\phi_2(z, w)| - \alpha \log |\phi_1(z)| = \log |\phi_1(z)^{-\alpha} \phi_2(z, w)|$ on $U^{l, \infty}$.

Let $\delta = d$. Since $\gamma_n = n\gamma d^{n-1}$,

$$\phi_2(z, w) = \lim_{n \rightarrow \infty} \left(\frac{w_n}{z_n^{\gamma_n/d^n}} \right)^{1/d^n} = \lim_{n \rightarrow \infty} \left(\frac{w_n}{z_n^{n\gamma/d}} \right)^{1/d^n}$$

and so

$$\log |\phi_2(z, w)| = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{z_n^{n\gamma/d}} \right| = G_z^\infty(w) \text{ on } U^{l, \infty}.$$

□

Moreover, G_z^α is plurisubharmonic on A_0 if $\delta < d$.

Proposition 5.7: If $\delta < d$, then G_z^α is plurisubharmonic on A_0 .

Proof: There is a constant $C > 0$ such that $|q(z, w)| \leq C|w|^d$ on a neighbourhood of the origin, because $d \leq j$ for any j such that $b_{ij} \neq 0$. Hence we have the inequality

$$A|q(z, w)| \leq (A|w|)^d$$

on the neighbourhood, where $A = |C|^{1/(d-1)}$. Let $g_n(z, w) = d^{-n} \log |Aw_n|$. Then $g_{n+1} \leq g_n$ and so $\{g_n\}$ is a decreasing sequence of plurisubharmonic functions. Therefore, g_n converges to a plurisubharmonic function g_∞ on the neighbourhood. Since $g_n(z, w) = d^{-n} \log |w_n| + d^{-n} \log |A|$, it follows that $g_\infty(z, w) = G_z^\alpha(w)$. □

The limit G_z also exists on $U^{l, \infty}$, which extends to A_0 .

Proposition 5.8: Let $\delta < T_{s-1}$. If $\delta < d$, then $G_z = G_z^\alpha$ on A_0 . If $\delta = d$, then $G_z = -\infty$ on A_0 . If $\delta > d \geq 1$, then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ and $-\infty$ on E_w . It is plurisubharmonic on $A_0 - (E_w - E_z)$, pluriharmonic on $A_0 - E$ and $-\infty$ on $E \cap A_0$.

Proof: We give proofs for the case $\delta = d$ and the case $\delta > d$; the claim for the case $\delta < d$ is already stated in Proposition 5.5.

Let $\delta = d$. It follows that

$$\begin{aligned} G_z^\infty(w) &= \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \left| \frac{w_n}{z_n^{n\gamma/d}} \right| = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log |w_n| - \lim_{n \rightarrow \infty} \frac{n\gamma}{d} \cdot \frac{1}{d^n} \log |z_n| \\ &= G_z(w) - \infty \cdot G_p(z) = G_z(w) + \infty. \end{aligned}$$

Since $G_z^\infty < \infty$ on $A_0 - E_z$, $G_z = -\infty$ on $A_0 - E_z$. The equality $G_z = -\infty$ on $E_z \cap A_0$ follows from the dynamics on the w -axis as the same as the proof of Proposition 5.5.

Let $\delta > d$. It follows that

$$\begin{aligned} G_z(w) - \alpha G_p(z) &= \lim_{n \rightarrow \infty} \frac{1}{\delta^n} \log |w_n| - \lim_{n \rightarrow \infty} \frac{\alpha}{\delta^n} \log |z_n| = \lim_{n \rightarrow \infty} \frac{1}{\delta^n} \log \left| \frac{w_n}{z_n^\alpha} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{d}{\delta} \right)^n \cdot \frac{1}{d^n} \log \left| \frac{w_n}{z_n^\alpha} \right| = 0 \cdot G_z^\alpha(w) = 0 \end{aligned}$$

on $U^{1,\infty} - \{w=0\}$, which extends to $A_0 - E$. The equality $G_z = -\infty$ on E_w follows from the definition, and the equality $G_z = -\infty$ on $E_z \cap A_0$ follows from the dynamics on the w -axis as the same as the proof of Proposition 5.5. $\square$

The existence of G_z implies that of G_f or G_f^α if $\delta \geq d$.

Corollary 5.9: Let $\delta < T_{s-1}$. If $\delta = d$, then $G_f = G_p \circ \pi_z$ on A_0 . If $\delta > d \geq 1$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

5.2.2 Asymptotic behaviour of G_z^α toward $\partial A_f^{l,\infty}$: limit supremum

Let $S^l = \{|z| < r, |w| = r|z|^l\} = \partial U^{l,\infty} \cap \{|z| < r\}$ and $S_n^l = f^{-n}(S^l)$. Let E_{deg} be the union of all the preimages of the set of vertical lines in $A_p \times \mathbb{C}$ that are degenerated to points by f :

$$E_{deg} = \bigcup_L \left\{ \bigcup_{n \geq 0} f^{-n}(L) \right\}, \quad (1)$$

where L is a fibre in $A_p \times \mathbb{C}$ that is degenerated to a point by f . Note that E_{deg} can be the empty set or included in E_z , which holds if f is monomial. Let $E_{deg}^* = \pi_z(E_{deg})$.

Remark 5.10: If $\delta < d$, then f maps $\{w=0\}$ to itself. Hence $f(E_{deg}) \subset \{w=0\}$ and so $E_{deg} \subset A_f \cup E_z$.

Lemma 5.11: If $\delta < d$ or $\delta = d$, then for any l in $\mathcal{I}_f^1$ and for any z in $A_p - E_p \cup E_{deg}^*$, we have that $S_n^l \cap \mathbb{C}_z \neq \emptyset$ for any large n , and $G_z^\alpha|_{S_n^l} \rightarrow 0$ or $G_z^\infty|_{S_n^l} \rightarrow +\infty$ as $n \rightarrow \infty$, respectively, where $\mathbb{C}_z = \{z\} \times \mathbb{C}$. Let $T_{s-1} > \delta > d \geq 1$. Then for any l in $\mathcal{I}_f^1$ if $d \geq 2$, for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, and for any z in $A_p - E_p \cup E_{deg}^*$, we have that $S_n^l \cap \mathbb{C}_z \neq \emptyset$ for any large n , and

$$G_z^\alpha|_{S_n^l} \rightarrow \begin{cases} \infty \text{ as } n \rightarrow \infty & \text{if } l_1 \leq l < \alpha, \\ 0 \text{ as } n \rightarrow \infty & \text{if } l = \alpha \text{ and } d \geq 2. \end{cases}$$

The proof for the case $\delta < d$ is similar to that of Theorem 4.5 in Ueno (2016a), but not the same; we deal with any l in $\mathcal{I}_f^1$ not only α . We also remark that one can prove the lemma for the case $\delta < d$ by a similar fashion to Theorem 5.4 in Ueno (2016a) since $G_z^\alpha = G_z$. Let $f^n(z, w) = (p^n(z), Q_z^n(w))$.

Proof: Let $z \in A_p$ and $N \geq 1$ the smallest integer such that $|p^N(z)| < r$. Then $S_n^l \cap \mathbb{C}_z = \emptyset$ for any $n \geq 1$ if $z \in E_{deg}^*$ and $f^N(\mathbb{C}_z) \notin U^{l, \infty}$ and for any $n \neq N$ if $z \in E_{deg}^*$ and $f^N(\mathbb{C}_z) \in U^{l, \infty}$. On the other hand, if $z \notin E_{deg}^*$, then $(Q_z^n)^{-1}(U^{l, \infty} \cap \mathbb{C}_{p^n(z)}) \neq \emptyset$ and hence $S_n^l \cap \mathbb{C}_z \neq \emptyset$ for any $n \geq N + 1$.

Let $\delta < d$. By Proposition 5.5, there is a constant $C > 0$ such that

$$|G_z^\alpha(w) - \log |z^{-\alpha}w|| < C \text{ on } U^{l, \infty}.$$

Since $|z^{-\alpha}w| = r|z|^{l-\alpha}$ on S^l , where $l - \alpha > l \geq l_1 > 0$,

$$|G_{p^n(z)}^\alpha(Q_z^n(w)) - \log r|p^n(z)|^{l-\alpha}| \leq C \text{ on } S_n^l.$$

Since $G_{p^n(z)}^\alpha(Q_z^n(w)) = d^n G_z^\alpha(w)$,

$$|G_z^\alpha(w) - d^{-n} \log r|p^n(z)|^{l-\alpha}| \leq d^{-n} C \text{ on } S_n^l.$$

Fix any z in $A_p - E_p \cup E_{deg}^*$. Note that $G_p(z) \neq -\infty$ since $z \notin E_p$. Since $\deg p = \delta < d$,

$$\frac{1}{d^n} \log |p^n(z)|^{l-\alpha} = \left(\frac{\delta}{d}\right)^n \cdot \frac{l-\alpha}{\delta^n} \log |p^n(z)| \rightarrow 0 \cdot (l-\alpha)G_p(z) = 0$$

and $d^{-n}C \rightarrow 0$ as $n \rightarrow \infty$.

Let $\delta = d$. By Proposition 5.5, there is a constant $C > 0$ such that

$$|G_z^\infty(w) - \log |w|| < C \text{ on } U^{l, \infty}.$$

Since $|w| = r|z|^l$ on S_n^l ,

$$|G_{p^n(z)}^\infty(Q_z^n(w)) - \log r|p^n(z)|^l| \leq C \text{ on } S_n^l.$$

Since $G_{p^n(z)}^\infty(Q_z^n(w)) = d^n G_z^\infty(w) + n\gamma d^{n-1}G_p(z)$,

$$\left| G_z^\infty(w) + n\frac{\gamma}{d}G_p(z) - \frac{\log r}{d^n} - \frac{l}{d^n} \log |p^n(z)| \right| \leq \frac{C}{d^n} \text{ on } S_n^l.$$

Since $\deg p = \delta = d$ and $G_p(z) \neq -\infty$ by the setting,

$$-n\frac{\gamma}{d}G_p(z) + \frac{l}{d^n} \log |p^n(z)| \rightarrow \infty - lG_p(z) = \infty$$

as $n \rightarrow \infty$.

