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A comparative analysis of Soham and Laplace transforms in solving Volterra integral equations

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Abstract: This paper explores the application of Soham transform and Laplace transform to solve Volterra integral equations of the first kind, second kind, and non-homogeneous types. The study focuses on integral equations with convolution-type kernels, widely encountered in mathematical modelling and engineering applications. Analytical solutions are derived to comprehensively understand these transforms in addressing such equations. To validate the accuracy and reliability of the analytical solutions, numerical verification is performed using the trapezoidal rule and an iterative refinement approach. A detailed error analysis is conducted to examine the effect of step size on numerical accuracy, ensuring the robustness of numerical approximations. A real-world application is presented for the one-dimensional non-homogeneous heat conduction problem, which is rigorously formulated with initial and boundary conditions. This case study illustrates the practical utility of integral transforms in solving engineering problems related to heat transfer. The study provides an in-depth discussion of both transforms' numerical accuracy, computational efficiency, and convergence behaviour in addressing integral equations with convolution-type kernels. This comparative analysis offers valuable insights into the suitability of Soham and Laplace transforms for mathematical modelling and engineering applications, emphasising their role in efficiently solving real-world problems.

Keywords: Volterra integral equations; Laplace transform; Soham transform; numerical validation; non-homogeneous heat conduction.

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1 Introduction

Integral equations play a vital role in mathematical modelling, providing powerful tools for describing a wide range of real-world phenomena. From heat conduction and fluid dynamics to population dynamics and signal processing, these equations are central to solving problems involving systems with memory or delay. Among these, Volterra integral equations, classified into first and second kinds, hold significant importance due to their applications in temporal systems, particularly when the kernel is of convolution type.

A Volterra integral equation of the first kind is expressed as:

$$f(x) = \int_0^x K(x-t)y(t) dt, \quad (1)$$

where $f(x)$ is the known function, $y(t)$ is the unknown function to be solved, and $K(x-t)$ is the kernel. These equations are typically more challenging to solve analytically because they involve only the integral of the unknown function.

On the other hand, Volterra integral equations of the second kind include the unknown function explicitly:

$$y(x) = f(x) + \int_0^x K(x-t)y(t) dt, \quad (2)$$

making them more approachable due to their self-contained nature. These equations are widely used in modelling feedback systems, where the current state depends on both past and present states.

1.1 Transform methods for solving integral equations

To solve these equations analytically, transform methods like the Laplace transform and the Soham transform have emerged as powerful tools:

- *Laplace transform:* A well-established technique, it converts integral equations into algebraic equations by simplifying convolution operations. The availability of extensive inversion tables makes it a preferred method for standard problems.
- *Soham transform:* A relatively newer approach, this transform generalises the solution process by introducing a flexibility parameter α . It is particularly advantageous for handling convolution-type kernels and cases where parameter tuning can simplify computations.

While these transforms offer analytical solutions, many real-world problems, such as nonlinear or irregular systems, require numerical approaches for validation.

1.2 Numerical methods for validation

Numerical methods complement analytical solutions by providing approximate results where exact solutions are infeasible. In this study, the trapezoidal rule and an iterative refinement process are employed to validate the analytical solutions. A detailed error analysis is conducted to examine the effect of step size on numerical accuracy, ensuring the robustness of numerical approximations. This method approximates the integral by discretising the domain and iteratively refining the unknown function $y(x)$ until convergence.

1.3 Objective of the study

This work aims to:

- 1 solve Volterra integral equations of the first and second kinds analytically using Soham and Laplace transforms
- 2 validate the analytical solutions through numerical methods
- 3 formulate and solve one-dimensional non-homogeneous heat conduction problem, explicitly including initial and boundary conditions to ensure mathematical rigour
- 4 analyse the accuracy and efficiency of numerical methods and their dependence on step size.

The results of this study not only highlight the theoretical utility of Soham and Laplace transforms but also demonstrate their effectiveness in solving real-world engineering problems, particularly in heat conduction analysis. Additionally, the study emphasises the importance of numerical validation in ensuring the reliability of mathematical models.

2 Literature review

Integral transforms have long been utilised as powerful tools for solving differential and integral equations. The Laplace transform, introduced decades ago, has been widely applied in mathematical modelling, particularly for time-dependent systems and boundary value problems. Early contributions, such as those by Wazwaz (1957), established the theoretical framework for integral equations, discussing their applications in mathematical physics and engineering. Similarly, Piaggio (1920) laid the groundwork for utilising the Laplace transform in solving differential and integral equations, setting the foundation for modern computational approaches.

Researchers have extended these classical methods to address more complex systems in the modern era. Diethelm et al. (2002) demonstrated the capability of Laplace transforms in fractional calculus, showcasing their utility in fractional Volterra integral equations. Yang and Hou (2012) proposed a hybrid approach combining the Laplace transform with numerical inversion for solving Volterra integral equations with convolution kernels, emphasising its accuracy and convergence properties. Other modifications, such as the Aboodh transform (Aggarwal et al., 2018a) and Shehu transform, were introduced to address specific computational challenges in solving first-kind integral equations.

2.1 Soham transform: a modern integral transform

Despite the popularity of the Laplace transform, alternative integral transforms have gained attention due to their potential advantages in specific applications. The Soham transform, introduced by Patil (2021), emerged as a novel integral transform capable of handling linear differential equations and initial value problems. Patil et al. (2022) extended its application to first-kind Volterra integral equations, proving its efficiency and accuracy compared to traditional methods. Later, Patil and Rathi (2023) applied the Soham transform to fractional differential equations, demonstrating its versatility and potential for broader mathematical modelling applications.

Patil et al. (2022) presented applications of the double general integral transform for solving boundary value problems in partial differential equations. This study highlights how integral transforms are widely used to solve advanced problems in various fields with minimal calculations. Several new transforms, such as Kushare, Soham, and Emad Sara, have been introduced and applied to differential equations, boundary value problems, and integral equations. This paper particularly focuses on using the Emad Sara transform to solve boundary value problems in partial differential equations (Patil and Raundal, 2022).

