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A multi-period inventory model with price, time and service level dependent demand under preservation technology investment

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Abstract: Price and time are two important parameters having significant impact on market demand, especially for fashion items, newly launched electronic products, etc. After-sale service facility offered by the retailers is seen to boost demand while investing in preservation technology reduces product spoilage. All these issues are taken into consideration while developing a multi-period inventory model where market demand depends on all three of the above-mentioned factors. The replenishment cycles are all of equal length, but due to the time-dependent nature of demand, the stock-in (and consequently stock-out) periods in the cycles are allowed to vary. The policy of planned shortages followed by replenishment in each cycle is adopted and seen to be fruitful indeed. Learning effect in holding and ordering costs are taken into account. The effects of limited capital and warehousing space are investigated. Numerical examples are employed to demonstrate the developed model and gain managerial insights from it. [Received: 23 December 2022; Accepted: 3 August 2023]

Keywords: price dependent demand; learning effect; preservation technology; service level; multi-period inventory model.

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1 Introduction

It is about a century ago when Harris et al. (1913) first published his work on inventory management and laid out the principle of economic order quantity (EOQ). Since then intention of the researchers was to develop inventory models and bring them more and more close to reality. In this process, Whitin (1957) introduced the phenomenon ‘deterioration’ in his model. Deterioration of a product can be defined as degradation of that item such that it fails to serve its original purpose. Commodities such as foodstuffs, vegetables, etc. suffer from declination by direct spoilage while holding in stores. Highly volatile liquids such as gasoline, turpentine, etc. undergo physical depletion over time through the process called evaporation. Electronic goods, radioactive substances, photographic film, grain, etc. deteriorate through gradual loss of potential or utility with the passage of time. Deterioration is a very common and inseparable part for most of the inventories. According to the Worldcount report (<https://www.theworldcounts.com/>), roughly a third of the world’s food is wasted which is about 1.3 billion tons a year. According to CBC news, “A whopping 58% of all foods produced in Canada is lost or wasted and about a third of that wasted food could be ‘rescued’ and sent to communities in need across the country” (<https://www.cbc.ca/news/canada/toronto/food-waste-report-second-harvest-1.4981728>). The introduction of preservation technology in inventory management research by Hsu et al. (2010) has an important role in slowing down deterioration. Any kind of product, no matter how it is treated, starts deteriorating naturally over time. Though

the process is natural, sometimes excessive use of some chemicals, faulty packaging, and abnormal storage condition may also stimulate the process. For fruits, flowers or vegetables, it is seen that excessive use of urea in plants while growth stimulate the rotting process. For fruits, when carbide powder is used to ripen them, they usually possess a very short shelf life. It is seen in reality that the products do not show any noticeable deteriorating changes until a certain point of time which is referred to as 'non-deterioration period', and this period varies from product to product. A high rate of deterioration not only causes a huge inventory loss but also brings a huge disposal cost with it. As a result of it, the inventory suffers with its profit level. That is why any kind of firm management usually prefers to invest a good amount of money in preservation technology when required. Usually the preservation technology investment not only reduces the deterioration rate significantly but also delays the non-deterioration period.

Price is one of the most fundamental factors to control market demand. More and more consumers are finding it increasingly challenging to afford the goods as prices rise. There are many other factors that affect demand as well, the time being one among the most important ones. The advancement in today's technology produces more sophisticated, higher-quality and more user-friendly products which draw customers' attraction as soon as they enter into market, and prevail over its earlier versions. Examples include fashion items, electronic gadgets such as mobile phone, processors, semiconductors, etc. People also reject a product when they do not find it worth buying or not enough valued for money. Another factor that regulates the demand and usually gets unnoticed is sales service. As pointed out by Tsay and Agrawal (2000), service is taken to broadly represent all forms of demand-enhancing effort, which includes investments in brand promotion or promotion of a newly launched product emphasising its unique features, providing free samples to a section of customers, and after-sale customer service including free replacement or return, free or discounted servicing up to certain period, and the overall quality of the shopping experience. In reality, the circumstances of the mall to the service they provide matters a lot for the customers to pick up products from there.

When it comes to performing a job, it is obvious that very few in the world are as much genius to master it in one trial; it improves with experience. With each cycle, the whole system gets improved along with its services. Not only service improvement but learning effect also helps to reduce various costs of the firm such as holding and ordering costs.

The current work encompasses all the issues described above. The sensibility as well as stability of the system with optimal pricing, service and preservation technology investment are investigated numerically. A multi-period inventory system facing price, time and service dependent demand is modelled. Planned shortages are allowed and are partially backlogged. With non-instantaneous deterioration, investments in preservation technology is made to combat deterioration. Such investment not only moderates the deterioration process but also delays it. Learning effect is taken into account to reduce ordering and holding cost as well as to improve the service level. Clear advantages of having planned shortages is evidenced. The retailer tends to divide the entire time horizon into a greater number of business cycles in order to reap the benefits of the learning effect in subsequent cycles. The study may help the supermarket managers selling products where demand increases with time initially but eventually starts

reducing as the product loses its value over time. The sensitivity analysis may help the manager to adjust optimal strategies in changed business situations.

The rest of the paper is designed as follows: Section 2 provides a related literature review and positions the present work with regard to the existing literature. Section 3 presents the notations used throughout the paper and the assumptions made for developing the model. Section 4 describes the proposed model. Section 5 illustrates the model through numerical examples, provides two extensions under warehouse space and capital constraints, and derives valuable managerial insights from it. Finally, Section 6 concludes the paper indicating some future research directions.

2 Literature review

In this section, the relevant literature in the field of preservation technologies, various demand patterns, learning effect, and space and capital constraints will be reviewed, and the motivation and contribution of the present work will be explained.

2.1 Deterioration and investment in preservation

Deterioration has managed to maintain the attention of many researchers for decades after the notion was initially introduced in inventory modelling by Whitin (1957). There are a huge amount of related research works, details of which can be found in Goyal and Giri (2001) and Bakker et al. (2012). At the beginning, the deterioration process was considered to be of instantaneous type, i.e., items are subject to deterioration from the moment they are produced. Wu et al. (2006) first introduced the idea of ‘non-instantaneous’ deterioration. Hsu et al. (2010) assumed a constant cost estimate for preservation technology, which was independent of the length of the cycle. Dye and Hsieh (2012) proposed it to be dependent on the length of the replenishment cycle. At the earlier stages, it was assumed that the preservation technology investment controls only the rate of deterioration and not the non-deterioration period, which may be found in Hsieh and Dye (2013) and Mishra et al. (2017), to name a few. It was Li et al. (2019) who introduced the idea of delaying the non-deterioration period with the preservation technology investment and maximised the profit function. Bardhan et al. (2019) developed a model on deteriorating inventory with preservation technology investment and found a lower bound of t_d beyond which preservation technology investment is really propitious. Khanna et al. (2020) proposed a cost minimisation model by finding the optimal investment in preservation technology and optimal cycle length. Ahmad and Benkherouf (2020) discussed how the deterioration rate impacts the optimal policies. Mishra et al. (2021) considered controllable deterioration in production inventory and developed their work to find out an optimal production policy. Mahata and Debnath (2022) worked on maximising the profit level considering the deterioration to be an increasing function of preservation technology investment.