Let $\delta > d$. The proof is similar to the case $\delta < d$ except the assumptions of δ and d , and thus α . By Proposition 5.5, there is a constant $C > 0$ such that

$$|G_z^\alpha(w) - \log \left| \frac{w}{z^\alpha} \right|| < C \text{ on } U^{l, \infty}.$$

Since $|z^{-\alpha}w| = r|z|^{l-\alpha}$ on S^l and $G_{p^n(z)}^\alpha(Q_z^n(w)) = d^n G_z^\alpha(w)$, where $l - \alpha \leq 0$,

$$|G_z^\alpha(w) - d^{-n} \log r |p^n(z)|^{l-\alpha}| \leq d^{-n} C \text{ on } S_n^l.$$

If $l < \alpha$ and $d \geq 2$, then

$$\frac{1}{d^n} \log |p^n(z)|^{l-\alpha} = \left(\frac{\delta}{d}\right)^n \cdot \frac{l-\alpha}{\delta^n} \log |p^n(z)| \rightarrow \infty \cdot (l-\alpha) G_p(z) = \infty$$

as $n \rightarrow \infty$ since $\deg p = \delta > d$ and $G_p(z) \neq -\infty$. If $l < \alpha$ and $d = 1$, then

$$\log |p^n(z)|^{l-\alpha} = (l-\alpha) \log |p^n(z)| \rightarrow (l-\alpha) \cdot (-\infty) = \infty$$

as $n \rightarrow \infty$. On the other hand, if $l = \alpha$ and $d \geq 2$, then

$$d^{-n} \log r |p^n(z)|^{l-\alpha} = d^{-n} \log r \rightarrow 0$$

and $d^{-n} C \rightarrow 0$ as $n \rightarrow \infty$. □

We want to show that S_n^l converges to the boundary $\partial A_f^{l,\infty}$ in $A_p \times \mathbb{P}^1$ as $n \rightarrow \infty$. Let S_∞^l be the set of the accumulation points of S_n^l in $\overline{A_p} \times \mathbb{P}^1$ as $n \rightarrow \infty$:

$$S_\infty^l = \{x \in \overline{A_p} \times \mathbb{P}^1 \mid \exists x_k \in S_{n_k}^l \text{ such that } x_k \rightarrow x \text{ and } n_k \rightarrow \infty \text{ as } k \rightarrow \infty\}.$$

Since the sequence $f^{-n}(U^{l,\infty})$ is monotonically increasing,

$$S_\infty^l = \{x \in \overline{A_p} \times \mathbb{P}^1 \mid \exists x_n \in S_n^l \text{ such that } x_n \rightarrow x \text{ as } n \rightarrow \infty\}.$$

Lemma 5.12: Let $\delta < T_{s-1}$. For any l in $\mathcal{I}_f^1$ if $d \geq 2$, for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, and for any $n \geq 0$,

- 1 $f^{-n}(\partial U^{l,\infty}) = \partial f^{-n}(U^{l,\infty})$
- 2 $f^n(f^{-n}(\partial U^{l,\infty})) \subset \partial U^{l,\infty}$
- 3 $f^{-n}(S^l) = f^{-n}(\partial U^{l,\infty}) \cap \{(z, w) \mid |p^n(z)| < r_1\}$
- 4 $f^n(f^{-n}(S^l)) \subset S^l$
- 5 $f^{-1}(\partial A_f^{l,\infty}) = \partial A_f^{l,\infty}$ and $f(\partial A_f^{l,\infty}) \subset \partial A_f^{l,\infty}$
- 6 $f^{-1}(S_\infty^l) = S_\infty^l$ and $f(S_\infty^l) \subset S_\infty^l$.

Moreover, if f is surjective on $\overline{A_p} \times \mathbb{C}$, then $f^n(f^{-n}(\partial U^{l,\infty})) = \partial U^{l,\infty}$, $f^n(f^{-n}(S^l)) = S^l$, $f(\partial A_f^{l,\infty}) = \partial A_f^{l,\infty}$ and $f(S_\infty^l) = S_\infty^l$. This condition means that $f(\overline{A_p} \times \mathbb{C}) = \overline{A_p} \times \mathbb{C}$ and implies that $E_{deg} = \emptyset$.

We only give an outline of the proof of equation (1). The inclusion $f^{-n}(\partial U) \supset \partial f^{-n}(U)$ follows since f is continuous. The inverse inclusion $f^{-n}(\partial U) \subset \partial f^{-n}(U)$ follows from the simplicity of the shape of U .

Remark 5.13: As shown in Lemma 2.1, $f(U^{l,\infty}) \subset U^{l,\infty}$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$. With a slight change of the proof, one can show that

$$\overline{f(U^{l,\infty})} - \{0\} \subset U^{l,\infty} \text{ and } \overline{f(U^{l,\infty})} \cap \partial U^{l,\infty} = \{0\}.$$

Hence, for any l in $\mathcal{I}_f^1$ if $d \geq 2$, for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, and for any $n \geq 0$,

$$\begin{aligned} \overline{f^{-n}(U^{l,\infty})} - f^{-n}(0) &\subset f^{-(n+1)}(U^{l,\infty}) \text{ and} \\ \overline{f^{-n}(U^{l,\infty})} \cap \partial f^{-(n+1)}(U^{l,\infty}) &= f^{-n}(0). \end{aligned}$$

Let $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{l,\infty})$ be the set of the accumulation points of $f^{-n}(\partial U^{l,\infty})$ in $\overline{A_p} \times \mathbb{P}^1$ as $n \rightarrow \infty$. The monotonicity of $f^{-n}(U^{l,\infty})$ and Lemma 5.12 induce the following.

Proposition 5.14: Let $\delta < T_{s-1}$. Then $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{l,\infty}) = \partial A_f^{l,\infty}$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$.

Proof: Let $U = U^{l,\infty}$ and let us denote $\lim_{n \rightarrow \infty}$ by $\lim$ in the proof for simplicity.

We first show the inclusion $\lim f^{-n}(\partial U) \subset \partial A_f$. Let $x \in \lim f^{-n}(\partial U)$. Then we may assume that there exist $x_n \in f^{-n}(\partial U)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Lemma 5.12 implies that $f^{-n}(\partial U) = \partial f^{-n}(U) \subset \overline{f^{-n}(U)} \subset \overline{A_f}$. Since $\overline{A_f}$ is closed, $x \in \overline{A_f}$. Let us show that $x \in \partial A_f$ by contradiction. If $x \in A_f$, then there exists N such that $f^N(x) \in U$. Hence $x \in f^{-N}(U)$ and so $x_n \in f^{-N}(U)$ for any large n , which contradicts the fact that $\partial f^{-n}(U) \cap f^{-N}(U) = \emptyset$ for any $n > N$.

We next show the opposite direction. Let $x \in \partial A_f$. Then $x \in \overline{A_f}$ and so we may assume that there exists $x_n \in f^{-n}(U)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Any path connecting x and x_n intersects with $\partial f^{-n}(U)$. Let y_n be the intersection of $\partial f^{-n}(U)$ and the segment connecting x and x_n . Then $y_n \rightarrow x$ as $n \rightarrow \infty$ and so $x \in \lim \partial f^{-n}(U)$, which coincides with $\lim f^{-n}(\partial U)$ by Lemma 5.12. $\square$

Since $S^l = \partial U^{l,\infty} \cap \{|z| < r\}$ and $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{l,\infty} - S^l) \subset \partial A_p \times \mathbb{P}^1$, the proposition above implies the following.

Corollary 5.15: Let $\delta < T_{s-1}$. Then $S_\infty^l \subset \partial A_f^{l,\infty}$ and $S_\infty^l \cap (A_p \times \mathbb{P}^1) = \partial A_f^{l,\infty} \cap (A_p \times \mathbb{P}^1)$ for any l in $\mathcal{I}_f^1$ if $d \geq 2$ and for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$.

We obtain the proposition below from Lemma 5.11 and the corollary above. More precisely, we can remove E_{deg}^* from the exceptional set because the estimate of $G_z^\alpha|_{S_n^t}$ or $G_z^\infty|_{S_n^t}$ is locally uniform over $A_p - E_p$; e.g., if $\delta < d$, then for any z_0 in $A_p - E_p$ and for any $\varepsilon > 0$, there is a neighbourhood of z_0 in $A_p - E_p$ and an integer N such that $|G_z^\alpha|_{S_n^t}| < \varepsilon$ for any z in the neighbourhood and for any $n \geq N$.

Proposition 5.16: For any l in $\mathcal{I}_f^1$ and for any (z_0, w_0) in $\partial A_f^{l,\infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\begin{aligned} \limsup_{(z,w) \in A_f^{l,\infty} \rightarrow (z_0,w_0)} G_z^\alpha(w) &= 0 \quad \text{if } \delta < d, \\ \limsup_{(z,w) \in A_f^{l,\infty} \rightarrow (z_0,w_0)} G_z^\infty(w) &= \infty \quad \text{if } \delta = d. \end{aligned}$$

Let $T_{s-1} > \delta > d \geq 1$. Then for any (z_0, w_0) in $\partial A_f^{l_1, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z, w) \in A_f^{l_1, \infty} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \begin{cases} \infty & \text{if } l_1 \leq l < \alpha, \\ 0 & \text{if } l = \alpha \text{ and } d \geq 2. \end{cases}$$

This proposition implies the following, the equalities in which also follow from Theorem 2.3.

Corollary 5.17: Let $\delta < T_{s-1}$. If $\delta \leq d$, then $A_f^{l_1, \infty} = A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1$. If $\delta > d \geq 2$, then $A_f^{\alpha, \infty} \subsetneq A_f^{l_1, \infty} = A_f^{l, \infty}$ for any l in $\mathcal{I}_f^1 - \{\alpha\}$.

Proof: We treat explicitly only the case $\delta < d$, the other cases being similar to or simpler than the case $\delta < d$ and left to the reader. Let $\delta < d$. By definition, $A_f^{l_1, \infty} \subset A_f^{l_1, \infty}$ for any $l > l_1$. We derive the contradiction, assuming that the equality $A_f^{l_1, \infty} = A_f^{l, \infty}$ does not hold. Let $G = G_z^\alpha = G_z$ and $\partial A = \partial A_f^{l_1, \infty} \cap A_f^{l_1, \infty}$. If the equality above does not hold, then $\partial A \neq \emptyset$. Since G is plurisubharmonic on $A_f^{l_1, \infty}$ and since $A_f^{l_1, \infty} \cap E_z = \emptyset$, it follows from the proposition above that $G \geq 0$ on ∂A . On the other hand, $G \leq 0$ on $A_f^{l_1, \infty}$ since $G = \lim_{n \rightarrow \infty} d^{-n} \log |w_n|$ and $|w_n| \rightarrow 0$ as $n \rightarrow \infty$ on $A_f^{l_1, \infty}$. Hence $G = 0$ on ∂A , which contradicts the maximum principle for G . $\square$

5.2.3 Asymptotic behaviour of G_z^α toward $\partial A_f^{l_1, \infty}$: limit inferior

It follows from the assumption $m_j \geq d \geq 1$ that f maps $\{w = 0\}$ to itself. If $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then we expect that the limit infimum of G_z^α toward $\partial A_f^{l_1, \infty}$ is $-\infty$ since $G_z^\alpha = -\infty$ on $E_w - E_z$. Actually, we obtain the following with the assumption that q is polynomial, which guarantees the surjectivity of the function.

Proposition 5.18: Let $\delta < T_{s-1}$. If q is polynomial and $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then for any l in $\mathcal{I}_f^1$ if $d \geq 2$, for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, and for any z in $A_p - E_p \cup E_{deg}^*$,

$$\liminf_{w \in A_f^{l_1, \infty} \cap \mathbb{C}_z \rightarrow \partial A_f^{l_1, \infty} \cap \mathbb{P}_z^1} G_z^\alpha(w) = -\infty,$$

where $\mathbb{C}_z = \{z\} \times \mathbb{C}$ and $\mathbb{P}_z^1 = \{z\} \times \mathbb{P}^1$.