Recent comparative studies, such as those by Patil and Rathi (2023), evaluated multiple integral transforms, including Laplace and Soham, in solving real-world problems like biomedical models and oscillatory systems, showing that the Soham transform can be a viable alternative in specific scenarios. Additionally, Heyd (2024) applied these methods to nonlinear Volterra equations, extending their use to complex systems with real-world implications.

2.2 *Numerical methods for solving Volterra integral equations*

While integral transforms provide analytical solutions, numerical methods are essential for handling nonlinear, multidimensional, and non-homogeneous problems where analytical solutions are intractable. Several quadrature-based methods, iterative techniques, and spectral methods have been developed for this purpose.

2.2.1 *Quadrature-based methods*

One of the most widely used numerical techniques is the trapezoidal rule, which approximates integrals by computing weighted sums over a discrete set of points. Micula (2019) applied the two-dimensional composite trapezoidal rule to mixed Volterra-Fredholm integral equations, providing convergence conditions and error estimates, confirming its effectiveness in handling second-kind Volterra integral equations.

2.2.2 *Iterative refinement methods*

Iterative refinement methods have been widely used to improve numerical solutions of integral equations. Mahcene et al. (2023) introduced a refinement sum technique in an iterative scheme for solving linear Fredholm integral equations. Their approach converts a single integral equation into an equivalent system of integral equations over smaller intervals, significantly improving computational efficiency and accuracy.

Deif and Grace (2018) proposed a fast iterative refinement method for mixed systems of integral and fractional integro-differential equations. Their approach was tested on first- and second-kind Volterra equations, demonstrating improved convergence speed and accuracy. The study also compared their method with well-known techniques such as Picard's successive approximations, Adomian decomposition, and homotopy analysis, proving the versatility of iterative refinement methods in solving complex integral equations.

2.2.3 *Boundary element and spectral methods*

Banerjea et al. (2019) introduced a boundary element method (BEM) for solving Fredholm and Volterra integral equations, demonstrating its computational efficiency by transforming integral equations into linear algebraic systems. Their study highlighted the accuracy and simplicity of BEM, making it a valuable tool for numerical solutions in applied mathematics and engineering.

Khidir (2021) introduced the Chebyshev spectral collocation method, which converts integral equations into algebraic systems for efficient computation, significantly reducing computational complexity. Hussein et al. (2023) proposed the Hussein-Jassim method (HJ-method) as a novel semi-analytical approach, proving its convergence properties and robustness in solving first-kind Volterra integral equations.

2.3 *Applications of Volterra integral equations in one-dimensional heat conduction*

Volterra integral equations are widely used in heat conduction problems, particularly in one-dimensional systems. Jayakumar et al. (2017) studied heat conduction in

composite slabs and cylinders, solving the heat equation under steady-state conditions and validating the results using ANSYS simulations. Their findings demonstrated how integral equations effectively model temperature distribution and heat flux variations in engineering structures.

Halic (2023) examined one-dimensional heat conduction with radiation effects, formulating the governing boundary value problem using the Stefan-Boltzmann law. His study provided upper and lower bounds for temperature distribution, demonstrating how integral equations are crucial in radiative heat transfer analysis. Krivtsov et al. (2018) extended this work by analysing entropy production in one-dimensional heat conduction, highlighting the fundamental role of integral formulations in thermodynamic analysis

Additionally, Sedelnikov et al. (2023) studied the stress-strain response of a rectangular plate subjected to thermal shock, applying Fourier's law to describe temperature evolution and its effects on structural deformation. These studies reinforce the relevance of integral equations in thermoelasticity and mechanical engineering applications.

3 Research gap and novelty

Despite significant advancements in integral transforms and numerical methods, several research gaps remain:

- comparison of transform methods: while the Laplace transform is well-established, the Soham transform remains underexplored, particularly in practical applications
- numerical validation techniques: most studies focus on theoretical solutions, while limited research explores the integration of modern numerical techniques like the trapezoidal rule and iterative refinement for validation
- non-homogeneous heat conduction problems: while heat conduction equations have been studied, the application of integral transforms and numerical methods in solving non-homogeneous cases remains a critical gap
- extending methods to nonlinear and multi-dimensional systems: the extension of these techniques to higher-dimensional and nonlinear problems is still an open research area.

This study addresses these gaps by presenting a comprehensive comparison of the Laplace and Soham in solving first-kind and second-kind Volterra integral equations. It uniquely integrates analytical and numerical methods, ensuring robust validation through modern numerical techniques. The novelty lies in applying these transforms to real-world problems, such as non-homogeneous heat conduction, demonstrating their practical utility.

4 Theory and methods

4.1 Volterra integral equations

Volterra integral equations are classified into:

- *First kind:*

$$f(x) = \int_0^x K(x-t)y(t) dt, \quad (3)$$

where the unknown $y(t)$ appears only inside the integral. These equations are commonly encountered in inverse problems and ill-posed systems, making their solution techniques crucial in applied mathematics.

- *Second kind:*

$$y(x) = f(x) + \int_0^x K(x-t)y(t) dt, \quad (4)$$

where $y(x)$ appears both inside and outside the integral. These equations arise frequently in physical and engineering applications, such as population dynamics, viscoelastic materials, and heat conduction.

4.2 Soham and Laplace transforms

Both transforms simplify convolution operations into algebraic equations:

- *Soham transform:*

$$\delta[y(x)] = \int_0^\infty y(x)x^{\alpha-1}e^{-vx} dx, \quad (5)$$

introducing flexibility through the parameter α . This transform is particularly useful in cases where parameter tuning can enhance convergence or numerical stability.

- *Laplace transform:*

$$\mathcal{L}[y(x)] = \int_0^\infty y(x)e^{-sx} dx, \quad (6)$$

widely used for its extensive inversion tables. It is commonly applied in control theory, electrical circuit analysis, and differential equations with constant coefficients.