2.2 Demand patterns

For a profit maximisation problem, the demand is one of the most important factors. Yang et al. (2015) worked on preservation technology with time and credit length

dependent demand. Liu et al. (2015) considered the demand pattern to depend on both price and product quality. Shah et al. (2016) considered time and price sensitive demand. Mishra et al. (2017) discussed how to address price and stock dependent demand under controllable lead time in presence of investment in preservation. Shah et al. (2019) extended the work of Zhang et al. (2016) and considered the demand to be dependent on price, time, service and inventory level. Chen et al. (2019) determined the optimal price and cycle time based on price and time-dependent demand. Khan et al. (2019) developed an inventory model that takes into account price-dependent demand, and identified the optimal price, cycle time, and replenishment amount to maximise overall profit. Li et al. (2019) worked on price dependent demand. Singh (2019) assumed the demand to be price, stock and time dependent. Bardhan et al. (2019) worked on inventory level dependent demand. Hanukov et al. (2021) worked on a multi-server queuing model with perishability effect, and provided two case studies to establish the applicability of their model in real-life with the example of a bike store and a pizza store while considering price and stock dependent demand. Pal et al. (2021a, 2021b) developed two-echelon supply chain models facing quality and promotional effort dependent demand. Xie et al. (2021) considered inventory control problems for deteriorating items assuming maximum serviceable lifetime under mixed sales situation, considering both demand and deterioration rates to be time-dependent. Bajegani and Gholamian (2022) studied an inventory control model and determined the optimal number of replenishment and the optimal price for obsolete products whose demand is dependent on both time and price.

2.3 *Learning effect*

Learning may be defined as an improvement in quality or quantity over time without any change in investment (Wright, 1936). Jaber et al. (2008) assumed 'S' shaped learning curve that reduces number of defective items per lot. Khan et al. (2010) constructed an inventory model with a screening rate in which the production cost was determined by the effect of learning. Konstantaras et al. (2012) developed an inventory model allowing shortages and applying learning in the screening process. Kumar et al. (2015) used learning effect that reduced ordering cost. Aggarwal et al. (2017) developed a perishable inventory model that allows for shortages with partial backlog and used learning to reduce holding costs. Shah and Naik (2018) extended the learning application to reduce both ordering and holding cost for the inventory. Afshari et al. (2019) studied a supply chain model to investigate the effect of learning and forgetting on the feasibility of adopting additive manufacturing. Jayaswal et al. (2019) discussed an EOQ model with perishable items considering learning to reduce holding and ordering costs as well as to screen out defective items, whereas Jayaswal et al. (2021b) considered demand to depend on credit period and learning effect for deterioration cost. Jayaswal et al. (2021a) discussed the effect of learning and credit financing on lot size and the corresponding costs in a fuzzy environment. Considering learning effect and carbon emission, Alamri et al. (2022) developed an EOQ model under inflationary circumstances. Ye et al. (2022) analysed a PC supply chain and applied demand learning to find out the optimal ordering quantity and established integrated decision-making model for dynamic pricing and inventory control. Yadav et al. (2022) applied learning-forgetting phenomena on setup cost in a two-echelon supply chain and found out the ideal order quantity, lead time, reorder point, and shipment quantity values that maximise the profit level.

2.4 Capital and space constraints

It is impractical to assume that companies may store as much as they want, or they may spend as much as needed as initial investments. Zhang et al. (2016) and Shah et al. (2019) considered service-dependent demand with preservation technology investment under capital constraint. Emtehani et al. (2021) established the superiority of the coordinated model in supply chain taking into account the capital constraint, and formed a bi-objective optimisation problem that simultaneously optimises system costs as well as service level. Bhuniya et al. (2021) worked on cost minimisation problem with fuzzy demand and service level constraint. Zhi et al. (2022) examined the effect of in-transit inventory financing on a logistic supply chain considering capital constraint.

Figure 1 Changes in demand pattern with time (see online version for colours)

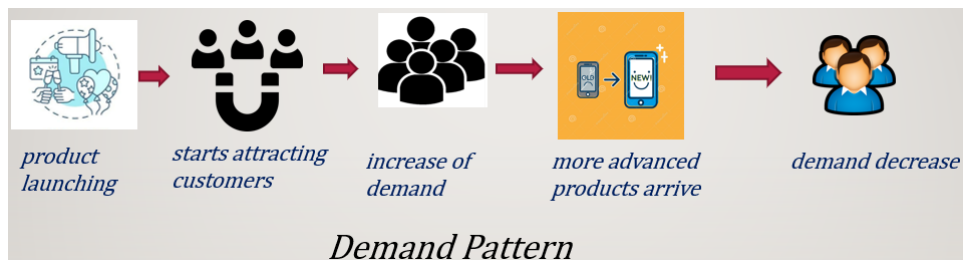


Figure 2 Improvement of service level with learning effect and demand over time (see online version for colours)



Huang (2006) extended the model of Teng (2002) with the consideration of space limitation. Alimardani et al. (2013) studied a three-echelon supply chain with a continuous review policy, identical retailers, and a limited central warehouse. Ghosh et al. (2015) developed a perishable inventory model with stock-dependent demand considering limited warehouse space. Hoseini Shekarabi et al. (2019) modelled a multi-product, multi-wholesaler, multi-level, and integrated supply chain under shortage and limited warehouse space. Agustiani et al. (2021) developed a single-retailer multi-buyer single item supply chain model with capital, service and space constraint and analysed the model to minimise the joint total expected cost. Silitonga et al. (2022) developed a multi-item EOQ model with perishable items that permits payment deferral, and extended it by taking into account space constraints.