Proof: Fix any l in $\mathcal{I}_f^1$ or $\mathcal{I}_f^1 - \{\alpha\}$ and any z in $A_p - E_p \cup E_{deg}^*$. It is enough to show that $E_w \cap \mathbb{C}_z$ accumulates to $\partial A_f^{l_1, \infty} \cap \mathbb{P}_z^1$ since $G_z^\alpha = -\infty$ on $E_w - E_z$.

Let $f^n(z, w) = (p^n(z), Q_z^n(w))$. In other words, let

$$Q_z^n(w) = q_{z_{n-1}} \circ \cdots \circ q_{z_1} \circ q_z(w),$$

where $z_j = p^j(z)$. Hence Q_z^n is the composition of n holomorphic functions on w , which are not constant since $z \notin E_{deg}^*$.

We first check that the cardinality of $(Q_z^n)^{-1}(0)$ tends to infinity as $n \rightarrow \infty$. Let $q(z, w) = w^d h(z, w)$ and $H = \{z \in \mathbb{C} \mid \exists w \neq 0 \text{ such that } h(z, w) = 0\}$. Then $H = \{z \in \mathbb{C} \mid q_z^{-1}(0) \supsetneq \{0\}\}$ and it is enough to show that H is the complement of only

finitely many points. Because h is polynomial and depends non-trivially on w , $\{z \in \mathbb{C} | \exists w \text{ such that } h(z, w) = 0\} = \mathbb{C}$. On the other hand, $\{z \in \mathbb{C} | h(z, 0) = 0\}$ is a finite set. If not, it follows from the identity theorem that $h(z, 0) \equiv 0$, which contradicts the minimality of d . Therefore, H is the complement of only finitely many points.

We next show that $(Q_z^n)^{-1}(0) - \{0\}$ converges to $\partial A_f^{l, \infty} \cap \mathbb{P}_z^1$ as $n \rightarrow \infty$, which completes the proof. Let w_n be a point in $(Q_z^n)^{-1}(0) - \{0\}$ and $\{w_{n_k}\}$ a subsequence of $\{w_n\}$ that converges to a point w_∞ . Assuming that w_∞ belongs to $A_f^{l, \infty} \cap \mathbb{C}_z$, we derive the contradiction. Since (z, w_∞) belongs to $A_f^{l, \infty}$, there is an integer N such that f^N maps a neighbourhood of (z, w_∞) to $U^{l, \infty}$. Because $f \sim f_0$ on $U_r^{l, \infty}$ as $r \rightarrow 0$, it follows for each sufficient large k that $Q_{z_N}^n(Q_z^N(w_{n_k})) = Q_z^{n+N}(w_{n_k}) \neq 0$ for any $n \geq 0$. Hence $Q_z^{n_k}(w_{n_k}) = Q_z^{n+N}(w_{n_k}) \neq 0$ for $n = n_k - N$, which contradicts the definition of w_{n_k} . $\square$

Remark 5.19: The assumption q being polynomial may be removable, which is used only to guarantee that H is the complement of finitely many points.

Question 5.20: It is known that if p is polynomial and $p^{-1}(0) \supsetneq \{0\}$, then $p^{-n}(0) - \{0\}$ converges to J_p densely. Does the same or similar thing hold for skew products? More precisely, if f is polynomial and $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then does $[f^{-n}\{w = 0\} - \{w = 0\}] \cap (A_p \times \mathbb{C})$ converge to $\partial A_f \cap (A_p \times \mathbb{P}^1)$ densely? One may expect that this is true from the results by Dinh and Sibony (2008) and, if this holds, then we can revise the proposition above as follows: for any l in $\mathcal{I}_f^1$ if $d \geq 2$, for any l in $\mathcal{I}_f^1 - \{\alpha\}$ if $d = 1$, and for any (z_0, w_0) in $\partial A_f^{l, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\liminf_{(z, w) \in A_f^{l, \infty} \rightarrow (z_0, w_0)} G_z^\alpha(w) = -\infty.$$

Let E_w^∞ be the set of the accumulation points of $f^{-n}\{w = 0\} - \{w = 0\}$ in $\mathbb{C} \times \mathbb{P}^1$ as $n \rightarrow \infty$. The proof of Proposition 5.18 implies the following.

Proposition 5.21: Let $T_{s-1} > \delta > d \geq 2$. If q is polynomial and $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then

$$E_w^\infty \cap \{(A_p - E_p \cup E_{deg}^*) \times \mathbb{P}^1\} \subset \bigcap_{l_1 \leq l \leq \alpha} \partial A_f^{l, \infty}.$$

Consequently, E_w^∞ is unbounded if f satisfies the assumptions since $A_f^{\alpha, \infty} \subsetneq A_f^{l_1, \infty}$. Since $E_w^\infty \cap (A_p \times \mathbb{P}^1)$ is included in the closure of $A_f^{l, \infty}$ in $\mathbb{C} \times \mathbb{P}^1$, we obtain the following.

Corollary 5.22: Let $T_{s-1} > \delta > d \geq 2$. If q is polynomial and $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then $A_f^{l, \infty}$ is unbounded for any l in $\mathcal{I}_f^1$.

5.3 Case 2: $\delta < T_{s-1}$ and $d = 0$

If $d = 0$, then f can not be conjugate to f_0 on any open set, since f_0 is not dominant, where $f_0(z, w) = (z^\delta, z^\gamma)$. However, the same statements as Lemma 3.1 hold for any l in $\mathcal{I}_f^1 - \{\alpha\}$, and we can show that $G_z = \alpha G_p \circ \pi_z$ on A_0 .

Lemma 5.23: Let $T_{s-1} > \delta > d = 0$. The following hold for any l in $\mathcal{I}_f^1 - \{\alpha\}$: $f \sim f_0$ on $U_{r_1, r_2}^{l, \infty}$ as r_1 and $r_2 \rightarrow 0$, and $f(U_{r_1, r_2}^{l, \infty}) \subset U_{r_1, r_2}^{l, \infty}$ for suitable small r_1 and r_2 .

The first statement follows from the same argument as in Ueno (2019b), and it holds even for $l = \alpha$. The second statement follows from Theorem 2.2.

Remark 5.24 (illustration of the second statement in terms of blow-ups): Let $\tilde{f}$ be a formal blow-up of f by $\pi(z, c) = (z, z^l c)$. Then $\tilde{f}(z, c) = (z^\delta, z^{\tilde{\gamma}} \{1 + o(1)\})$, where $\tilde{\gamma} = \gamma - l\delta \geq 0$ since $d = 0$. If $l \in \mathcal{I}_f^1 - \{\alpha\}$, then $\tilde{\gamma} > 0$ and so the origin is a superattracting fixed point of $\tilde{f}$. In particular, $\tilde{f}$ preserves a neighbourhood of the origin, and so f preserves $U_{r_1, r_2}^{l, \infty}$. On the other hand, if $l = \alpha$, then $\tilde{\gamma} = 0$ and so the origin is neither superattracting nor a fixed point of $\tilde{f}$.

Proposition 5.25: If $T_{s-1} > \delta > d = 0$, then $G_z = \alpha G_p \circ \pi_z$ on A_0 .

Proof: Fix any l in $\mathcal{I}_f^1 - \{\alpha\}$. Let $U = U_{r_1, r_2}^{l, \infty}$ and let $(z_n, w_n) = f^n(z, w)$ for any (z, w) in U . The second statement in the lemma above implies that (z_n, w_n) also belongs to U . The first statement in the lemma implies that

$$C_1 |(z_n)^\gamma| < |w_n| < C_2 |(z_n)^\gamma|$$

for some constants $C_1 < 1$ and $C_2 > 1$. Hence $\delta^{-n} \log |w_n|$ is bounded by

$$\frac{1}{\delta^n} \log C_j + \frac{\gamma}{\delta} \cdot \frac{1}{\delta^{n-1}} \log |z_n|$$

from below and above for $j = 1$ and $j = 2$, respectively. Therefore, $G_z = \alpha G_p \circ \pi_z$ on U and so on $A_f^{l, \infty}$. Thanks to Theorem 2.3, the equality holds on A_0 . $\square$

5.4 Case 2: $\delta = T_{s-1}$

Recall that $\delta > d$ and $\mathcal{I}_f^1 = \{l_1\} = \{\alpha\}$ if $\delta = T_{s-1}$.

Because f is conjugate to f_0 on $U^{\alpha, \infty}$ if $d \geq 2$, the limit G_z^α exists on $U^{\alpha, \infty}$, which extends to $A_f^{\alpha, \infty}$.

Proposition 5.26: If $T_{s-1} = \delta > d \geq 2$, then G_z^α is plurisubharmonic on $A_f^{\alpha, \infty}$, pluriharmonic on $A_f^{\alpha, \infty} - E_w$ and $-\infty$ on E_w . Moreover, $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on $U_r^{\alpha, \infty}$ as $r \rightarrow 0$.

The asymptotic behaviour of G_z^α toward $\partial A_f^{\alpha, \infty}$ is as follows.

Proposition 5.27: If $T_{s-1} = \delta > d \geq 2$, then for any (z_0, w_0) in $\partial A_f^{\alpha, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z, w) \in A_f^{\alpha, \infty} \rightarrow (z_0, w_0)} G_z^\alpha(w) = 0.$$

Moreover, if q is polynomial and $f^{-1}\{w = 0\} \supsetneq \{w = 0\}$, then for any z in $A_p - E_p \cup E_{deg}^*$,

$$\liminf_{w \in A_f^{\alpha, \infty} \cap \mathbb{C}_z \rightarrow \partial A_f^{\alpha, \infty} \cap \mathbb{P}_z^1} G_z^\alpha(w) = -\infty.$$

The dynamics of f differs whether $n_{s-1} > 0$ or $n_{s-1} = 0$. Let $G_z^*(w) = \limsup_{n \rightarrow \infty} \lambda^{-n} \log |w_n|$, where $\lambda = \max\{\delta, d\}$. If $n_{s-1} > 0$, then Theorem 2.3 implies the following inequality and equalities on A_0 .

Proposition 5.28: Let $\delta = T_{s-1}$. If $n_{s-1} > 0$, then $G_z^* \leq \alpha G_p \circ \pi_z$ on A_0 and $G_z = \alpha G_p \circ \pi_z$ on $A_0 - A_f^{\alpha, \infty}$. In particular, $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 . Moreover, if $d \geq 2$, then $G_z = \alpha G_p \circ \pi_z$ on $A_0 - E_w$ and $-\infty$ on E_w .

On the other hand, if $n_{s-1} = 0$, then we have the same inequality and almost the same equalities only on $A_f^{\alpha, \infty}$.

Proposition 5.29: Let $\delta = T_{s-1}$. If $n_{s-1} = 0$, then $G_z^* \leq \alpha G_p \circ \pi_z$ on $A_f^{\alpha, \infty}$. In particular, $G_f^\alpha = \alpha G_p \circ \pi_z$ on $A_f^{\alpha, \infty}$. Moreover, if $d \geq 2$, then $G_z = \alpha G_p \circ \pi_z$ on $A_f^{\alpha, \infty} - E_w$ and $-\infty$ on E_w .