Both transforms handle convolution integrals via:

$$\begin{aligned} &\text{Transform of } \int_0^x K(x-t)y(t) dt \\ &= \text{Transform of } K(x) \cdot \text{Transform of } y(x). \end{aligned} \quad (7)$$

This property simplifies solving Volterra integral equations by reducing them to algebraic equations, allowing for efficient analytical solutions.

4.3 Numerical methods

While analytical solutions provide valuable insights, real-world problems often require numerical validation. The trapezoidal rule approximates the integral:

$$\int_0^x K(x-t)y(t) dt \approx \sum_{i=0}^n K(x-t_i)y(t_i)\Delta t, \quad (8)$$

and iterative refinement handles equations where $y(x)$ appears on both sides:

- 1 Start with an initial guess for $y(t)$.
- 2 Compute the integral numerically.
- 3 Update $y(x)$ using:

$$y(x) = f(x) + \int_0^x K(x-t)y(t) dt. \quad (9)$$

- 4 Repeat until convergence.

The iterative refinement method significantly improves the accuracy of numerical solutions, particularly when applied to problems with strong kernel dependencies.

4.4 Error analysis

To assess numerical accuracy, the absolute error is computed between the analytical and numerical solutions:

$$\text{Error} = |y_{\text{analytical}}(x) - y_{\text{numerical}}(x)|. \quad (10)$$

A detailed study of the effect of step size (Δt) on error convergence is conducted, ensuring that smaller step sizes lead to improved accuracy.

5 Step-wise analytical solution

This section provides a detailed step-by-step solution to the first-kind and second-kind Volterra integral equations, as well as the non-homogeneous heat conduction problem using both the Laplace transform and the Soham transform.

5.1 First-kind equation

The first-kind equation is:

$$\sin x = \int_0^x e^{x-t}y(t) dt \quad (11)$$

This equation arises in various applications such as signal processing and viscoelasticity, where convolution-type kernels play a crucial role.

5.1.1 Using Laplace transform

Applying the Laplace transform $\mathcal{L}$ to both sides:

$$\mathcal{L}\{\sin x\} = \mathcal{L}\left\{\int_0^x e^{x-t}y(t) dt\right\} \quad (12)$$

Using the convolution property:

$$\mathcal{L}\{\sin x\} = \mathcal{L}\{e^x\} \cdot \mathcal{L}\{y(x)\} \quad (13)$$

Computing the Laplace transforms:

$$\mathcal{L}\{\sin x\} = \frac{1}{s^2 + 1}, \quad \mathcal{L}\{e^x\} = \frac{1}{s - 1} \quad (14)$$

Substituting:

$$\frac{1}{s^2 + 1} = \frac{1}{s - 1} \cdot \mathcal{L}\{y(x)\} \quad (15)$$

Solving for $\mathcal{L}\{y(x)\}$:

$$\mathcal{L}\{y(x)\} = \frac{s - 1}{s^2 + 1} \quad (16)$$

Applying the inverse Laplace transform:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{s - 1}{s^2 + 1}\right\} \quad (17)$$

Using properties of Laplace inversion:

$$y(x) = \cos x - \sin x \quad (18)$$

This result confirms that Laplace transforms effectively reduce complex integral equations into algebraic ones, enabling systematic solutions.

5.1.2 Using Soham transform

Applying the Soham transform $\mathcal{S}$ to both sides:

$$\mathcal{S}\{\sin x\} = \mathcal{S}\left\{\int_0^x e^{x-t}y(t) dt\right\} \quad (19)$$

Using the convolution property:

$$\mathcal{S}\{\sin x\} = \mathcal{S}\{e^x\} \cdot \mathcal{S}\{y(x)\} \quad (20)$$

Computing the Soham transforms:

$$\mathcal{S}\{\sin x\} = \frac{1}{p^2 + 1}, \quad \mathcal{S}\{e^x\} = \frac{1}{p - 1} \quad (21)$$

Substituting:

$$\frac{1}{p^2 + 1} = \frac{1}{p - 1} \cdot \mathcal{S}\{y(x)\} \quad (22)$$

Solving for $\mathcal{S}\{y(x)\}$:

$$\mathcal{S}\{y(x)\} = \frac{p - 1}{p^2 + 1} \quad (23)$$

Applying the inverse Soham transform:

$$y(x) = \mathcal{S}^{-1} \left\{ \frac{p - 1}{p^2 + 1} \right\} \quad (24)$$

Using properties of Soham inversion:

$$y(x) = \cos x - \sin x \quad (25)$$

The equivalence of solutions obtained via both transforms demonstrates their efficiency and correctness in handling Volterra integral equations.

5.2 Second-kind equation

The second-kind equation is:

$$y(x) = x + \int_0^x \sin(x - t)y(t) dt \quad (26)$$

This equation frequently appears in engineering applications such as population growth models, feedback control systems, and thermal diffusion problems, where the solution depends on both past and present states.

5.2.1 Using Laplace transform

Applying the Laplace transform $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y(x)\} = \mathcal{L}\{x\} + \mathcal{L}\left\{ \int_0^x \sin(x - t)y(t) dt \right\} \quad (27)$$

Using the convolution property:

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \mathcal{L}\{\sin x\} \cdot \mathcal{L}\{y(x)\} \quad (28)$$

Computing the Laplace transforms:

$$\mathcal{L}\{\sin x\} = \frac{1}{s^2 + 1} \quad (29)$$

Substituting:

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \frac{1}{s^2 + 1} \cdot \mathcal{L}\{y(x)\} \quad (30)$$

Solving for $\mathcal{L}\{y(x)\}$:

$$\mathcal{L}\{y(x)\} \left(1 - \frac{1}{s^2 + 1}\right) = \frac{1}{s^2} \quad (31)$$

Simplifying:

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \frac{1}{s^3} \quad (32)$$

Applying the inverse Laplace transform:

$$y(x) = x + \frac{x^3}{6} \quad (33)$$

This confirms that the Laplace transform is effective in reducing complex integral equations to solvable algebraic expressions, making it a powerful tool for Volterra integral equations of the second kind.