2.4.1 *Motivation of the current work*

While reviewing the existing literature, the necessity of developing and studying a model that represents newly launched products – especially edible and electronic goods which can be bought in supermarkets – was felt. Demand of such products admits typical characteristics: the demand rate increases with time at initial stages during launching of the product, but decreases afterwards when newer models with improved features become available. Figure 1 depicts such a situation. Such a demand pattern is best represented by a time-quadratic pattern. More and more people are keen on buying things at a discount. Before and after sales services are inseparable issue associated with sales policy of the companies now-a-days to sustain in the market. Such servicing usually improves with time through expertise in customer handling, analysing and implementing the feedback from customers, updated training schedules, etc. Learning effect encompasses all these aspects which is pertinent while dealing with such products. Customers frequently find a product to be reputable or trustworthy if it includes after-sale support because they are not familiar with the technical aspects of the products. Customers become interested in the product through pre-sale promotions. Figure 2 depicts how business organisations improve their service to generate more demand. Aiming at considering all the important parameters associated with a newly launched product, the authors consider the demand pattern to be dependent on time, price, and service level. A part of the demand pattern is similar to Shah et al. (2016); however, unlike them the authors in the present paper consider the price dependency part to be linear, and the consideration of dependency on service level is surely an extension. This extension allows our model to represent the business dynamics of a wide range of products such as fashion goods, high-tech accessories, electronic equipment, for which preservation is necessary for smooth performance or quality control of the product as well as extracting benefit from the items by selling them rather than salvaging, and time-dependency part of the demand of which admits an ‘inverted U’ shape. It is clear that businesses let advance bookings even before a product is introduced, therefore it could be worthwhile to think about the business cycle beginning with a planned shortage and ending with a replenishment at some point in the cycle. Our approach also represents the idea of a multi-periodic model, in contrast to their single periodic one, as it might not be economical to accumulate all the things at the beginning and let them deteriorate throughout the entire business cycle. Further, the time varying demand may compel the retailer to align to different strategies in terms of replenishment timing during later part of the business cycle. In such a scenario where jobs are repeated, the effect of learning in consecutive periods indeed helps to reduce cost. Furthermore, service level improves in every cycle with learning, resulting in stimulating demand in consecutive cycles, thus fighting against the negative effect of price or time to some extent. With an increasing demand the stock-in time also varies and so does the ordering quantity. To the best of our knowledge, this is the first effort towards optimising a multi-period inventory pricing, service and preservation investment problem with learning effect and constraints, thus securing a position in the existing literature. Two possible realistic extension of the present model considering space and capital constraints have also been provided to identify how the manager should realign his strategies under such scenarios. Since there is a limited amount of shelf space in supermarkets for a particular type of product, consideration of space limitation is worth studying.

3 Notations and assumptions

The notations in Table 1 are used throughout the paper.

Table 1 Notations

H	Time horizon (months)
n	Number of replenishment periods (a decision variable)
T	Length of one cycle, $H = nT$ (months)
p	Unit selling price (\$/unit, a decision variable)
s_0	Service infrastructure at initial level, $0 \leq s_0 \leq 1$, depending directly on service investment (ratings, a decision variable)
s_i	Service level at i^{th} cycle, $i = 1, 2, \dots, n$, ($0 \leq s_i \leq 1$)
t	Time
$D(p, s_i, t)$	Demand rate during i^{th} replenishment period, $i = 1, 2, \dots, n$
c_1	Purchasing cost per unit (\$/ unit)
ξ	Preservation technology investment per unit time (\$/unit, a decision variable)
$t_d(\xi)$	Non-deterioration period at preservation technology investment ξ (months)
t_i	Replenishment time point in i^{th} cycle, $i = 1, 2, \dots, n$ (decision variable) (months)
$B(x)$	Backlogging rate, where x is the waiting time to the next replenishment
θ	Constant deterioration rate per unit time, $0 \leq \theta < 1$
$m(\xi)$	Proportion of reduced deterioration rate with preservation investment, $0 \leq m(\xi) < 1$
$I(t)$	Inventory level at any time t , $t \in [0, nT]$
Q_i	Ordering quantity at the i^{th} replenishment, $i = 1, 2, \dots, n$
A_1	Constant component of the ordering cost (\$/order)
$\frac{A_2}{i^{\alpha_1}}$	Variable component of the ordering cost, decreasing in each cycle due to learning effect (\$/order)
h_1	Constant component of holding cost (\$/unit/month)
$\frac{h_2}{i^{\alpha_2}}$	Variable component of holding cost which decreases in each cycle due to learning effect (\$/unit/month)
c_s	Shortage cost (\$/unit)
c_d	Disposal cost of deteriorated items (\$/unit)
α	Operating efficiency of the staff
B_i	Total sold item in i^{th} period, $i = 1, 2, \dots, n$
S_i	Total backlog in i^{th} period, $i = 1, 2, \dots, n$
k	Parameter associated with the cost per unit service level

The following assumptions are made for developing the proposed model:

- The horizon H is divided into n equal shipment intervals, each being of time length T . Each interval starts with planned shortages; the backlog is mitigated and storage is filled at time points t_i , $i = 1, 2, \dots, n$.

- Khanra et al. (2011) and Shah et al. (2016) assumed quadratic time dependent demand and price dependent demand, respectively and Sarkar and Pal (2022) considered demand to depend on service. Li et al. (2019) developed their model assuming the demand pattern to be price dependent. Combining them together, the demand rate in the i^{th} interval is assumed to be of the form $D(p, s_i, t) = a + bt - ct^2 - \beta p + \delta s_i$, which is a function of time, selling price and service level; $a > 0$ denotes the total market potential demand, $b > 0$ and $c > 0$ denote respectively the linear and quadratic rates of change of demand with respect to time. The parameter-values indicate that the demand increases with time at initial levels, but subsequently falls down. The authors impose $a + bH - cH^2 > 0$ to ensure that business does not collapse within its horizon. It is worth mentioning that demand eventually reaches zero at some point of time, indicating that the product has become completely obsolete or useless thereafter. The parameters $\beta > 0$ and $\delta > 0$ indicate sensitivity towards price and service level, respectively.
- Following Li et al. (2019), this paper assumes that the preservation technology investment not only mitigates the deterioration rate but also delays deterioration. The inventory system involves single non-instantaneous deteriorating items. $t_d(\xi)$ is assumed to be an increasing function of preservation technology investment ξ . There is no repair or replacement of the deteriorated inventory. The authors use $t_d(\xi)$ and t_d interchangeably, when no confusion arises.
- As is seen, a higher investment in preservation ensures more reduction in deterioration rate. However, deterioration can not be completely eliminated in any way, i.e., no finite amount of investment is able to wipe out the effect of deterioration completely. Hence, the relationship between preservation investment and reduction in deterioration has to be such that when the investment tends to be very large, the deterioration rate tends to be very small, near zero. A concave pattern of the reduction function justifies this behaviour. Further, the concave pattern ensures the existence of an optimal level of preservation investment. Hence, the proportion of reduced deterioration rate $m(\xi)$ is assumed to be a continuous, concave, increasing function of preservation investment ξ , with $m(0) = 0$ and $\lim_{\xi \rightarrow \infty} m(\xi) = 1$.
- The warehouse is of infinite capacity, and lead time is constant, so that without any loss of generality it is assumed to be zero. If the lead time is not zero, the retailer just needs to place an order in advance by a period of the lead time and the result holds.
- Shortages are allowed and partially backlogged. The fraction of shortages back-ordered is a decreasing function $B(x)$, where x is the waiting time up to the next replenishment, $0 \leq B(x) \leq 1$ with $B(0) = 1$. To guarantee the existence of an optimal solution, it is assumed that $B(x) + HB(x) \geq 0$, where $B(x)$ is the first derivative of $B(x)$. Following Pal et al. (2017) and Pal and Adhikari (2019), in this paper, a specific form is considered to be $B(t_i - t) = e^{-\eta(t_i - t)}$.
- In an ever changing business world, no matter how much one learn about service management that can never be considered as complete learning. The market, customers and even the concept of ‘best service’ is always transforming. But with time, one get accustomed to those changes somewhat. Keeping all the arguments