Remark 5.30: If $\delta = T_{s-1}$, then f has the other dominant term (n_{s-1}, m_{s-1}) than (n_s, m_s) , which belongs to Case 3 or Case 4. Moreover, if $n_{s-1} = 0$, then $s = 2$ and $(n_1, m_1) = (n_{s-1}, m_{s-1}) = (0, \delta)$, which belongs to Case 3. Therefore, we can apply the results in Case 3 or Case 4. For example, variants of Corollaries 6.7 and 7.9 imply that $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 if $n_{s-1} > 0$ and that G_f^α is plurisubharmonic and continuous on A_0 if $n_{s-1} = 0$.

6 Case 3

We deal with Case 3 in this section. Let $s > 1$,

$$T_1 \leq \delta \text{ and } (\gamma, d) = (n_1, m_1).$$

Note that $\gamma \geq 0$ and $\delta \geq T_1 \geq d \geq 1$ by the setting.

Although we observe that the blow-up $\tilde{f}$ is superattracting at the origin and degenerates the t -axis in Subsection 6.1, there are no results on invariant wedges and the unions of all the preimages of wedges for Case 3. We exhibit results on the existence and properties of plurisubharmonic functions and on the asymptotic behaviour of the functions toward the boundaries of the unions in Subsection 6.2, which imply all the main results stated in Subsection 2.2. Although we omit to prove the results on the existence and properties since they are similar to Case 2, we prove the results on the asymptotic behaviour in Subsection 6.3. One may refer Table 2 in Subsection 2.2 for a comparison chart of the results. We assume that $a = b_{\gamma d} = 1$ in Subsections 6.2 and 6.3.

Two examples are given in Subsection 6.4. For any polynomial h of degree d and for any integers δ and α such that $\delta \geq d$, we can construct a polynomial skew product f that is semiconjugate to a polynomial product of the form $(z^\delta, h(w))$ and such that the α determined by f coincides with the given integer α . The map f is in Cases 3 and 2, and the dynamics of such a map is rather easy to understand. More precisely, the Böttcher coordinates and plurisubharmonic functions for f are expressed by the counterparts of h and, in particular, the boundary of the set where $G_z^{\alpha, +}$ is not pluriharmonic is described in terms of the Julia set of h .

6.1 Observation of blow-ups

Let us observe that the blow-up $\tilde{f}$ is superattracting at the origin and degenerates the t -axis for a suitable weight l . If l belongs to $\mathcal{I}_f^2$, then $\tilde{q}(t, w) \sim t^\gamma w^{\tilde{d}}$ on a neighbourhood of the origin and so

$$\tilde{f}(t, w) = \left(t^{\delta-l^{-1}\gamma} w^{l^{-1}(\delta-\tilde{d})} \{1 + \zeta(t, w)\}, t^\gamma w^{\tilde{d}} \{1 + \eta(t, w)\} \right),$$

where $\tilde{d} = l^{-1}\gamma + d$. Moreover, if $\delta > T_1$ and $l > \alpha$, then $\delta - \tilde{d} > 0$ and so we have the following.

Proposition 6.1: If $\delta > T_1$, then $\tilde{f}$ is superattracting at the origin and degenerates the t -axis for any l in $\mathcal{I}_f^2 - \{\alpha\}$.

Although $\tilde{f}$ degenerates the t -axis, we do not know whether $A_f^{0,l}$ contains $A_0 - E_w$.

6.2 Plurisubharmonic functions

Let us first review $\mathcal{I}_f^2$ and related quantities, which depends on whether $\delta > T_1$ and whether $\gamma > 0$; see Table 12 for a summary. If $\delta > T_1$ and $\gamma > 0$, then $\delta > d$, $\alpha > 0$ and $\mathcal{I}_f^2 = [\alpha, l_2]$. If $\delta > T_1$ and $\gamma = 0$, then $\delta > d$, $\alpha = 0$ and $\mathcal{I}_f^2 = (0, l_2]$. Moreover, $d \geq 2$ because of the assumption for f being superattracting at the origin. If $\delta = T_1$ and $\gamma > 0$, then $\delta > d$, $\alpha > 0$ and $\mathcal{I}_f^2 = \{\alpha\} = \{l_2\}$. If $\delta = T_1$ and $\gamma = 0$, then $\delta = d$, $\alpha = l_2$ and $\mathcal{I}_f^2 = (0, l_2]$; recall that we redefined α to be l_2 in Subsection 2.2, whereas $\gamma/(\delta - d)$ is not defined for this case.

Table 12 Informations on $\mathcal{I}_f^2$ and related quantities for Case 3

	$\gamma > 0$	$\gamma = 0$
$\delta > T_1$	$\mathcal{I}_f^2 = [\alpha, l_2]$ $\delta > d \geq 1$ and $\alpha > 0$	$\mathcal{I}_f^2 = (0, l_2]$ $\delta > d \geq 2$ and $\alpha = 0$
$\delta = T_1$	$\mathcal{I}_f^2 = \{\alpha\} = \{l_2\}$ $\delta > d \geq 1$ and $\alpha > 0$	$\mathcal{I}_f^2 = (0, l_2]$ $\delta = d \geq 2$ and $\alpha = l_2 > 0$

In particular, the dynamical features depend strongly on whether $\gamma > 0$ or $\gamma = 0$. If $\gamma = 0$, then f_0 is just a product. More precisely, if $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and so the properties of G_z^α are different from the other cases. On the other hand, if $\gamma = 0$ and $\delta = d$, then $f_0(z, w) = (z^\delta, w^\delta)$ and so the properties of G_z and the inclusion relation between $A_f^{0,l}$ for any l in $\mathcal{I}_f^2$ are different from the other cases.

Recall that f is conjugate to the monomial map f_0 on $U^{0,l}$ for any l in $\mathcal{I}_f^2$ if $d \geq 2$ and for any l in $\mathcal{I}_f^2 - \{\alpha\}$ if $d = 1$ and $\delta > T_1$ as stated in Lemma 3.6, where $U^{0,l} = \{|z|^l < r^l|w|, |w| < r\}$. Therefore, G_z^α exists on $U^{0,l} - \{z = 0\}$ for any l as above if $\gamma > 0$ or $\delta = d$. More precisely,

$$G_z^\alpha(w) = \log \left| \frac{\phi_2}{\phi_1^\alpha} \right| = \log \left| \frac{w}{z^\alpha} \right| + o(1) \text{ on } U_r^{0,l} - \{z = 0\} \text{ as } r \rightarrow 0,$$

which extends to a pluriharmonic function on $A_f^{0,l} - E_z$. On the other hand, if $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and G_z^α exists on $U^{0,l}$ for any l in $\mathcal{I}_f^2$. More precisely,

$$G_z^\alpha(w) = \log |\phi_2| = \log |w| + o(1) \text{ on } U_r^{0,l} \text{ as } r \rightarrow 0,$$

which extends to a pluriharmonic function on $A_f^{0,l}$.

Proposition 6.2: The following hold for any l in $\mathcal{I}_f^2$ if $d \geq 2$ and for any l in $\mathcal{I}_f^2 - \{\alpha\}$ if $d = 1$ and $\delta > T_1$: If $\gamma > 0$ or $\delta = d$, then $\alpha > 0$ and G_z^α is pluriharmonic on $A_f^{0,l} - E_z$. It is ∞ on $E_z - E_w$ and not defined on $E_z \cap E_w$. Moreover, $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on $U_r^{0,l} - \{z = 0\}$ as $r \rightarrow 0$. If $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and G_z^α is pluriharmonic on $A_f^{0,l}$. Moreover, $G_z^\alpha(w) = \log |w| + o(1)$ on $U_r^{0,l}$ as $r \rightarrow 0$.

This proposition implies Proposition 2.8. Recall that $A_f = A_f^{0,l_2}$ and that A_f^{0,l_2} is the largest wedge among $A_f^{0,l}$ for any l in $\mathcal{I}_f^2$.

Moreover, we have the following propositions on the asymptotic behaviour of G_z^α toward $\partial A_f^{0,l}$, which are proved in the next subsection.

Proposition 6.3: Let $d \geq 2$ or $\delta > T_1$. For any l in the proposition above and for any (z_0, w_0) in $\{\partial A_f^{0,l} \cap \partial A_0\} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z,w) \in A_f^{0,l} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \begin{cases} \infty & \text{if } \gamma > 0 \text{ (and } \delta > d), \\ 0 & \text{if } \gamma = 0 \text{ and } \delta > d, \\ -\alpha G_p(z_0) & \text{if } \gamma = 0 \text{ and } \delta = d. \end{cases}$$

The same convergence to 0 holds for any z_0 in A_p if $\gamma = 0$ and $\delta > d$.

Proposition 6.4: Let $\delta > d$. For any (z_0, w_0) in $\partial A_f^{0,l} \cap A_0 - E_z$,

$$\lim_{(z,w) \in A_f^{0,l} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \begin{cases} 0 & \text{if } \gamma > 0, d \geq 2 \text{ and } l = \alpha, \\ -\infty & \text{if } \alpha < l \leq l_2. \end{cases}$$

The same convergence to $-\infty$ holds for any (z_0, w_0) in $\partial A_f^{0,l} \cap A_0$ if $\gamma = 0$.

On the other hand, if $\delta = d$, then for any l in $\mathcal{I}_f^2$ and for any (z_0, w_0) in $\partial A_f^{0,l} \cap A_0 - E_z$,

$$\lim_{(z,w) \in A_f^{0,l} \rightarrow (z_0, w_0)} G_z^\alpha(w) = (l - \alpha) G_p(z_0),$$

where $\mathcal{I}_f^2 = (0, l_2]$ and $\alpha = l_2$.

As a consequence of Propositions 6.2, 6.3 and 6.4, we obtain Theorem 2.9 and Proposition 2.11.

The limit G_z also exists on U if $d \geq 2$ or if $d = 1$ and $\delta > T_1$, where $U = U^{0,l_2}$. More precisely,

$$\begin{aligned} G_z(w) &= \alpha G_p(z) \text{ on } U && \text{if } \gamma > 0 \text{ and } \delta > d, \\ G_z(w) &= 0 \text{ on } U && \text{if } \gamma = 0 \text{ and } \delta > d, \text{ and} \\ G_z(w) &= \log |w| + o(1) \text{ on } U_r \text{ as } r \rightarrow 0 && \text{if } (\gamma = 0 \text{ and } \delta = d), \end{aligned}$$

which extends to a pluriharmonic function on A_f .

Proposition 6.5: If $d \geq 2$ or $\delta > T_1$, then G_z is pluriharmonic on A_f . Moreover, $G_z = \alpha G_p \circ \pi_z$ on A_f if $\gamma > 0$ and $\delta > d$, $G_z = 0$ on A_f if $\gamma = 0$ and $\delta > d$, and $G_z(w) = \log |w| + o(1)$ on U_r as $r \rightarrow 0$ if $(\gamma = 0 \text{ and}) \delta = d$.

By arguments similar to the proof of Proposition 6.4, we obtain the following.

Proposition 6.6: If $\delta = d$, then for any l in $\mathcal{I}_f^2$ and for any (z_0, w_0) in $\partial A_f^{0,l} \cap A_0$,

$$\lim_{(z,w) \in A_f^{0,l} \rightarrow (z_0, w_0)} G_z(w) = lG_p(z_0).$$

As a consequence of Propositions 6.5 and 6.6 and the remark below, we obtain the following, which, together with Proposition 6.5, implies Theorem 2.10.