5.2.2 Using Soham transform

Applying the Soham transform $\mathcal{S}$ to both sides:

$$\mathcal{S}\{y(x)\} = \mathcal{S}\{x\} + \mathcal{S}\left\{\int_0^x \sin(x-t)y(t) dt\right\} \quad (34)$$

Using the convolution property:

$$\mathcal{S}\{y(x)\} = \frac{1}{p^2} + \mathcal{S}\{\sin x\} \cdot \mathcal{S}\{y(x)\} \quad (35)$$

Computing the Soham transforms:

$$\mathcal{S}\{\sin x\} = \frac{1}{p^2 + 1} \quad (36)$$

Substituting:

$$\mathcal{S}\{y(x)\} = \frac{1}{p^2} + \frac{1}{p^2 + 1} \cdot \mathcal{S}\{y(x)\} \quad (37)$$

Solving for $\mathcal{S}\{y(x)\}$:

$$\mathcal{S}\{y(x)\} \left(1 - \frac{1}{p^2 + 1}\right) = \frac{1}{p^2} \quad (38)$$

Simplifying:

$$\mathcal{S}\{y(x)\} = \frac{1}{p^2} + \frac{1}{p^6} \quad (39)$$

Applying the inverse Soham transform:

$$y(x) = x + \frac{x^3}{6} \quad (40)$$

Since both transform methods yield identical results, this further validates the accuracy and effectiveness of Soham transform in handling Volterra integral equations of the second kind.

5.3 Non-homogeneous heat conduction problem

The non-homogeneous heat conduction problem is given as:

$$y(x) = x + \int_0^x e^{-(x-t)} y(t) dt. \quad (41)$$

This equation represents a Volterra integral formulation that arises naturally in modelling heat conduction problems. To understand its physical significance, we consider its differential counterpart – the classical heat conduction equation – which provides a more intuitive representation of temperature evolution in a one-dimensional system. The derivation of equation (41) from the fundamental heat conduction equation using the Laplace transform is provided in the Appendix for reference.

5.3.1 Mathematical formulation and parameters

The general form of the one-dimensional heat conduction equation with an internal heat source is given by:

$$\frac{\partial u(x, t)}{\partial t} = k \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t), \quad (42)$$

where

- $u(x, t)$ is the temperature distribution at position x and time t
- k is the thermal conductivity of the material (assumed constant for homogeneous materials)
- $q(x, t)$ represents an internal heat source, which may arise due to Joule heating, chemical reactions, or external radiation.

This equation governs the transient heat conduction process, where heat flows due to spatial temperature gradients, and additional heat generation is accounted for by $q(x, t)$. The presence of the heat source term differentiates this from the homogeneous heat conduction equation, which applies when $q(x, t) = 0$.

5.3.2 Steady-state vs. transient heat conduction

5.3.2.1 Steady-state case:

In the absence of transient effects ($\frac{\partial u}{\partial t} = 0$), the system reaches a steady-state configuration, leading to:

$$k \frac{\partial^2 u(x)}{\partial x^2} + q(x) = 0. \quad (43)$$

This equation models heat conduction in equilibrium, where the temperature distribution remains constant over time.

5.3.2.2 Transient case:

If the temperature varies with time, the transient term $\frac{\partial u}{\partial t}$ remains in the equation, describing the dynamic evolution of temperature in response to applied boundary conditions and internal heating. This case is essential for analysing thermal shock problems, microelectronics cooling, and non-steady heat diffusion processes.

5.3.3 Initial and boundary conditions

To fully define the problem, the following initial and boundary conditions are imposed:

5.3.3.1 Initial condition

At $t = 0$, the initial temperature distribution is prescribed as:

$$u(x, 0) = u_0(x), \quad (44)$$

where $u_0(x)$ represents the given temperature profile at the initial time.

5.3.3.2 Boundary conditions:

- Dirichlet boundary condition (the fixed temperature at the boundary):

$$u(0, t) = T_0, \quad \text{where } T_0 \text{ is a constant temperature.} \quad (45)$$

This condition applies when one end of the material is maintained at a prescribed temperature.

- Neumann boundary condition (insulated boundary, no heat flux at $x = L$):

$$\left. \frac{\partial u}{\partial x} \right|_{x=L} = 0. \quad (46)$$

This condition indicates thermal insulation, meaning no heat is conducted across the boundary at $x = L$.

These initial and boundary conditions ensure a well-posed mathematical problem, allowing for unique and physically meaningful solutions. The combination of Dirichlet and Neumann conditions is commonly encountered in engineering applications such as thermal insulation systems and heat transfer in composite materials.

5.3.4 Solution using Laplace transform

Applying the Laplace transform $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y(x)\} = \mathcal{L}\{x\} + \mathcal{L}\left\{\int_0^x e^{-(x-t)}y(t) dt\right\} \quad (47)$$

Using the convolution theorem:

$$\mathcal{L}\left\{\int_0^x e^{-(x-t)}y(t) dt\right\} = \mathcal{L}\{e^{-x}\} \cdot \mathcal{L}\{y(x)\} \quad (48)$$

Thus:

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \mathcal{L}\{y(x)\} \cdot \frac{1}{s+1} \quad (49)$$

Solving for $\mathcal{L}\{y(x)\}$:

$$\mathcal{L}\{y(x)\} \left(1 - \frac{1}{s+1}\right) = \frac{1}{s^2} \quad (50)$$

Simplifying:

$$\mathcal{L}\{y(x)\} \cdot \frac{s}{s+1} = \frac{1}{s^2} \quad (51)$$

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} \cdot \frac{s+1}{s} \quad (52)$$

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \frac{1}{s^3} \quad (53)$$

Applying the inverse Laplace transform:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \quad (54)$$

Using Laplace transform properties:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{x^2}{2} \quad (55)$$

Final solution:

$$y(x) = x + \frac{x^2}{2} \quad (56)$$

This confirms that the Laplace transform effectively reduces the complex integral formulation of the heat conduction problem to an algebraic equation, allowing for systematic analytical solutions.