into consideration the service level at i^{th} cycle is defined as $s_i = (s_0)^{(\alpha+1)/i}$ (Jaber, 2006), α being a parameter. Learning effect also helps to reduce the ordering and holding costs significantly.

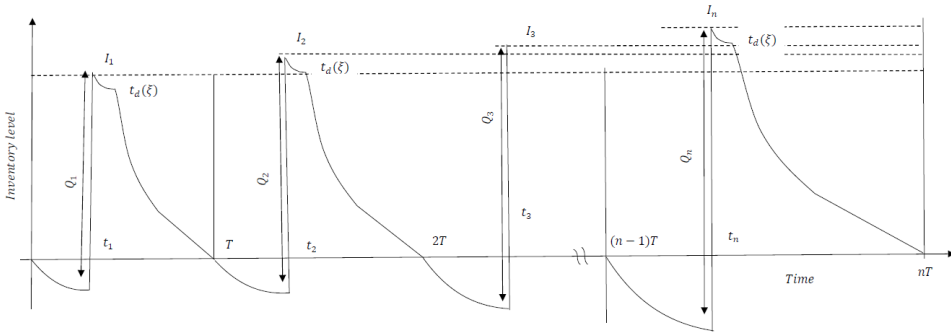
- The model is further extended in Subsections 5.2 and 5.3 considering two constraints – the supply chain manager's capital resources and warehouse space limits. With limited resources, the firm has to make joint investment decisions with respect to the preservation technology and service. It is assumed that the resource capacity is restricted to R . Further, with limited space W in warehouse, the retailer should optimise his decisions so that inventory level never exceeds W .

4 Model formulation

We now develop the proposed model in accordance with the assumptions stated in the previous section. A schematic diagram of the inventory system is depicted in Figure 3. The changes in inventory level at any time point t during the interval $[(i-1)T, iT]$ is given by

$$\frac{dI(t)}{dt} = \begin{cases} -e^{-\eta(t_i-t)}D(p, s_i, t) \text{ with } I((i-1)T) = 0 \\ \text{and } I(t_i) = -S_i, & (i-1)T \leq t \leq t_i \\ -D(p, s_i, t) \text{ with } I(t_i) = Q_i - S_i, & t_i < t \leq t_i + t_d \\ -D(p, s_i, t) - (1 - m(\xi))\theta I(t) & t_i + t_d < t \leq iT. \\ \text{with } I(iT) = 0, \end{cases} \quad (1)$$

Figure 3 Schematic diagram of the inventory system



Graphical Representation of the Inventory System

Solving the equations with the given initial and continuity conditions on inventory and the fact that Q_i amount is replenished at time point t_i , the inventory level during the entire period may be obtained as

$$I(t) = \begin{cases} -\frac{e^{-\eta(t_i-t)}}{\eta} \left\{ a + bt - ct^2 - \beta p + \delta s_i - \frac{(b-2ct)}{\eta} - \frac{2c}{\eta^2} \right\} + \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \\ \times \left\{ a + b(i-1)T - c((i-1)T)^2 - \beta p + \delta s_i - \frac{b-2c(i-1)T}{\eta} - \frac{2c}{\eta^2} \right\}, & (i-1)T \leq t \leq t_i, \\ Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i \\ + \delta s_i t_i - at - \frac{bt^2}{2} + \frac{ct^3}{3} + \beta pt - \delta s_i t, & t_i < t \leq (t_i + t_d), \\ \left(Q_i - S_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - at_d - \frac{b(t_i+t_d)^2}{2} + \frac{c(t_i+t_d)^3}{3} + \beta pt_d - \delta s_i t_d \right. \\ + \frac{(a+b(t_i+t_d)-c(t_i+t_d)^2-\beta p+\delta s_i)}{(1-m)\theta} \\ \left. - \frac{(b-2c(t_i+t_d))}{(1-m)^2\theta^2} - \frac{2c}{(1-m)^3\theta^3} \right) e^{-(1-m)\theta(t-(t_i+t_d))} \\ - \frac{(a+bt-ct^2-\beta p+\delta s_i)}{(1-m)\theta} + \frac{(b-2ct)}{(1-m)^2\theta^2} + \frac{2c}{(1-m)^3\theta^3}, & t_i + t_d < t \leq iT. \end{cases} \quad (2)$$

Detailed proof is given in Appendix A. We further derive the following obvious relations:

$$S_i = \frac{1}{\eta} \left\{ a + bt_i - ct_i^2 - \beta p + \delta s_i - \frac{(b-2ct_i)}{\eta} - \frac{2c}{\eta^2} \right\} - \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \\ \times \left\{ a + b(i-1)T - c((i-1)T)^2 - \beta p + \delta s_i - \frac{b-2c(i-1)T}{\eta} - \frac{2c}{\eta^2} \right\}, \quad (3)$$

for $i = 1, 2, \dots, n$, and from $I(iT) = 0$ and $H = nT$,

$$Q_i = S_i - \frac{bt_i^2}{2} + \frac{ct_i^3}{3} + at_d + \frac{b(t_i+t_d)^2}{2} - \frac{c(t_i+t_d)^3}{3} - \beta pt_d + \delta s_i t_d \\ + \frac{2c}{(1-m)^3\theta^3} - \frac{(a+b(t_i+t_d)-c(t_i+t_d)^2-\beta p+\delta s_i)}{(1-m)\theta} \\ + \frac{(b-2c(t_i+t_d))}{(1-m)^2\theta^2} + \left(\frac{(a+bt_i-ct_i^2-\beta p+\delta s_i)}{(1-m)\theta} \right. \\ \left. - \frac{(b-2ct_i)}{(1-m)^2\theta^2} - \frac{2c}{(1-m)^3\theta^3} \right) e^{(1-m)\theta(iT-(t_i+t_d))}. \quad (4)$$

Now we derive the cost function in each interval. The total number of items sold during the period (t_i, iT) is given by $B_i = \int_{t_i}^{iT} D(p, s_i, t) dt = a(iT - t_i) + \frac{b}{2}(iT^2 - t_i^2) - \frac{c}{3}(iT^3 - t_i^3) - \beta p(iT - t_i) + \delta s_i(iT - t_i)$, which, combined with total backlog S_i generates total revenue (SR_i) in the i^{th} interval as $p(S_i + B_i)$.