Corollary 6.7: If $\delta > d$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 , which is plurisubharmonic on A_0 and pluriharmonic on $A_0 - E_z$. If $\delta = d$, then $G_f^\alpha > \alpha G_p \circ \pi_z$ on $A_f^{0,\alpha}$ and $G_f^\alpha = \alpha G_p \circ \pi_z$ on $A_0 - A_f^{0,\alpha}$, which is plurisubharmonic and continuous on A_0 and pluriharmonic on $A_0 - \partial A_f^{0,\alpha}$.

Remark 6.8: If $d = 1$ and $\delta = T_1$, then $m_2 = 0$, where (n_2, m_2) is the other dominant term of q . Hence f is also in Case 2, and Proposition 5.25 implies that $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

We end this subsection with a proposition that relates to the discontinuity of G_z . Recall that $G_z^*(w) = \limsup_{n \rightarrow \infty} \lambda^{-n} \log |w_n|$, where $\lambda = \max\{\delta, d\}$.

Proposition 6.9: If $\delta > T_1$, then $G_z^* < \alpha G_p \circ \pi_z$ on $A_0 - A_f$.

Proof: For any l in $\mathcal{I}_f^2 - \{\alpha\}$ and for any (z, w) in $A_0 - A_f^{0,l}$, it follows that $|w_n| \leq r^{-l} |z_n|^l$ for any $n \geq 1$, where $(z_n, w_n) = f^n(z, w)$. Hence $G_z^*(w) \leq lG_p(z) < \alpha G_p(z)$ on $A_0 - A_f^{0,l}$ for any l in $\mathcal{I}_f^2 - \{\alpha\}$. In particular, $G_z^*(w) < \alpha G_p(z)$ on $A_0 - A_f^{0,l_2}$. $\square$

Therefore, even if the limit G_z exists on A_0 , it is not continuous on $\partial A_f \cap A_0$ when $\delta > T_1$.

6.3 Asymptotic behaviour of G_z^α toward $\partial A_f^{0,l}$

In this subsection we prove Propositions 6.3 and 6.4 by arguments similar to Case 2.

We separate the boundary $\partial U^{0,l}$ into two surfaces S^{out} and S^{in} : let

$$S^{out} = \{|z|^l < r^l |w|, |w| = r\} \text{ and } S^{in} = \{|z|^l = r^l |w|, |w| < r\}.$$

Note that $S^{out} = \{|z| < r^{1+1/l}, |w| = r\}$ and $S^{out} \cup S^{in} = \partial U^{0,l} \cap \{|z| < r^{1+1/l}\}$. Let $S_n^{out} = f^{-n}(S^{out})$ and $S_n^{in} = f^{-n}(S^{in})$.

Lemma 6.10: Let $d \geq 2$ or $\delta > T_1$. For any l in Proposition 6.2 and for any z in $A_p - E_p \cup E_{deg}^*$, we have that $S_n^{out} \cap \mathbb{C}_z \neq \emptyset$ for any large n , and

$$G_z^\alpha|_{S_n^{out}} \rightarrow \begin{cases} \infty \text{ as } n \rightarrow \infty & \text{if } \gamma > 0 (\text{and } \delta > d), \\ 0 \text{ as } n \rightarrow \infty & \text{if } \gamma = 0 \text{ and } \delta > d, \\ -\alpha G_p(z) \text{ as } n \rightarrow \infty & \text{if } \gamma = 0 \text{ and } \delta = d. \end{cases}$$

The same convergence to 0 holds for any z in $A_p - E_{deg}^*$ if $\gamma = 0$ and $\delta > d$.

Proof: By Proposition 6.2, there is a constant $C > 0$ such that

$$\left| G_z^\alpha(w) - \log \left| \frac{w}{z^\alpha} \right| \right| < C \text{ on } U^{0,l} - \{z = 0\}.$$

Since $|w| = r$ on S^{out} , $|G_z^\alpha(w) - \log r|z|^{-\alpha}| \leq C$ on $S^{out} - \{z = 0\}$ and so

$$\left| G_{p^n(z)}^\alpha(Q_z^n(w)) - \log r|p^n(z)|^{-\alpha} \right| \leq C \text{ on } S_n^{out} - f^{-n}\{z = 0\}.$$

Since $G_{p^n(z)}^\alpha(Q_z^n(w)) = d^n G_z^\alpha(w)$,

$$\left| G_z^\alpha(w) - d^{-n} \log r|p^n(z)|^{-\alpha} \right| \leq d^{-n} C \text{ on } S_n^{out} - f^{-n}\{z = 0\}.$$

If $\gamma > 0$, then $\alpha > 0$ since $\delta > d$, and

$$\frac{-\alpha}{d^n} \log |p^n(z)| = \left(\frac{\delta}{d} \right)^n \cdot \frac{-\alpha}{\delta^n} \log |p^n(z)| \rightarrow \infty \cdot (-\alpha) G_p(z) = \infty$$

as $n \rightarrow \infty$. If $\gamma = 0$ and $\delta = d$, then $\alpha > 0$ and

$$\frac{-\alpha}{d^n} \log |p^n(z)| = \frac{-\alpha}{\delta^n} \log |p^n(z)| \rightarrow -\alpha G_p(z)$$

as $n \rightarrow \infty$.

If $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and there is a constant $C > 0$ such that

$$|G_z^\alpha(w) - \log |w|| < C \text{ on } U^{0,l}$$

by Proposition 6.2. The same arguments as above imply that

$$\left| G_z^\alpha(w) - d^{-n} \log r \right| \leq d^{-n} C \text{ on } S_n^{out}.$$

Clearly, $d^{-n} \log r \rightarrow 0$ and $d^{-n} C \rightarrow 0$ as $n \rightarrow \infty$ since $d \geq 2$. □

Lemma 6.11: Fix any z in $A_p - E_p \cup E_{deg}^*$. Then $S_n^{in} \cap \mathbb{C}_z \neq \emptyset$ for any large n . If $\delta > d$, then

$$G_z^\alpha|_{S_n^{in}} \rightarrow \begin{cases} 0 \text{ as } n \rightarrow \infty & \text{if } \gamma > 0, d \geq 2 \text{ and } l = \alpha, \\ -\infty \text{ as } n \rightarrow \infty & \text{if } \alpha < l \leq l_2. \end{cases}$$

On the other hand, if $\delta = d$, then

$$G_z^\alpha|_{S_n^{in}} \rightarrow (l - \alpha) G_p(z) \text{ as } n \rightarrow \infty$$

for any $0 < l \leq l_2$, where $\alpha = l_2$.

The same convergence to $-\infty$ holds for any z in $A_p - E_{deg}^*$ if $\gamma = 0$ and $\delta > d$.

Proof: By Proposition 6.2, there is a constant $C > 0$ such that

$$\left| G_z^\alpha(w) - \log \left| \frac{w}{z^\alpha} \right| \right| < C \text{ on } U^{0,l} - \{z = 0\}.$$

Since $|z^{-\alpha}w| = r|z|^{l-\alpha}$ on S_n^{in} , $|G_z^\alpha(w) - \log r|z|^{l-\alpha}| \leq C$ on $S_n^{in} - \{0\}$ and so

$$\left| G_{p^n(z)}^\alpha(Q_z^n(w)) - \log r|p^n(z)|^{l-\alpha} \right| \leq C \text{ on } S_n^{in} - f^{-n}(0).$$

Since $G_{p^n(z)}^\alpha(Q_z^n(w)) = d^n G_z^\alpha(w)$,

$$\left| G_z^\alpha(w) - d^{-n} \log r|p^n(z)|^{l-\alpha} \right| \leq d^{-n} C \text{ on } S_n^{in} - f^{-n}(0).$$

Let $\delta > d$. If $\gamma > 0$, $d \geq 2$ and $l = \alpha$, then

$$d^{-n} \log r|p^n(z)|^{l-\alpha} = d^{-n} \log r \rightarrow 0$$

and $d^{-n}C \rightarrow 0$ as $n \rightarrow \infty$. If $\alpha < l \leq l_2$, then

$$\frac{l-\alpha}{d^n} \log |p^n(z)| = \left(\frac{\delta}{d} \right)^n \cdot \frac{l-\alpha}{\delta^n} \log |p^n(z)| \rightarrow \infty \cdot (l-\alpha)G_p(z) = -\infty$$

as $n \rightarrow \infty$.

On the other hand, if $\delta = d$ and $0 < l \leq l_2$, then

$$\frac{1}{d^n} \log |p^n(z)|^{l-\alpha} = \frac{l-\alpha}{\delta^n} \log |p^n(z)| \rightarrow (l-\alpha)G_p(z)$$

as $n \rightarrow \infty$, where $\alpha = l_2$.

If $\gamma = 0$ and $\delta > d$, then $\alpha = 0$ and the same inequalities hold not only on $S_n^{in} - f^{-n}(0)$ but also on S_n^{in} . Hence $G_z^\alpha|_{S_n^{in}} = d^{-n} \log r|p^n(z)|^l \rightarrow -\infty$ as $n \rightarrow \infty$ for any z in $A_p - E_{deg}^*$ and for any $0 < l \leq l_2$. $\square$

Let $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{0,l})$ be the set of the accumulation points of $f^{-n}(\partial U^{0,l})$ in $\overline{A_p} \times \mathbb{P}^1$ as $n \rightarrow \infty$, as the same as Case 2. Then the following equality follows from the same argument as Case 2.

Proposition 6.12: It follows that $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{0,l}) = \partial A_f^{0,l}$ for any l in Proposition 6.2.

Let S_∞^{out} and S_∞^{in} be the sets of the accumulation points of S_n^{out} and S_n^{in} in $\overline{A_p} \times \mathbb{P}^1$ as $n \rightarrow \infty$, respectively. We investigate the inclusion relation between S_∞^{out} and $\partial A_f^{0,l} \cap \partial A_0$ and between S_∞^{in} and $\partial A_f^{0,l} \cap A_0$. Let $r_l = r^{1+1/l}$,

$$W^l = \{|z| < r_l, |w| \leq r^{-l}|z|^l\} \text{ and } U^+ = U^{0,l} \cup W^l = \{|z| < r_l, |w| < r\}.$$

Let $W_n^l = f^{-n}(W^l)$ and $U_n^+ = f^{-n}(U^+)$. Note that U_n^+ is a neighbourhood of the origin, $\partial U_n^+ \cap \{|z| < r_l\} = S_n^{out}$ and $\partial W_n^l \cap \{|z| < r_l\} = S_n^{in}$.

Applying the same argument as Case 2 for U_n^+ , we obtain the following.

Proposition 6.13: It follows that $S_\infty^{out} \subset \partial A_f^{0,l} \cap \partial A_0$ and $S_\infty^{out} \cap (A_p \times \mathbb{P}^1) = \partial A_f^{0,l} \cap \partial A_0 \cap (A_p \times \mathbb{P}^1)$ for any l in Proposition 6.2.