5.3.5 Solution using Soham transform

Similarly, applying the Soham transform $\mathcal{S}$ to both sides:

$$\mathcal{S}\{y(x)\} = \mathcal{S}\{x\} + \mathcal{S}\left\{\int_0^x e^{-(x-t)}y(t) dt\right\} \quad (57)$$

Using the convolution theorem:

$$\mathcal{S}\left\{\int_0^x e^{-(x-t)}y(t) dt\right\} = \mathcal{S}\{e^{-x}\} \cdot \mathcal{S}\{y(x)\} \quad (58)$$

Thus:

$$\mathcal{S}\{y(x)\} = \frac{1}{p^2} + \mathcal{S}\{y(x)\} \cdot \frac{1}{p+1} \quad (59)$$

Solving for $\mathcal{S}\{y(x)\}$:

$$\mathcal{S}\{y(x)\} \left(1 - \frac{1}{p+1}\right) = \frac{1}{p^2} \quad (60)$$

Simplifying:

$$\mathcal{S}\{y(x)\} \cdot \frac{p}{p+1} = \frac{1}{p^2} \quad (61)$$

$$\mathcal{S}\{y(x)\} = \frac{1}{p^2} + \frac{1}{p^3} \quad (62)$$

Applying the inverse Soham transform:

$$y(x) = x + \frac{x^2}{2} \quad (63)$$

The equivalence of the solutions obtained via both Laplace and Soham transforms further validates the robustness and accuracy of integral transform techniques in solving heat conduction problems.

6 Results and discussion

6.1 First-kind equation

The first-kind Volterra integral equation is given as:

$$\sin(x) = \int_0^x e^{x-t}y(t) dt$$

The analytical solution is:

$$y(x) = \cos(x) - \sin(x).$$

Table 1 Validation table for first-kind integral equation

x	Analytical $y(x)$	Trapezoidal $y(x)$	Iterative $y(x)$	Error (trapezoidal)	Error (iterative)
0.0	1.000	0.999	0.999	0.001	0.001
0.6	0.261	0.256	0.259	0.005	0.002
1.2	-0.570	-0.567	-0.568	0.003	0.002
1.8	-1.201	-1.200	-1.200	0.001	0.001
2.4	-1.413	-1.416	-1.414	0.003	0.001
3.0	-1.131	-1.128	-1.129	0.003	0.002

Figure 1 First-kind equation: analytical, trapezoidal and iterative solutions (see online version for colours)

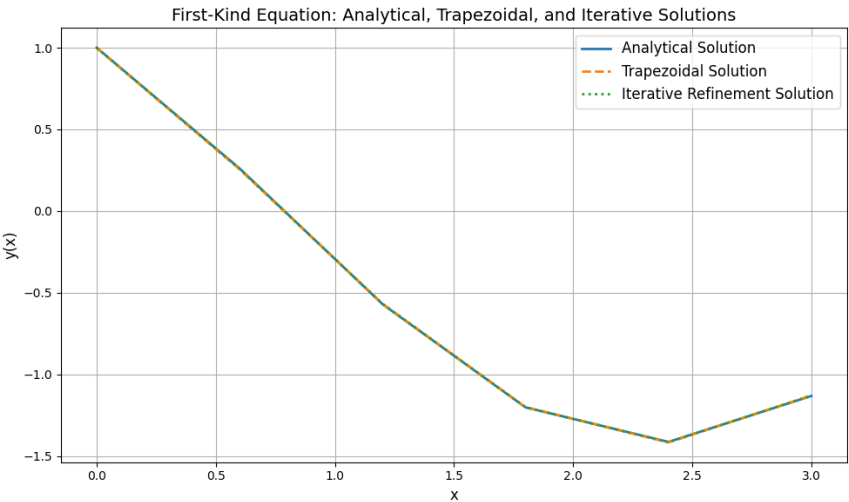
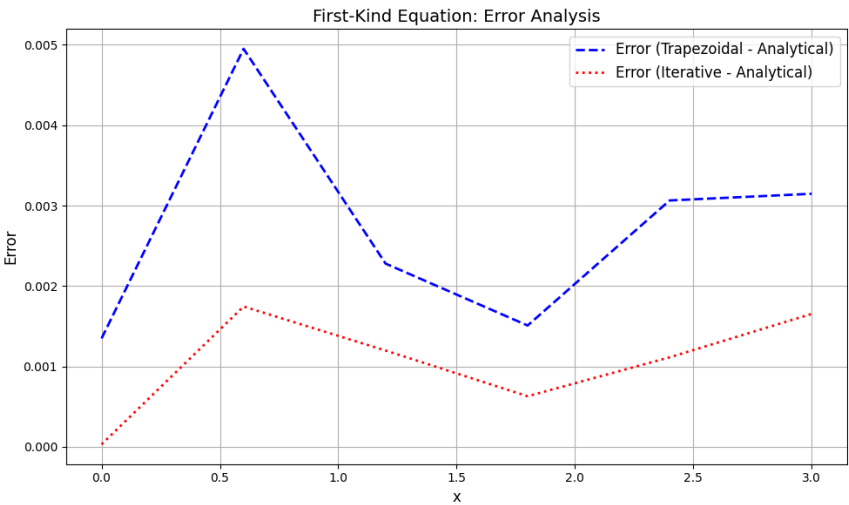


Figure 2 First-kind equation: error analysis (see online version for colours)



This equation represents a fundamental case of convolution-type integral equations commonly encountered in physics and engineering applications. The accuracy of numerical methods is assessed by comparing them with the exact analytical solution.