The ordering cost is $(OC_i) = (A_1 + \frac{A_2}{i^{\alpha_1}})$.

The holding cost in each interval is given by

$$(HC_i) = \left(h_1 + \frac{h_2}{i^{\alpha_2}} \right) \int_{t_i}^{iT} I(t) dt \\ = \left(h_1 + \frac{h_2}{i^{\alpha_2}} \right) \left[\left(Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i + \delta s_i t_i \right) t_d \right.$$

$$\begin{aligned}
& - \left(\frac{a((t_i + t_d)^2 - t_i^2)}{2} + \frac{b((t_i + t_d)^3 - t_i^3)}{6} - \frac{c((t_i + t_d)^4 - t_i^4)}{12} \right. \\
& - \frac{\beta p((t_i + t_d)^2 - t_i^2)}{2} + \frac{\delta s_i((t_i + t_d)^2 - t_i^2)}{2} \left. \right) + \left(Q_i - S_i + \frac{bt_i^2}{2} \right. \\
& - \frac{ct_i^3}{3} - at_d - \frac{b(t_i + t_d)^2}{2} + \frac{c(t_i + t_d)^3}{3} + \beta p t_d - \delta s_i t_d \\
& - \frac{(b - 2c(t_i + t_d))}{(1 - m)^2 \theta^2} + \frac{(a + b(t_i + t_d) - c(t_i + t_d)^2 - \beta p + \delta s_i)}{(1 - m)\theta} \\
& - \frac{2c}{(1 - m)^3 \theta^3} \left. \right) \left(\frac{1 - e^{-(1-m)\theta(iT - (t_i + t_d))}}{(1 - m)\theta} \right) - \frac{a(iT - (t_i + t_d))}{(1 - m)\theta} \\
& - \frac{b(i^2 T^2 - (t_i + t_d)^2)}{2(1 - m)\theta} + \frac{c(i^3 T^3 - (t_i + t_d)^3)}{3(1 - m)\theta} + \frac{\beta p(iT - (t_i + t_d))}{(1 - m)\theta} \\
& - \frac{\delta s_i(iT - (t_i + t_d))}{(1 - m)\theta} + \frac{b(iT - (t_i + t_d))}{(1 - m)^2 \theta^2} - \frac{c(i^2 T^2 - (t_i + t_d)^2)}{(1 - m)^2 \theta^2} \\
& + \frac{2c}{(1 - m)^3 \theta^3} (iT - (t_i + t_d)) \left. \right].
\end{aligned}$$

The costs related to preservation technology investment ((PTC_i)) and purchase ((PC_i)) are $\xi(iT - t_i)$ and $c_1 Q_i$, respectively.

The deterioration cost ((DC_i)) may be obtained as

$$\begin{aligned}
& (c_d)(1 - m)\theta \left[\left(Q_i - S_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - at_d - \frac{b(t_i + t_d)^2}{2} + \frac{c(t_i + t_d)^3}{3} \right. \right. \\
& + \beta p t_d - \delta s_i t_d - \frac{2c}{(1 - m)^3 \theta^3} + \frac{(a + b(t_i + t_d) - c(t_i + t_d)^2 - \beta p + \delta s_i)}{(1 - m)\theta} \\
& - \frac{(b - 2c(t_i + t_d))}{(1 - m)^2 \theta^2} \left. \right) \left(\frac{1 - e^{-(1-m)\theta(iT - t_i - t_d)}}{(1 - m)\theta} \right) - \frac{a(iT - (t_i + t_d))}{(1 - m)\theta} \\
& - \frac{b(i^2 T^2 - (t_i + t_d)^2)}{2(1 - m)\theta} + \frac{c(i^3 T^3 - (t_i + t_d)^3)}{3(1 - m)\theta} + \frac{\beta p(iT - (t_i + t_d))}{(1 - m)\theta} \\
& - \frac{\delta s_i(iT - (t_i + t_d))}{(1 - m)\theta} + \frac{b(iT - (t_i + t_d))}{(1 - m)^2 \theta^2} - \frac{c(i^2 T^2 - (t_i + t_d)^2)}{(1 - m)^2 \theta^2} \\
& + \frac{2c}{(1 - m)^3 \theta^3} (iT - (t_i + t_d)) \left. \right].
\end{aligned}$$

The shortage cost is derived as

$$\begin{aligned}
SC_i = c_s \left\{ a(t_i - (i - 1)T) + \frac{b(t_i^2 - ((i - 1)T)^2)}{2} - \frac{c(t_i^3 - ((i - 1)T)^3)}{3} \right. \\
\left. - \beta p(t_i - (i - 1)T) + \delta s_i(t_i - (i - 1)T) - S_i \right\}.
\end{aligned}$$

Noting that the service cost, i.e., investment in maintaining service level is an one time investment for the entire business period, and the service level will automatically be improved as the effect of learning, the total profit during the entire time horizon is given by

$$\Pi = \sum_{i=1}^n \left(SR_i - (OC_i + HC_i + PTC_i + PC_i + DC_i + SC_i) \right) - \frac{ks_0^2}{2}. \quad (5)$$