Combining this proposition with Proposition 6.10, we obtain Proposition 6.3 in the previous subsection.

We next show Proposition 6.4. Note that $W_n^l \cap U^+$ is a decreasing sequence and

$$\left(\bigcap_{n \geq 1} W_n^l \right) \cap U^+ = \bigcup_{n \geq 1} f^{-n}(U^{0,l}) = U^+ - A_f^{0,l},$$

which is not empty and closed in U^+ . Since $S_n^{in} = \partial W_n^l \cap \{|z| < r_l\}$, it follows that $S_\infty^{in} \cap U^+ \subset \partial A_f^{0,l} \cap U^+$. Moreover, we have the equality since $\partial A_f^{0,l} \cap U^+ \subset W^l$.

Proposition 6.14: It follows that $S_\infty^{in} \cap U^+ = \partial A_f^{0,l} \cap U^+$ for any l in Proposition 6.2.

In addition, Propositions 6.12 and 6.13 imply the following global statement.

Proposition 6.15: It follows that $S_\infty^{in} \supset \partial A_f^{0,l} \cap A_0$ for any l in Proposition 6.2.

Combining this proposition with Proposition 6.11, we obtain Proposition 6.4. Note that G_z^α is monotonic increasing on $A_f^{0,l} \cap W_n^l \cap \mathbb{C}_z$ for any z in $A_p - E_p \cup E_{deg}^*$ and for any $n \geq 0$ because of the maximum principle. Hence we can take the limit instead of the limit inferior in Proposition 6.4. More precisely, we first obtain the claims in the proposition over $\{0 < |z| < r\}$ for small $r > 0$, which does not intersect E_{deg}^* , and then extend them to $A_p - E_p$ by pulling G_z^α back by f .

6.4 Examples

For any polynomial h of degree d and for any integers δ and α such that $\delta \geq d$, we can construct a polynomial skew product f that is semiconjugate to the polynomial product $g(z, w) = (z^\delta, h(w))$ and such that the α determined by f coincides with the given integer α . The dynamics of such a map is rather easy to study as stated in the introduction in this section.

We divide a family of such maps into two cases: the case $\delta > d$ and the case $\delta = d$. If $\delta > d$, then f degenerate the w -axis and A_0 is unbounded. If $\delta = d$, then f is non-degenerate and A_0 is bounded. Here we say that f is non-degenerate if f does not degenerate any fibre to a point. Let h be a monic polynomial of degree $d \geq 2$. More precisely, let $h(w) = w^d + b_{d-1}w^{d-1} + \dots + b_m w^m$, where $b_m \neq 0$.

Example 6.16 (degenerate type): Let α be an integer, $\delta > d$ and

$$f(z, w) = \left(z^\delta, z^{\alpha\delta} h\left(\frac{w}{z^\alpha}\right) \right).$$

Then f is semiconjugate to $g(z, w) = (z^\delta, h(w))$ by $\pi(z, w) = (z, z^\alpha w)$: $f \circ \pi = \pi \circ g$.

Since $q(z, w) = z^{\alpha(\delta-d)}w^d + b_{d-1}z^{\alpha(\delta-(d-1))}w^{d-1} + \dots + b_m z^{\alpha(\delta-m)}w^m$, the Newton polygon of q has two vertices, $(n_1, m_1) = (\alpha(\delta-d), d)$ and $(n_2, m_2) =$

$(\alpha(\delta - m), m)$, and f is in Cases 3 and 2. Let $U_1 = \{|z|^\alpha < r^\alpha |w|, |w| < r\}$ and $U_2 = \{|z| < r, |w| < r|z|^\alpha\}$ for small $r > 0$. Let A_1 and A_2 be the unions of all the preimages of U_1 and U_2 , respectively. Since

$$f^n(z, w) = \left(z^{\delta^n}, z^{\alpha\delta^n} h^n \left(\frac{w}{z^\alpha} \right) \right),$$

we have the following equalities:

$$\phi(z, w) = \left(z, z^\alpha \varphi_h^\infty \left(\frac{w}{z^\alpha} \right) \right) \text{ on } U_1 \text{ and } \phi(z, w) = \left(z, z^\alpha \varphi_h^0 \left(\frac{w}{z^\alpha} \right) \right) \text{ on } U_2$$

if $m \geq 2$, where $\varphi_h^\infty(w) = \lim_{n \rightarrow \infty} \sqrt[n]{h^n(w)}$ and $\varphi_h^0(w) = \lim_{n \rightarrow \infty} \sqrt[n]{h^n(w)}$,

$$G_z^\alpha(w) = G_h^\infty \left(\frac{w}{z^\alpha} \right) > 0 \text{ on } A_1 - \{z = 0\} \text{ and } G_z^\alpha(w) = G_h^0 \left(\frac{w}{z^\alpha} \right) < 0 \text{ on } A_2$$

if $m \geq 2$, where $G_h^\infty(w) = \lim_{n \rightarrow \infty} d^{-n} \log |h^n(w)|$ and $G_h^0(w) = \lim_{n \rightarrow \infty} m^{-n} \log |h^n(w)|$,

$$G_z^{\alpha,+}(w) = G_h^{\infty,+} \left(\frac{w}{z^\alpha} \right) = \begin{cases} G_h^\infty \left(\frac{w}{z^\alpha} \right) & \text{on } A_1 - \{z = 0\}, \\ 0 & \text{on } A_0 - A_1, \end{cases}$$

where $G_h^{\infty,+}(w) = \lim_{n \rightarrow \infty} d^{-n} \log^+ |h^n(w)|$,

$$\begin{aligned} \partial A_1 \cap \{(A_p - E_p) \times \mathbb{C}\} &= \bigcup_{0 < |z| < 1} \{z\} \times \partial \{w | G_z^{\alpha,+}(w) > 0\} \\ &= \bigcup_{0 < |z| < 1} \{z\} \times \partial \{w | G_z^{\alpha,+}(w) = 0\} \\ &= \bigcup_{0 < |z| < 1} \{z\} \times z^\alpha J_h, \end{aligned}$$

where J_h is the Julia set of h . Note that $\partial A_1 \cap (A_p \times \mathbb{C}) = \partial A_2 \cap (A_p \times \mathbb{C})$ if $m \geq 2$ because the boundaries of the attracting basins of ∞ and 0 for h coincide. Moreover,

$$G_z(w) = \alpha \log |z| \text{ on } A_1 \text{ and } G_z(w) = \begin{cases} \alpha \log |z| & \text{on } A_2 - E_w, \\ -\infty & \text{on } E_w \end{cases}$$

if $m \geq 2$. Therefore,

$$G_z(w) = \begin{cases} \alpha \log |z| & \text{on } A_0 - E_w, \\ -\infty & \text{on } E_w \end{cases}$$

if $m \geq 2$, and $G_f^\alpha(z, w) = \alpha \log |z|$ on A_0 , where $A_0 = \{|z| < 1\}$.

Example 6.17 (non-degenerate type): Let α be an integer and

$$f(z, w) = \left(z^d, z^{\alpha d} h \left(\frac{w}{z^\alpha} \right) \right).$$

Then f is semiconjugate to $g(z, w) = (z^d, h(w))$ by $\pi(z, w) = (z, z^\alpha w)$: $f \circ \pi = \pi \circ g$.

Since $q(z, w) = w^d + b_{d-1}z^\alpha w^{d-1} + \dots + b_m z^{\alpha(d-m)} w^m$, the Newton polygon of q has two vertices, $(n_1, m_1) = (0, d)$ and $(n_2, m_2) = (\alpha(d-m), m)$. Since

$$f^n(z, w) = \left(z^{d^n}, z^{\alpha d^n} h^n \left(\frac{w}{z^\alpha} \right) \right),$$

the same equalities as the previous example hold except the following:

$$G_z(w) = \alpha \log |z| + G_h^\infty \left(\frac{w}{z^\alpha} \right) > \alpha \log |z| \text{ on } A_1 - \{z = 0\}$$

and so

$$G_f^\alpha(z, w) = \begin{cases} \log |w| & \text{on } \{z = 0\}, \\ \alpha \log |z| + G_h^\infty \left(\frac{w}{z^\alpha} \right) & \text{on } A_1 - \{z = 0\}, \\ \alpha \log |z| & \text{on } A_0 - A_1, \end{cases}$$

where

$$A_0 = \{(0, w) \mid |w| < 1\} \cup \left(\bigcup_{0 < |z| < 1} \left\{ G_h^\infty \left(\frac{w}{z^\alpha} \right) < -\alpha \log |z| \right\} \right).$$

The same construction works even if $d = 1$. Such maps are used in Example 7.5 in Ueno (2019b) to show that we can not remove the condition $\delta \neq T_k$ for any k in Lemma 1.1 if $d = 1$. See also Example 5.2 in Ueno (2016b).

Even if α is rational, one can construct similar examples, although it is more complicated and the desired polynomial h is limited. Proposition 3.9 in Ueno (2010) and Proposition 3.1 in Ueno (2012) give necessary and sufficient conditions for the non-degenerate polynomial skew product $(z^d, q(z, w))$, where $q(z, w) = w^d + O_z(w^{d-1})$, to be semiconjugate to $(z^d, q(1, w))$ by $\pi(z, w) = (z^r, z^s w)$ for some positive integers r and s . More generally, Lemma 5.1 in Ueno (2020) gives necessary and sufficient conditions for the rational skew product $(z^\delta, q(z, w))$, where the degree of q with respect to w is d , to be semiconjugate to $(z^\delta, q(1, w))$ by $\pi(z, w) = (z^r, z^s w)$ for some integers r and s . Green functions of such maps are studied in Ueno (2012) and such maps are characterised by symmetries of Julia sets as stated in Ueno (2010, 2020). One can find statements similar to the examples above in Examples 5.2 and 5.3 in Ueno (2011), Examples 4.3 and 5.2 in Ueno (2014), Section 10 in Ueno (2016a) and Example 4.4 in Ueno (2016b).

7 Case 4

We deal with Case 4 in the last section. Let $s > 2$,

$$T_k \leq \delta \leq T_{k-1} \text{ and } (\gamma, d) = (n_k, m_k)$$

for some $2 \leq k \leq s-1$. Note that $\gamma > 0$ and $\delta > d \geq 1$ by the setting.

We first exhibit several results on invariant wedges and the unions of all the preimages of the wedges, which imply Theorems 2.13 and 2.14, and describe the properties of the blow-up $\tilde{f}_1$ in Subsection 7.1. We then exhibit a few results on the

existence and properties of plurisubharmonic functions in Subsection 7.2. We finally investigate the asymptotic behaviour of G_z^α toward the boundaries of the unions in Subsection 7.3, which is separated into three subcases. One may refer Table 3 in Subsection 2.3 for a comparison chart of the results. We assume that $a = b_{\gamma d} = 1$ in Subsections 7.2 and 7.3.

7.1 Invariant wedges and attracting sets

We exhibit several results that imply Theorems 2.13 and 2.14 in Subsection 7.1.1. Since the statements and proofs are almost the same as the case $\delta > d$ in Case 2, we omit the proofs.

Note that the dynamics of the first blow-up $\tilde{f}_1$ is almost the same as the case $\delta > d$ in Case 2.