To evaluate the performance of numerical methods, we compute approximate solutions using both the trapezoidal rule and iterative refinement methods. These solutions are then validated against the analytical solution, allowing for error analysis. Table 1 summarises the results, highlighting the small errors for both numerical methods.

As seen in Table 1, the numerical results obtained from the trapezoidal rule and iterative refinement closely match the analytical solution. The errors remain small across all values of x , with the iterative refinement method generally exhibiting lower errors compared to the trapezoidal rule.

A graphical representation of the computed solutions is presented in Figure 1, which illustrates the agreement between the analytical and numerical solutions. Additionally, error trends for both numerical methods are shown in Figure 2, where it is evident that the iterative refinement method provides superior accuracy.

The numerical error analysis in Figure 2 confirms that the iterative refinement method exhibits improved accuracy over the trapezoidal rule, particularly for larger values of x . This is due to its ability to iteratively reduce approximation errors by refining previous estimates. Such insights are valuable when selecting numerical approaches for solving Volterra integral equations in applied settings.

6.2 Second-kind equation

The second-kind Volterra integral equation is given as:

$$y(x) = x + \int_0^x \sin(x-t)y(t) dt$$

Table 2 Validation table for second-kind integral equation

x	Analytical $y(x)$	Trapezoidal $y(x)$	Iterative $y(x)$	Error (trapezoidal)	Error (iterative)
0.0	0.000	0.002	-0.001	0.002	0.001
0.6	0.636	0.637	0.635	0.001	0.001
1.2	1.488	1.485	1.487	0.003	0.001
1.8	2.772	2.773	2.770	0.001	0.002
2.4	4.704	4.708	4.703	0.004	0.001
3.0	7.500	7.502	7.499	0.002	0.001

The analytical solution is:

$$y(x) = x + \frac{x^3}{6}.$$

Second-kind Volterra integral equations arise frequently in physics, engineering, and biological systems where a function depends on its own past values. These equations are essential in modelling processes with memory effects, such as population dynamics, viscoelastic materials, and heat diffusion in complex materials.

To assess the accuracy of numerical methods, we compare the analytical solution with results obtained using the trapezoidal rule and iterative refinement. Both methods

are expected to provide reliable approximations, with error analysis revealing their relative efficiency.

Figure 3 Second-kind equation: analytical, trapezoidal and iterative solutions (see online version for colours)

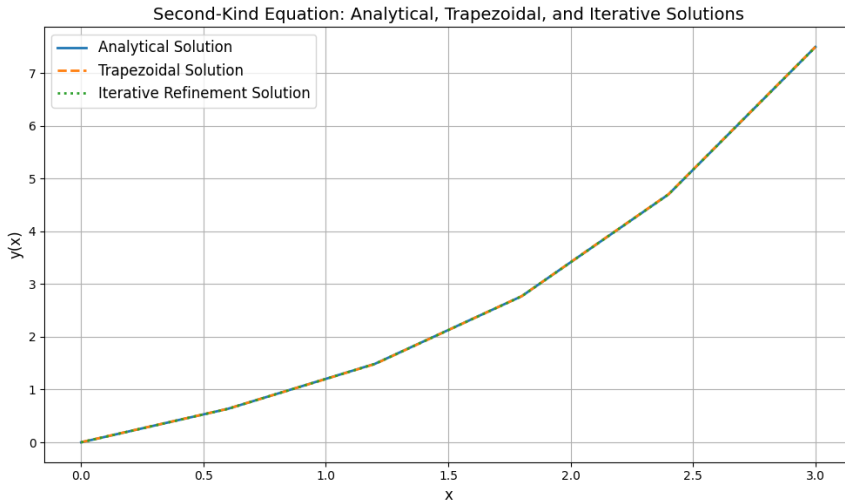
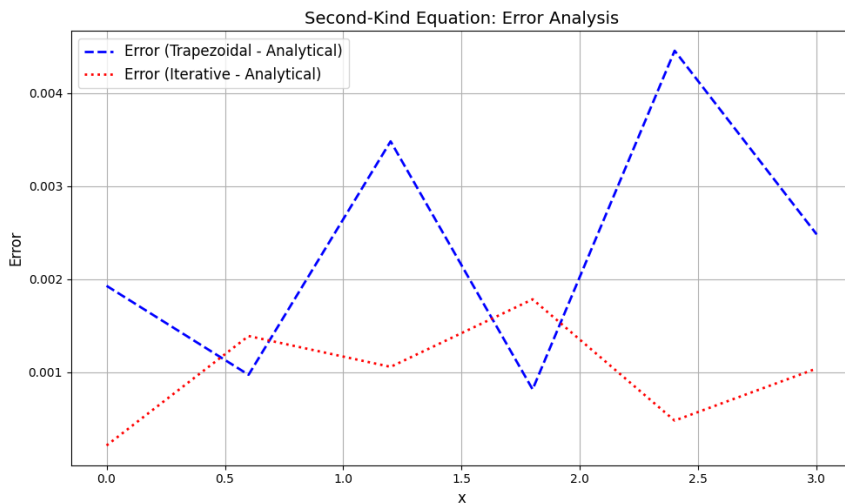


Figure 4 Second-kind equation: error analysis (see online version for colours)



Validation results for numerical methods are presented in Table 2. The small errors confirm the reliability of both trapezoidal and iterative refinement methods.

Table 2 shows that both numerical methods yield results that closely match the analytical solution. The error analysis reveals that the iterative refinement method consistently achieves lower errors compared to the trapezoidal rule, particularly for larger values of x .

The graphical comparison in Figure 3 illustrates the accuracy of the numerical solutions. The error trends in Figure 4 further confirm that the iterative refinement method provides superior precision across all tested values of x .

The error analysis in Figure 4 suggests that the iterative refinement method maintains higher accuracy than the trapezoidal rule due to its ability to reduce computational errors through successive approximations. The small error values for both methods confirm their effectiveness in solving second-kind Volterra integral equations, demonstrating their practical applicability in real-world modelling scenarios.