The aim is now to maximise the total profit with respect to the decision variables p , ξ , n , s_0 , and t_i , $i = 1, 2, \dots, n$; n being a positive integer, subject to the set of equations (3) and (4), $0 \leq t_i < iT$ and $t_d < iT - t_i$, $i = 1, 2, \dots, n$. For each n , the necessary and sufficient condition for existence of optimal maxima is that the Hessian matrix

$$H = \begin{vmatrix} \frac{\partial^2 \Pi}{\partial t_1^2} & \frac{\partial^2 \Pi}{\partial t_1 \partial t_2} & \cdots & \frac{\partial^2 \Pi}{\partial t_1 \partial t_n} & \frac{\partial^2 \Pi}{\partial t_1 \partial p} & \frac{\partial^2 \Pi}{\partial t_1 \partial s_0} & \frac{\partial^2 \Pi}{\partial t_1 \partial \xi} \\ \frac{\partial^2 \Pi}{\partial t_1 \partial t_2} & \frac{\partial^2 \Pi}{\partial t_2^2} & \cdots & \frac{\partial^2 \Pi}{\partial t_2 \partial t_n} & \frac{\partial^2 \Pi}{\partial t_2 \partial p} & \frac{\partial^2 \Pi}{\partial t_2 \partial s_0} & \frac{\partial^2 \Pi}{\partial t_2 \partial \xi} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 \Pi}{\partial t_1 \partial t_n} & \frac{\partial^2 \Pi}{\partial t_1 \partial t_n} & \cdots & \frac{\partial^2 \Pi}{\partial t_n^2} & \frac{\partial^2 \Pi}{\partial t_n \partial p} & \frac{\partial^2 \Pi}{\partial t_n \partial s_0} & \frac{\partial^2 \Pi}{\partial t_n \partial \xi} \\ \frac{\partial^2 \Pi}{\partial t_1 \partial p} & \frac{\partial^2 \Pi}{\partial t_2 \partial p} & \cdots & \frac{\partial^2 \Pi}{\partial t_n \partial p} & \frac{\partial^2 \Pi}{\partial p^2} & \frac{\partial^2 \Pi}{\partial s_0 \partial p} & \frac{\partial^2 \Pi}{\partial p \partial \xi} \\ \frac{\partial^2 \Pi}{\partial t_1 \partial s_0} & \frac{\partial^2 \Pi}{\partial t_2 \partial s_0} & \cdots & \frac{\partial^2 \Pi}{\partial t_n \partial s_0} & \frac{\partial^2 \Pi}{\partial p \partial s_0} & \frac{\partial^2 \Pi}{\partial s_0^2} & \frac{\partial^2 \Pi}{\partial s_0 \partial \xi} \\ \frac{\partial^2 \Pi}{\partial t_1 \partial \xi} & \frac{\partial^2 \Pi}{\partial t_2 \partial \xi} & \cdots & \frac{\partial^2 \Pi}{\partial t_n \partial \xi} & \frac{\partial^2 \Pi}{\partial p \partial \xi} & \frac{\partial^2 \Pi}{\partial s_0 \partial \xi} & \frac{\partial^2 \Pi}{\partial \xi^2} \end{vmatrix}$$

should have the following properties: $|H_1| < 0$, $|H_2| > 0$, ..., $(-1)^{n+3}|H_{n+3}| > 0$, where $|H_i|$ denotes i^{th} order minor. Although we are unable to derive the optimal conditions explicitly due to the profit function's complex form, we are able to deduce the concavity criterion justifying $|H_1| < 0$ in the following theorem.

Theorem 1: The total profit function is concave in p for each n .

The proof is provided in Appendix B. In order to get managerial insight from it, the subsequent analytical observations are combined in the form of a proposition.

Proposition 1:

- a $\frac{\partial S_i}{\partial a} > 0$, $\frac{\partial Q_i}{\partial a} > 0$, $\frac{\partial S_i}{\partial \beta} < 0$, $\frac{\partial Q_i}{\partial \beta} < 0$, $\frac{\partial S_i}{\partial \delta} > 0$, $\frac{\partial Q_i}{\partial \delta} > 0$, $\frac{\partial S_i}{\partial c} < 0$, $\frac{\partial Q_i}{\partial c} < 0$
- b if $t_i - (i-1)T > \frac{1}{1+\eta(i-1)T}$, $\frac{\partial S_i}{\partial b} > 0$ and $\frac{\partial Q_i}{\partial b} > 0$
- c there is a time point t^* such that $\frac{\partial S_i}{\partial t_i} > 0$ or < 0 according as $t_i < 0$ or $> t^*$.

We refer the readers to Appendix C for proof. The proposition provides some valuable managerial insights as given below.

- The demand parameters have mostly similar effect on total backlog and order quantities in each interval. A higher base demand or higher sensitivity/dependency towards service level naturally stimulates more demand, so that both the backlog amount as well as order quantity would be higher in each interval. The price sensitivity parameter has exactly the opposite effect. A higher price sensitivity compels the manager to reduce price to some extent, thereby generating lower

per unit profit and consequently total profit too. To reduce this effective loss, the manager should reduce business volume, resulting in mitigating lesser demand. In a similar manner, when the demand is highly sensitive with respect to the quadratic-time-parameter (i.e., it attains higher values), the manager should reduce both backlog amount and order quantity in each interval, since higher values of that parameter results in faster reduction in demand as time moves on.

- The cases with time dependency parameters are not so straightforward. It is derived that an increment in linear time sensitivity parameter b compels to fulfill more backlog and higher demand only if the shortage period $t_i - (i - 1)T$ remains higher than a pre-calculated value. On the other hand, if the shortage period is sufficiently small, the effect of time during that small period in generating demand is also negligible, so that it practically does not affect the demand. In that case, a lesser amount of backlog is piled up due to smaller shortage period, so the manager has lesser backlog to fulfill and subsequently lesser order quantity to order. Since that pre-calculated lower limit as mentioned above is decreasing in i , we may conclude that in later intervals (having higher values of i), higher b generates more backlog and replenishment. However, it is further observed that there are values of b for which $\frac{\partial Q_i}{\partial b} > 0$ even if $\frac{\partial S_i}{\partial b} < 0$, so it may be concluded that there are situations in later intervals where higher linear price dependency results in lower backlog but higher replenishment, mostly due to higher demand generated during later periods (for higher values of t).
- The backlogged amount increases during initial consecutive intervals, and thereafter decreases, the reason being that the demand increases with time during initial periods, but the trend is reversed subsequently. A lower demand at retailer would surely have a negative impact on storage and on overall business performance.

5 Numerical illustration

5.1 The unconstrained model

We now examine the developed model using a numerical example. The parameter-values are set as follows: $a = 500$, $b = 5$, $c = 0.001$, $\alpha_1 = 2$, $\alpha_2 = 2$, $h_1 = 3$, $h_2 = 1$, $A_1 = 3,000$, $A_2 = 100$, $\theta = 0.05$, $c_s = 7$, $\delta = 100$, $k = 1$, $\beta = 10$, $\eta = 0.2$, $H = 10$, $c_1 = 10$, $c_d = 3$, $\alpha = 2$, $\omega = 3$, $m(\xi) = 1 - e^{-\gamma\xi}$ with $\gamma = 0.1$, $t_d(\xi) = t_0 + \omega(1 - e^{-\psi\xi})$ with $t_0 = 0$, $\psi = 0.005$, and $\omega = 2$, in appropriate units. Since the concavity of the profit function with respect to T (or n) could not be established analytically, a line search method is applied on n to find the optimal decisions. The model is solved by built-in multi-objective optimisation function of genetic algorithm in MATLAB 2018b. Table 2 establishes that the model indeed attains its maxima for some value of n .