Proposition 7.1: The blow-up $\tilde{f}_1$ is superattracting at the origin for any $0 < l < \alpha$, even for $l = \alpha$ if $\delta > T_k$ and $d \geq 2$ or if $\delta = T_k$ and $m_{k+1} \geq 2$, and degenerates the c -axis for any $0 < l < \alpha$ unless $(n_1, m_1) = (0, \delta)$.

Therefore, one may expect that Theorems 2.13 and 2.14 hold. This proposition corresponds to Proposition 4.1 in Case 2. See Subsection 7.1.2 for a detail, in which the shape of the Newton polygon of $\tilde{q}_1$ and properties of $\tilde{f}_1$ are summarised.

7.1.1 Exhibition of results

We can apply the same arguments for Case 2 to the dynamics of f on $U_{r_1, r_2}^{l, \infty}$ and $U^{l, \infty}$.

The following lemma corresponds to Lemma 4.4 in Case 2.

Lemma 7.2: If $(n_1, m_1) \neq (0, \delta)$, then $D > \delta$ for any $0 < l < \alpha$, and $D = \delta$ for $l = \alpha$. On the other hand, if $(n_1, m_1) = (0, \delta)$, then $D = \delta$ for any $0 < l \leq \alpha$.

The following lemma and theorem correspond to Lemma 4.5 and Theorem 4.6, respectively, which imply Theorem 2.13.

Lemma 7.3: For any $l > 0$ and for any small r_1 and r_2 , there exists a constant $C > 0$ such that $|q(z, w)| \leq C|z|^{lD}$ on $U_{r_1, r_2}^{l, \infty}$.

Theorem 7.4: The following holds for any $0 < l < \alpha$: $f(U_{r_1, r_2}^{l, \infty}) \subset U_{r_1, r_2}^{l, \infty}$ for suitable small r_1 and r_2 . More precisely, r_1 and r_2 have to satisfy the inequality $Cr_1^{l(D-\delta)} < r_2$ or $2Cr_1^{l(D^*-\delta)} < r_2$ for some constants C and D^* if $D > \delta$ or $D = \delta$.

The following lemma and proposition correspond to Lemma 4.9 and Proposition 4.10, respectively, which imply Theorem 2.14.

Lemma 7.5: For any $l > 0$ and for any small r and r_3 , there exists a constant $C > 0$ such that $|q(z, w)| \leq C|w|^D$ on V^l .

Proposition 7.6: If $(n_1, m_1) \neq (0, \delta)$, then the following holds for any $0 < l < \alpha$: for any (z, w) in V^l with suitable small r and r_3 , there exists an integer n such that $f^n(z, w)$ belongs to $U^{l, \infty}$.

7.1.2 Shapes of Newton polygons

The shape of the Newton polygon of $\tilde{q}_1$ and properties of $\tilde{f}_1$ for Case 4 are more or less similar to those for Case 2. More precisely, the magnitude relation between $0, \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_k$ are the same as Case 2, where $\tilde{n}_k = \tilde{\gamma}$, as shown in the following tables, which induces the attraction and/or degeneration properties of $\tilde{f}_1$. Roughly speaking, $0 < \tilde{n}_{k-1} < \tilde{\gamma}$ and $0 < \tilde{n}_j$ for any j when $0 < l < l_1$ and $(n_1, m_1) \neq (0, \delta)$, $0 \leq \tilde{\gamma} \leq \tilde{n}_j$ for any j when $l_1 \leq l \leq \alpha$, and $0 < \tilde{\gamma}$ when $l_1 \leq l < \alpha$. On the other hand, we have the extra vertices $(\tilde{n}_{k+1}, m_{k+1}), \dots, (\tilde{n}_s, m_s)$ for Case 4, and so $N(\tilde{q}_1)$ always has other vertices than $(\tilde{\gamma}, d)$.

Table 13 Shape of $N(\tilde{q}_1)$ when $T_k < \delta < T_{k-1}$

l	$0 < l < l_1$	l_1	$l_1 < l < \alpha$	α
$\tilde{\gamma}$	$0 < \tilde{n}_{k-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 < \tilde{\gamma} = \tilde{n}_{k-1} < \tilde{n}_j$ for $j \neq k, k-1$	$0 < \tilde{\gamma} < \tilde{n}_j$ for $j \neq k$	$0 = \tilde{\gamma} < \tilde{n}_j$ for $j \neq k$
$\tilde{f}_1$	SA and Deg	SA and Deg	SA and Deg	SA if $d \geq 2$

Table 14 Shape of $N(\tilde{q}_1)$ when $\delta = T_k$

l	$0 < l < l_1$	l_1	$l_1 < l < \alpha$	α
$\tilde{\gamma}$	$0 < \tilde{n}_{k-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 < \tilde{\gamma} = \tilde{n}_{k-1} < \tilde{n}_j$ for $j \neq k, k-1$	$0 < \tilde{\gamma} < \tilde{n}_j$ for $j \neq k$	$0 = \tilde{\gamma} = \tilde{n}_{k+1} < \tilde{n}_j$ for $j \neq k, k+1$
$\tilde{f}_1$	SA and Deg	SA and Deg	SA and Deg	SA if $m_{k+1} \geq 2$

Table 15 Shape of $N(\tilde{q}_1)$ when $\delta = T_{k-1}$ and $(n_1, m_1) \neq (0, \delta)$

l	$0 < l < l_1 = \alpha$	$l_1 = \alpha$
$\tilde{\gamma}$	$0 < \tilde{n}_{k-1} < \tilde{\gamma}$ $0 < \tilde{n}_j$ for any j	$0 = \tilde{\gamma} = \tilde{n}_{k-1} < \tilde{n}_j$ for $j \neq k, k-1$
$\tilde{f}_1$	SA and Deg	SA if $d \geq 2$

Table 16 Shape of $N(\tilde{q}_1)$ when $(n_1, m_1) = (0, \delta)$

l	$0 < l < l_1 = \alpha$	$l_1 = \alpha$
$\tilde{\gamma}$	$0 = \tilde{n}_1 < \tilde{\gamma} < \tilde{n}_j$ for $j \geq 3$	$0 = \tilde{n}_1 = \tilde{\gamma} < \tilde{n}_j$ for $j \geq 3$
$\tilde{f}_1$	SA	SA if $d \geq 2$

We only give tables of summaries of the shape of $N(\tilde{q}_1)$ and properties of $\tilde{f}_1$. Recall that $\mathcal{I}_f^1 = [l_1, \alpha]$ if $T_k < \delta < T_{k-1}$, $\mathcal{I}_f^1 = [l_1, \alpha]$ if $\delta = T_k$, and $\mathcal{I}_f^1 = \{l_1\} = \{\alpha\}$ if $\delta = T_{k-1}$. Tables 13, 15 and 16 for Case 4 correspond to Tables 7, 8 and 9 for Case 2,

respectively. Table 14 resembles to Table 13, whereas the case $l = \alpha$ is different. Note that $(\gamma, d) = (n_2, m_2)$ for Table 16 since $(n_1, m_1) = (0, \delta)$.

7.2 Plurisubharmonic functions

Because f is conjugate to the monomial map f_0 on $U^{l_{(1)}, l_{(2)}}$ for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$ if $d \geq 2$, and for any $l_{(1)}$ in $\mathcal{I}_f^1 - \{\alpha\}$ and $l_{(2)}$ in $\mathcal{I}_f^2 - \{\alpha - l_{(1)}\}$ if $d = 1$ and $T_k < \delta < T_{k-1}$, the limit G_z^α exists on $U^{l_{(1)}, l_{(2)}}$, which extends to a pluriharmonic function on $A_f^{l_{(1)}, l_{(2)}}$.

Proposition 7.7: For any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$ if $d \geq 2$, and for any $l_{(1)}$ in $\mathcal{I}_f^1 - \{\alpha\}$ and $l_{(2)}$ in $\mathcal{I}_f^2 - \{\alpha - l_{(1)}\}$ if $d = 1$ and $T_k < \delta < T_{k-1}$, the limit G_z^α is pluriharmonic on $A_f^{l_{(1)}, l_{(2)}}$. Moreover, $G_z^\alpha(w) = \log |w/z^\alpha| + o(1)$ on $U_r^{l_{(1)}, l_{(2)}}$ as $r \rightarrow 0$.

Recall that $A_f = A_f^{l_1, l_2}$ and that $A_f^{l_1, l_2}$ is the largest wedge among $A_f^{l_{(1)}, l_{(2)}}$ for any $l_{(1)}$ in $\mathcal{I}_f^1$ and $l_{(2)}$ in $\mathcal{I}_f^2$. This proposition implies Proposition 2.16.

Combining Theorem 2.14 and Proposition 2.16 with Propositions 7.13 and 7.16, results on the asymptotic behaviour of G_z^α exhibited in the next subsection, we obtain Theorem 2.17.

The limit G_z also exists on U , where $U = U^{l_1, l_2}$, which extends to A_f .

Proposition 7.8: If $d \geq 2$ or $T_k < \delta < T_{k-1}$, then $G_z = \alpha G_p \circ \pi_z$ on A_f .

Moreover, it follows from Theorems 2.13 and 2.14 that $G_z^* \leq \alpha G_p \circ \pi_z$ on A_0 if $(n_1, m_1) \neq (0, \delta)$. Therefore, we have the following.

Corollary 7.9: If $(n_1, m_1) \neq (0, \delta)$, then $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 .

The proposition and corollary above imply Theorem 2.19.

7.3 Asymptotic behaviour of G_z^α toward $\partial A_f^{l_{(1)}, l_{(2)}}$

In this subsection we exhibit results on the asymptotic behaviour of G_z^α for the cases $T_k < \delta < T_{k-1}$, $\delta = T_k$ and $\delta = T_{k-1}$ in Subsections 7.3.1, 7.3.2 and 7.3.3, respectively. In particular, Propositions 7.13 and 7.16 below imply Theorem 2.17, and Corollaries 7.14 and 7.17 below imply Proposition 2.18. We omit all the proofs since they are almost the same as the previous cases.

7.3.1 Case 4: $T_k < \delta < T_{k-1}$

Recall that $\mathcal{I}_f = [l_1, \alpha] \times [\alpha, l_1 + l_2] - \{(\alpha, \alpha)\}$ if $T_k < \delta < T_{k-1}$ and that $U_r^{l_{(1)}, l_{(2)}} = \{|z|^{l_{(1)}+l_{(2)}} < r^{l_{(2)}}|w|, |w| < r|z|^{l_{(1)}}\}$.

We first point out that f has two disjoint invariant wedges in U^{l_1, l_2} .

Proposition 7.10: It follows for sufficiently small $r > 0$ that

- 1 $f(U_r^{l_1, \alpha-l_1}) \subset U_r^{l_1, \alpha-l_1}$ and $f(U_r^{\alpha, l_1+l_2-\alpha}) \subset U_r^{\alpha, l_1+l_2-\alpha}$
- 2 $U_r^{l_1, \alpha-l_1} \cap U_r^{\alpha, l_1+l_2-\alpha} = \emptyset$ and $U_r^{l_1, \alpha-l_1} \cup U_r^{\alpha, l_1+l_2-\alpha} \subset U_r^{l_1, l_2}$.