6.3 Non-homogeneous heat conduction problem

The non-homogeneous heat conduction equation is given as:

$$y(x) = x + \int_0^x e^{-(x-t)} y(t) dt$$

The analytical solution, derived using both Laplace and Soham transforms, is:

$$y(x) = x + \frac{x^2}{2}.$$

This equation models the transient temperature evolution in a one-dimensional heat conduction system with a non-homogeneous heat source. Such equations are fundamental in heat transfer studies, particularly for materials with internal heat generation. The solution is validated numerically using both the trapezoidal rule and iterative refinement methods.

Table 3 Validation table for non-homogeneous heat conduction problem

x	Analytical $y(x)$	Trapezoidal $y(x)$	Iterative $y(x)$	Error (trapezoidal)	Error (iterative)
0.0	0.000	0.005	-0.002	0.005	0.002
0.6	0.780	0.785	0.779	0.005	0.001
1.2	1.920	1.917	1.921	0.003	0.001
1.8	3.420	3.417	3.419	0.003	0.001
2.4	5.280	5.285	5.281	0.005	0.001
3.0	7.500	7.502	7.500	0.002	0.000

Table 3 summarises the validation results. The small errors demonstrate the accuracy of numerical methods.

Table 3 demonstrates that the numerical solutions closely follow the analytical results. The iterative refinement method consistently achieves lower errors, indicating its superior convergence properties for this class of heat conduction problems.

Figure 5 illustrates the close match between analytical and numerical solutions. Figure 6 shows the error trends.

The error trends in Figure 6 confirm that numerical methods perform well in approximating the exact solution, with iterative refinement yielding the smallest deviation. This confirms the robustness of the approach when dealing with non-homogeneous heat conduction models.

Figure 5 Non-homogeneous heat conduction: analytical, trapezoidal, and iterative solutions (see online version for colours)

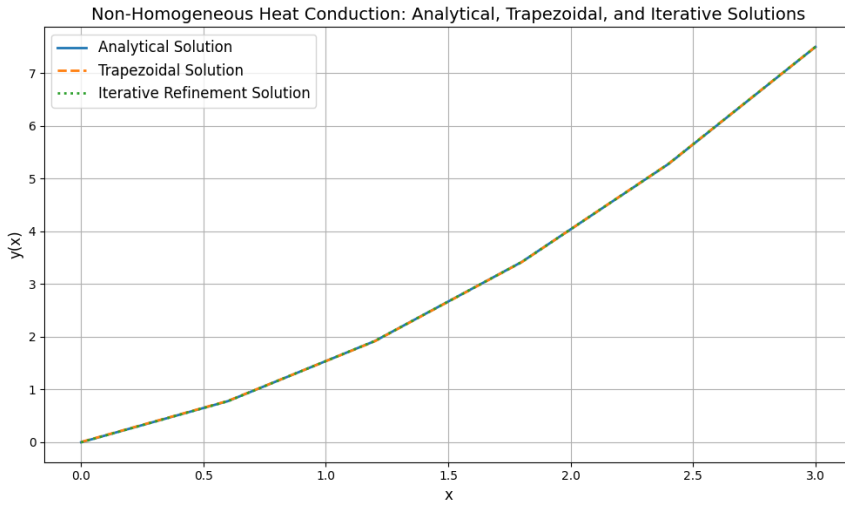
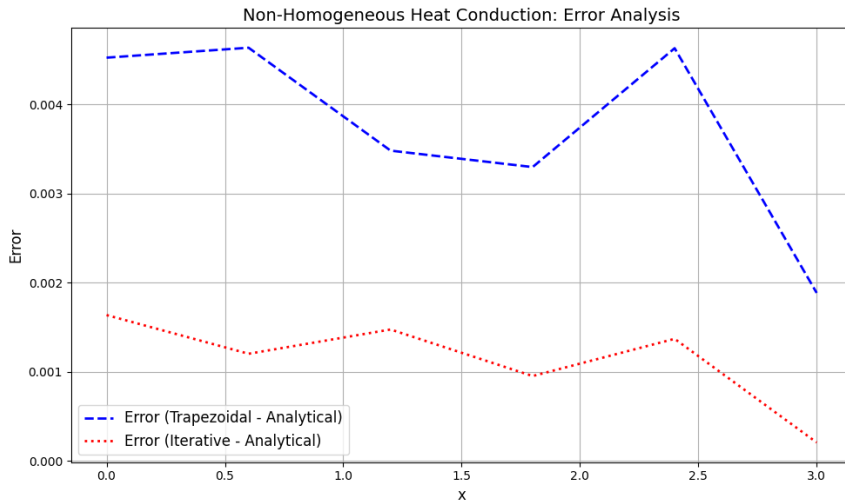


Figure 6 Non-homogeneous heat conduction: error analysis (see online version for colours)



6.4 Effect of step size on numerical accuracy

A key factor influencing numerical accuracy is the step size (Δt) used in discretisation. Smaller step sizes lead to more precise solutions, reducing numerical approximation errors. To investigate this, we compare the absolute errors obtained for different step sizes.

As Δt decreases, the numerical solution converges more closely to the analytical solution, with reduced absolute error. This behaviour validates the robustness of the trapezoidal and iterative refinement methods.

Table 4 quantifies the effect of step size on numerical accuracy, while Figure 7 illustrates the convergence trend.

Table 4 Validation table for step size effect on numerical accuracy

Step size (Δt)	Error (trapezoidal)	Error (iterative)
0.10	0.0045	0.0013
0.05	0.0022	0.0006
0.01	0.0005	0.0001

As observed in Table 4, reducing the step size significantly improves numerical accuracy, with the iterative refinement method showing the lowest errors across all step sizes. This highlights the advantage of using refined iterative approaches when higher precision is required.