Table 2 Optimal profit for different values of n

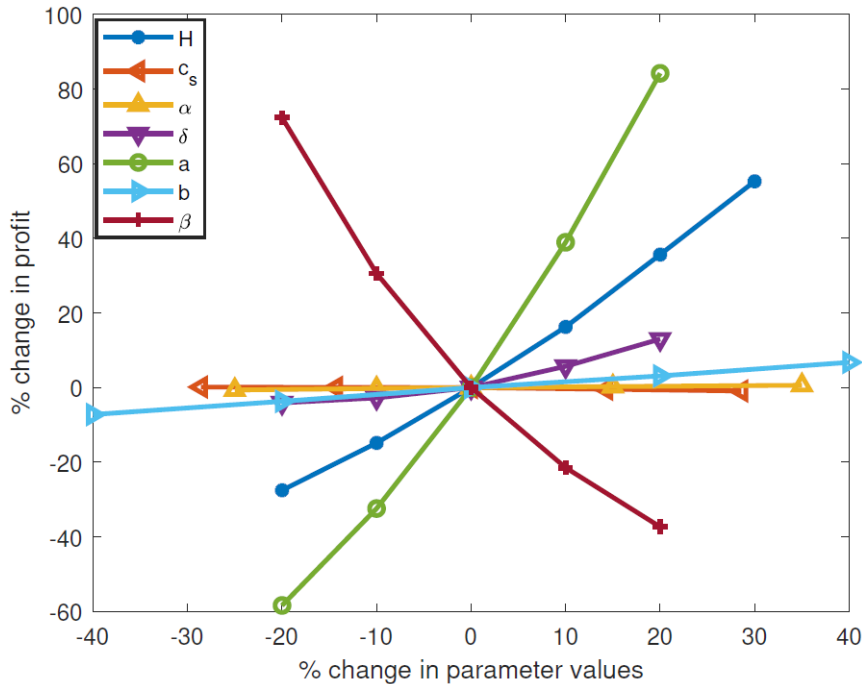
n	1	2	3	4	5	6	7	8	9	10
Total profit	21,558	28,219	30,175	30,581*	29,978	28,784	27,217	25,394	23,382	21,236

It is observed that for the set of chosen parameter-values, the optimal number of shipments is 4. We further see that the optimal values of t_i 's are as follows: $t_1 = 1.086$, $t_2 = 3.451$, $t_3 = 5.921$, and $t_4 = 8.409$, indicating that it is indeed beneficiary for the retailer to start with planned shortage, and the shortage period gets reduced in subsequent intervals. Optimal values of other decision variables are obtained as $p = 34.16$, $s_0 = 0.495$, and $\xi = 23.17$, in appropriate units. The Hessian matrix for the optimal case is a 7×7 matrix specified as

$$H = \begin{vmatrix} -1,606 & 0 & 0 & 0 & -30.4521 & -24.9159 & -0.7681 \\ 0 & -1,790 & 0 & 0 & -33.3963 & -28.0727 & -1.2887 \\ 0 & 0 & -1,993.4 & 0 & -37.2661 & -23.09 & -1.6218 \\ 0 & 0 & 0 & -2,167.6 & -40.6749 & -18.5852 & -1.8798 \\ -30.4521 & -33.3963 & -37.2661 & -40.6749 & -192.95 & 889.1556 & -0.2672 \\ -24.9159 & -28.0727 & -23.09 & -18.5852 & 889.1556 & -20,985 & 2.4845 \\ -0.7681 & -1.2887 & -1.6218 & -1.8798 & -0.2672 & 2.4845 & -159,040 \end{vmatrix}$$

from which it is straightforward to derive that $|H_1| = -1606 < 0$, $|H_2| = 2,874,740 > 0$, $|H_3| = -5,730,506,716 < 0$, $|H_4| = 12,421,446,357,601.6 > 0$, $|H_5| = -2,363,671,580,574,673.1702 < 0$, $|H_6| = 39,739,035,172,604,178,803 > 0$, and $|H_7| = -6.3200959721700257854 \times 10^{24} < 0$, thereby establishing the concavity numerically.

Figure 4 Sensitivity of profit with parameters (see online version for colours)



A sensitivity analysis is further carried out with respect to the key parameters, and the results are summarised in Table 3.

Table 3 Sensitivity analysis

Parameter	Value	p	s_0	ξ	t_i
a	400	28.15	0.234	7.62	1.76; 4.9; 8.2
	450	30.85	0.306	9.78	1.645; 4.783; 8.079
	500	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	550	40.17	0.98	29	1.31; 4.47; 7.77
	600	42.34	0.999	43.413	0.94; 3.309; 5.778; 8.264
b	3	23.555	0.467	10.76	1.127; 3.488; 5.961; 8.449
	4	33.969	0.484	12.62	1.106; 3.466; 5.948; 8.445
	5	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	6	34.479	0.509	29.042	1.083; 3.447; 5.915; 8.402
	7	34.8	0.524	26.128	1.082; 3.444; 5.913; 8.399
c	0.001	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	0.002	34.16	0.49	23.49	1.086; 3.45; 5.921; 8.409
	0.003	34.16	0.495	24.476	1.086; 3.449; 5.921; 8.409
	0.004	34.164	0.494	27.524	1.086 ; 3.45; 5.919; 8.408
	0.005	34.155	0.495	25.97	1.085; 3.45; 5.92; 8.407
β	8	45.2586	0.999	14.2518	1.22; 4.391; 7.698
	9	41.163	0.999	15.004	1.314; 4.484; 7.784
	10	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	11	30.844	0.306	25.373	1.587; 4.731; 8.02
	12	28.6	0.246	25.383	1.667; 4.81; 8.098
δ	80	33.091	0.254	12.135	1.543; 4.705; 7.991
	90	33.567	0.392	19.133	1.125; 3.503; 5.98; 8.452
	100	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	110	38.185	0.999	31.575	1.386; 4.568; 7.875
	120	38.639	0.999	31.568	1.34; 4.511; 7.803
α_1	1.5	34.233	0.495	24.721	1.086; 3.45; 5.92; 8.409
	1.8	34.153	0.495	33.579	1.088; 3.451; 5.919; 8.407
	2	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	2.3	34.151	0.494	23.504	1.086; 3.451; 5.992; 8.41
	2.7	34.172	0.495	28.411	1.084; 3.449; 5.919; 8.407
θ	0.01	34.175	0.495	6.721	1.079; 3.443; 5.917; 8.405
	0.03	34.159	0.494	14.926	1.088; 3.45; 5.926; 8.41
	0.05	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	0.075	34.144	0.487	30.22	1.16; 3.521; 5.99; 8.485
	0.1	34.184	0.494	41.845	1.168; 3.522; 6; 8.467
c_s	5	34.067	0.494	22.849	1.131; 3.495; 5.964; 8.452
	6	34.128	0.496	23.601	1.108; 3.471; 5.942; 8.432
	7	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	8	34.192	0.495	23.644	1.064; 3.43; 5.9; 8.408
	9	34.221	0.495	29.486	1.044; 3.411; 5.882; 8.37
H	8	32.706	0.26	23.663	1.197; 3.718; 6.352
	9	33.19	0.328	24.491	1.347; 4.181; 7.144
	10	34.16	0.495	23.17	1.086; 3.451; 5.921; 8.409
	11	37.762	0.999	28.645	1.538; 5.002; 8.625
	12	37.701	0.999	26.977	1.223; 4.; 7.067; 10.06
	13	37.93	0.999	42.836	1.334; 4.41; 7.665; 10.86
	15	38.07	0.999	28.203	1.22; 4.066; 7.031; 10.021; 13.012