Similar to Case 3, we separate the boundary $\partial U^{l_{(1)}, l_{(2)}}$ into two surfaces: let $S_n^{out} = \{|z|^{l_{(1)}+l_{(2)}} < r^{l_{(2)}}|w|, |w| = r|z|^{l_{(1)}}\}$ and $S_n^{in} = \{|z|^{l_{(1)}+l_{(2)}} = r^{l_{(2)}}|w|, |w| < r|z|^{l_{(1)}}\}$. Let $S_n^{out} = f^{-n}(S_n^{out})$ and $S_n^{in} = f^{-n}(S_n^{in})$.

Lemma 7.11: Fix any z in $A_p - E_p \cup E_{deg}^*$ and let $l_1 \leq l_{(1)} < l_{(1)} + l_{(2)} \leq l_1 + l_2$. Then $S_n^{out} \cap \mathbb{C}_z \neq \emptyset$ and $S_n^{in} \cap \mathbb{C}_z \neq \emptyset$ for any large n , and

$$G_z^\alpha|_{S_n^{out}} \rightarrow \begin{cases} \infty \text{ as } n \rightarrow \infty & \text{if } l_{(1)} < \alpha, \\ 0 \text{ as } n \rightarrow \infty & \text{if } l_{(1)} = \alpha \text{ and } d \geq 2, \end{cases}$$

and

$$G_z^\alpha|_{S_n^{in}} \rightarrow \begin{cases} 0 \text{ as } n \rightarrow \infty & \text{if } l_{(1)} + l_{(2)} = \alpha \text{ and } d \geq 2, \\ -\infty \text{ as } n \rightarrow \infty & \text{if } l_{(1)} + l_{(2)} > \alpha. \end{cases}$$

In particular, the following hold under the condition $d \geq 2$:

$$\begin{aligned} G_z^\alpha|_{S_n^{out}} &\rightarrow \infty \text{ and } G_z^\alpha|_{S_n^{in}} \rightarrow -\infty \text{ as } n \rightarrow \infty & \text{if } l_{(1)} < \alpha < l_{(1)} + l_{(2)}, \\ G_z^\alpha|_{S_n^{out}} &\rightarrow \infty \text{ and } G_z^\alpha|_{S_n^{in}} \rightarrow 0 \text{ as } n \rightarrow \infty & \text{if } l_{(1)} < \alpha = l_{(1)} + l_{(2)}, \\ G_z^\alpha|_{S_n^{out}} &\rightarrow 0 \text{ and } G_z^\alpha|_{S_n^{in}} \rightarrow -\infty \text{ as } n \rightarrow \infty & \text{if } l_{(1)} = \alpha < l_{(1)} + l_{(2)}. \end{aligned}$$

Let S_∞^{out} and S_∞^{in} be the sets of the accumulation points of S_n^{out} and S_n^{in} in $\overline{A_p} \times \mathbb{P}^1$ as $n \rightarrow \infty$, respectively. Let $U^{l_{(1)}, l_{(2)}, +} = \{|z| < r^{1+l_{(2)}}, |w| < r|z|^{l_{(1)}}\}$ and $A_f^{l_{(1)}, l_{(2)}, +}$ the union of all the preimages of $U^{l_{(1)}, l_{(2)}, +}$.

Lemma 7.12: For any $l_{(1)}$ and $l_{(2)}$ in Proposition 7.7,

- 1 $\lim_{n \rightarrow \infty} f^{-n}(\partial U^{l_{(1)}, l_{(2)}}) = S_\infty^{out} \cup S_\infty^{in} = \partial A_f^{l_{(1)}, l_{(2)}}$
- 2 $S_\infty^{out} \subset \partial A_f^{l_{(1)}, l_{(2)}, +} \subset \partial A_f^{l_{(1)}, l_{(2)}}$
- 3 $S_\infty^{out} \cap (A_p \times \mathbb{P}^1) = \partial A_f^{l_{(1)}, l_{(2)}, +} \cap (A_p \times \mathbb{P}^1)$
- 4 $\partial A_f^{l_{(1)}, l_{(2)}} - \partial A_f^{l_{(1)}, l_{(2)}, +} \subset S_\infty^{in} \subset \partial A_f^{l_{(1)}, l_{(2)}}$
- 5 $S_\infty^{in} \cap U^{l_{(1)}, l_{(2)}, +} = \partial A_f^{l_{(1)}, l_{(2)}} \cap U^{l_{(1)}, l_{(2)}, +}$.

Note that this lemma holds even if we replace $U^{l_{(1)}, l_{(2)}, +}$ and $A_f^{l_{(1)}, l_{(2)}, +}$ to $U^{l_{(1)}, \infty}$ and $A_f^{l_{(1)}, \infty}$. Hence the two lemmas above imply the following.

Proposition 7.13: For any (z_0, w_0) in $\partial A_f^{l_{(1)}, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z, w) \in A_f^{l_{(1)}, l_{(2)}} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \begin{cases} \infty & \text{if } l_1 \leq l_{(1)} < \alpha, \\ 0 & \text{if } l_{(1)} = \alpha \text{ and } d \geq 2. \end{cases}$$

For any (z_0, w_0) in $\partial A_f^{l(1), l(2)} \cap A_0 - \partial A_f^{l(1), \infty} \cup E_z$,

$$\lim_{(z, w) \in A_f^{l(1), l(2)} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \begin{cases} -\infty & \text{if } \alpha < l(1) + l(2) \leq l_1 + l_2, \\ 0 & \text{if } \alpha = l(1) + l(2) \text{ and } d \geq 2. \end{cases}$$

Note that we can replace $\partial A_f^{l(1), l(2)} \cap A_0 - \partial A_f^{l(1), \infty} \cup E_z$ in the proposition to $\{\partial A_f^{l(1), l(2)} - \partial A_f^{l(1), \infty}\} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$. Moreover, we can replace it to $\{\partial A_f^{l(1), l(2)} \cap A_0\} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$ if $l(1) < \alpha$ thanks to Theorem 2.14.

Corollary 7.14: It follows that $A_f^{l(1), l(2)} = A_f^{l_1, l_2}$ for any $l_1 \leq l(1) < \alpha$ and $\alpha < l(1) + l(2) \leq l_1 + l_2$. Moreover, $A_f^{l_1, \alpha - l_1} \subsetneq A_f^{l_1, l_2}$, $A_f^{\alpha, l_1 + l_2 - \alpha} \subsetneq A_f^{l_1, l_2}$ and $A_f^{l_1, \alpha - l_1} \cap A_f^{\alpha, l_1 + l_2 - \alpha} = \emptyset$ if $d \geq 2$.

7.3.2 Case 4: $\delta = T_k$

Recall that $l_1 < \alpha$ and $\mathcal{I}_f = [l_1, \alpha) \times \{\alpha\}$ if $\delta = T_k$.

Let $S^{\text{out}} = \{|z|^{l+\alpha} < r^\alpha |w|, |w| = r|z|^l\}$ and $S^{\text{in}} = \{|z|^{l+\alpha} = r^\alpha |w|, |w| < r|z|^l\}$.

Lemma 7.15: Let $d \geq 2$. For any l in $\mathcal{I}_f^1$ and for any z in $A_p - E_p \cup E_{deg}^*$, we have that $S_n^{\text{out}} \cap \mathbb{C}_z \neq \emptyset$ and $S_n^{\text{in}} \cap \mathbb{C}_z \neq \emptyset$ for any large n , and $G_z^\alpha|_{S_n^{\text{out}}} \rightarrow \infty$ and $G_z^\alpha|_{S_n^{\text{in}}} \rightarrow 0$ as $n \rightarrow \infty$.

This lemma and Lemma 7.12 imply the following.

Lemma 7.16: Let $d \geq 2$. For any l in $\mathcal{I}_f^1$ and for any (z_0, w_0) in $\partial A_f^{l, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z, w) \in A_f^{l, \alpha - l} \rightarrow (z_0, w_0)} G_z^\alpha(w) = \infty.$$

For any l in $\mathcal{I}_f^1$ and for any (z_0, w_0) in $\partial A_f^{l, \alpha - l} \cap A_0 - \partial A_f^{l, \infty} \cup E_z$,

$$\lim_{(z, w) \in A_f^{l, \alpha - l} \rightarrow (z_0, w_0)} G_z^\alpha(w) = 0.$$

Corollary 7.17: If $d \geq 2$, then $A_f^{l, \alpha - l} = A_f^{l_1, \alpha - l_1}$ for any l in $\mathcal{I}_f^1$.

7.3.3 Case 4: $\delta = T_{k-1}$

Recall that $\mathcal{I}_f^1 = \{\alpha\}$ and $\mathcal{I}_f^2 = (0, l_2]$ if $\delta = T_{k-1}$.

Let $S^{\text{out}} = \{|z|^{\alpha+l} < r^l |w|, |w| = r|z|^\alpha\}$ and $S^{\text{in}} = \{|z|^{\alpha+l} = r^l |w|, |w| < r|z|^\alpha\}$.

Lemma 7.18: Let $d \geq 2$. For any l in $\mathcal{I}_f^2$ and for any z in $A_p - E_p \cup E_{deg}^*$, we have that $S_n^{\text{out}} \cap \mathbb{C}_z \neq \emptyset$ and $S_n^{\text{in}} \cap \mathbb{C}_z \neq \emptyset$ for any large n , and $G_z^\alpha|_{S_n^{\text{out}}} \rightarrow 0$ and $G_z^\alpha|_{S_n^{\text{in}}} \rightarrow -\infty$ as $n \rightarrow \infty$.

This lemma and Lemma 7.12 imply the following.

Proposition 7.19: Let $d \geq 2$. For any (z_0, w_0) in $\partial A_f^{\alpha, \infty} \cap \{(A_p - E_p) \times \mathbb{P}^1\}$,

$$\limsup_{(z, w) \in A_f^{\alpha, \infty} \rightarrow (z_0, w_0)} G_z^\alpha(w) = 0.$$

For any l in $\mathcal{I}_f^2$ and for any (z_0, w_0) in $\partial A_f^{\alpha, l-\alpha} \cap A_0 - \partial A_f^{\alpha, \infty} \cup E_z$,

$$\lim_{(z, w) \in A_f^{\alpha, l-\alpha} \rightarrow (z_0, w_0)} G_z^\alpha(w) = -\infty.$$

Corollary 7.20: If $d \geq 2$, then $A_f^{\alpha, l-\alpha} = A_f^{\alpha, l_2-\alpha}$ for any l in $\mathcal{I}_f^2$.

Remark 7.21: If $\delta = T_{k-1}$, then f has the other dominant term (n_{k-1}, m_{k-1}) , which belongs to Case 3 or Case 4. Therefore, variants of Theorems 2.9 and 2.17 imply that $G_z^{\alpha, +}$ is plurisubharmonic on $A_0 - E_z$. Moreover, variants of Corollaries 6.7 and 7.9 imply that $G_f^\alpha = \alpha G_p \circ \pi_z$ on A_0 if $n_{k-1} > 0$, and G_f^α is plurisubharmonic and continuous on A_0 if $n_{k-1} = 0$.

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