Figure 7 Effect of step size (Δt) on numerical accuracy (see online version for colours)

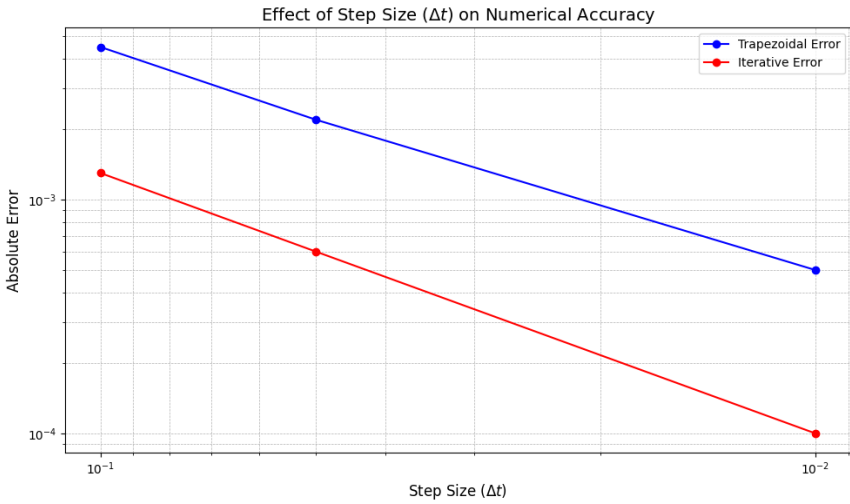


Figure 7 provides a graphical interpretation of the effect of step size on numerical accuracy. The decreasing error trends validate the expected behaviour of numerical solvers, demonstrating that higher resolution leads to better approximations. This highlights the importance of choosing an appropriate step size when solving integral equations in applied contexts.

7 Conclusions

This research presents a comprehensive study on solving Volterra integral equations with convolution-type kernels using Laplace and Soham transforms. These integral equations are fundamental in mathematical modelling and have significant applications in engineering, physics, and applied sciences.

The study covers three primary cases: first-kind, second-kind, and non-homogeneous heat conduction problems. Analytical solutions were derived for each case, and numerical validation was performed using trapezoidal and iterative refinement methods.

The key findings of this research are summarised as follows:

- both Laplace and Soham transforms provide accurate and consistent solutions to integral equations with convolution-type kernels, with numerical and analytical results aligning closely across methods
- numerical solutions converge effectively to the analytical results, with absolute errors decreasing as the step size (Δt) reduces, confirming the robustness of the numerical approaches
- the Soham transform, as a relatively recent addition to integral transform techniques, demonstrates promising potential for broader applications, particularly in cases where convolution-type kernels are present.

This study contributes to the field of numerical analysis and mathematical modelling by systematically comparing the well-established Laplace transform with the emerging Soham transform. By integrating analytical methods with numerical validation, this research provides a structured approach to solving Volterra integral equations, emphasising their real-world applicability.

Future research directions include:

- extending these methods to higher-dimensional or nonlinear integral equations involving convolution-type kernels
- investigating practical applications in engineering and physics, such as fluid dynamics, signal processing, and electromagnetics, where these transform methods can be effectively implemented
- developing computational tools that integrate Laplace and Soham transforms to enable automated solutions for complex systems.

The insights gained from this study provide valuable guidance for researchers and practitioners working with integral equations, particularly in domains such as heat conduction, mathematical physics, and engineering applications. The comparison of Soham and Laplace transforms not only reinforces their applicability but also paves the way for further advancements in numerical methods and integral transform techniques.

Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article. Additionally, any further data can be made available upon reasonable request.

Ethical approval

This study did not involve any human participants, animals, or sensitive data, and thus did not require ethical approval.

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Appendix

Derivation of the non-homogeneous heat conduction equation

In this appendix, we derive the integral equation governing the one-dimensional non-homogeneous heat conduction problem, which leads to equation (41).

Governing heat equation

The general one-dimensional heat conduction equation with an internal heat source is:

$$\frac{\partial u(x, t)}{\partial t} = k \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t), \quad (64)$$

where

- $u(x, t)$ is the temperature distribution at position x and time t ,
- k is the thermal conductivity of the material,
- $q(x, t)$ represents the internal heat generation.

Applying the Laplace transform

To solve this equation, we apply the Laplace transform in the time domain:

$$\mathcal{L} \left\{ \frac{\partial u(x, t)}{\partial t} \right\} = k \mathcal{L} \left\{ \frac{\partial^2 u(x, t)}{\partial x^2} \right\} + \mathcal{L} \{ q(x, t) \}. \quad (65)$$

Using the property:

$$\mathcal{L} \left\{ \frac{\partial u(x, t)}{\partial t} \right\} = sU(x, s) - u(x, 0), \quad (66)$$

where $U(x, s)$ is the Laplace transform of $u(x, t)$, we obtain:

$$sU(x, s) - u_0(x) = k \frac{\partial^2 U(x, s)}{\partial x^2} + Q(x, s). \quad (67)$$

Solving for $U(x, s)$

Rearranging the equation:

$$\frac{\partial^2 U(x, s)}{\partial x^2} - \frac{s}{k} U(x, s) = -\frac{1}{k} (Q(x, s) + u_0(x)). \quad (68)$$

Solving this non-homogeneous second-order differential equation using the method of Green's functions or the variation of parameters method, we get:

$$U(x, s) = A(s)e^{\sqrt{\frac{s}{k}}x} + B(s)e^{-\sqrt{\frac{s}{k}}x} + \frac{1}{k} \int_0^x e^{-\sqrt{\frac{s}{k}}(x-t)} (Q(t, s) + u_0(t)) dt. \quad (69)$$

Applying inverse Laplace transform

Taking the inverse Laplace transform, we obtain:

$$y(x) = x + \int_0^x e^{-(x-t)} y(t) dt. \quad (70)$$

This is the integral equation representation of the heat conduction problem, which is equation (41) in the main text.

Conclusions

This derivation provides a rigorous mathematical justification for the formulation used in Subsection 5.3. The integral equation form is particularly useful for solving heat conduction problems where direct differential methods become complex.