5.1.1 Managerial insights

From the table, the following managerial insights can be drawn.

- The demand parameter a impacts the profit positively. A higher base demand a allows the manager to charge a higher price to earn more revenue out of the extra demand. He also reduces shortage period to reduce lost sale and mitigate more demand, thereby generating more revenue. Both the optimal service level as well as preservation investment increase with higher a , indicating that the retailer has a prior planning of investing a portion of the extra earning back in the business again to generate more revenue. Similar arguments hold good for the parameter b too. As parameter value c increases, profit decreases since a greater c represents a more rapid reduction in demand over time. However, as smaller values of c were selected, it does not appear that the variation had a significant impact on the optimal decisions.
- Since the demand parameter β is associated with price, it affects the demand as well as the profit level in a negative way, as an one-unit increment in β results in p units' reduction in demand. The profit level witnesses a sharp downturn with the rise in β . The manager has to loose some share of profit by simultaneously reducing price to stimulate demand, and investing more in preservation as well as extending shortage period to reduce further loss from spoilage. To mitigate the loss generated by price cut to some extent, he further invests less in servicing, as is evident from reduced s_0 s. It also indicates that the manager should place *proper pricing* before *proper servicing*, as the demand is more sensitive to price than servicing.
- It is quite intuitive that a rise in δ will result in higher demand level and consequently higher profit, as is evident from the figure. One may notice from the table that a higher δ along with higher investment in preservation increases the certainty level of the manager to invest more in service level. So the manager may choose to generate some extra revenue by charging higher price.
- Higher values of α_1 lower the ordering cost without decreasing the quality, yet they may raise it to a certain level, which benefits profit. We assert the same result to be valid for α_2 . However, none of the decision variables seem to be affected with the variation. With higher α_2 , the manager orders higher lots in consecutive shipments to gain benefit from reduced holding cost.
- The increment in deterioration rate θ reduces the profit level, but that is not too high with the presence of preservation technology investment. Table 3 clearly shows that the increment in θ implies more preservation technology investment and lower service level compared to other parameters to retain the demand as undisturbed as possible. The manager further extends shortage periods to reduce spoilage.
- Greater shortage costs suggest either higher holding costs or more lost sales, both of which indicate a loss of profit. So, it is not surprising to see profit decreasing with the increase in c_s . Table 3 strengthens our claim by demonstrating a reduced shortage period with greater cost of shortage. The retailer would rather choose to accept loss by incurring more holding cost rather than losing goodwill and the

opportunity of earning revenue, thereby reducing shortage period. Holding the items for longer period results in investing more in preservation.

- The profit level increases with a longer business horizon H , which is due to the fact that the demand has not reached its maximum value with respect to time during the sensitivity range of H , indicating increasing demand with higher values of H ; however, it is seen that a higher H results in reduced profit once H crosses a threshold value, and subsequently the profit level would reach zero when H is much higher, i.e., when the product becomes obsolete. It is interesting to note that, with higher time horizon, the retailer divides the total horizon into more cycles when the time horizon is longer, benefiting from the learning effect on holding and ordering costs.

5.2 Space limitation

This subsection aims to study the effect of space limitation on the optimal decisions. Space limitation is a real life constraint for the business managers, since holding a large warehouse often involves large capitals, or sometimes it becomes impossible to arrange larger warehouse due to limited resource, manpower or other issues. For the above-mentioned numerical example, optimal value of n is obtained as 4 and the highest inventory level is derived as 419. If the retailer does not have enough space to hold 419 units at a time, he has to redesign his optimal strategy. Table 4 summarises the optimal values of the decision variables under different space limitations at the buyer.

Table 4 Effect of space limitations on optimal decisions

W	n	p	t_i	ξ	s_0	Total profit
200	4	38	1.2; 3.9; 6.54; 9.35	21	0.547	27,953
250	4	36.6	1.117; 3.755; 6.4; 8.993	42.97	0.591	29,256
300	4	34.1	1.094; 3.597; 6.256; 8.863	36.37	0.467	30,190
350	4	34.15	1.091; 3.455; 6.053; 8.675	27.19	0.493	30,481
400	4	34.19	1.09; 3.46; 5.92; 8.486	26.2	0.499	30,571

It is reasonable to believe that the more space a warehouse has, the faster its rent rises (the owner is assumed to pay the rent for owned warehouse to himself). According to the table, profit rises as inventory space increases. Space limitation has its effect mostly on the shortage period; the manager is unable to mitigate demand under space limitation, so he has to incur shortage. Optimal shortage periods continuously decrease with the increasing inventory space and so does the preservation technology investment. The preservation technology is assumed to be independent of inventory level in this paper, which signifies the event that the manager can afford more deterioration when he possesses larger quantity of items.

5.3 Capital constraint

This subsection addresses a crucial topic that business managers must deal with: the issue of capital constraint. It is reasonable to take resource limitations into consideration

because businesses frequently have a restricted or pre-specified budget to spend on particular streams. Overspending in one or more areas puts other decisions in jeopardy. Note that the total cost incurred in the unconstrained model is 42,767 units, the total capital R varies ranging from 20,000 units to 40,000 units to examine the effect on optimal decisions. It is revealed that financial constraint impacts every aspects of business decision – pricing, replenishment timings, servicing, and preservation investment. With lower capital at hand, the manager even reduces the number of shipments to reduce ordering cost. Shortage period is also extended as the manager faces hardship to store more amount; preservation investment is reduced, and so is service investment. All the decisions together reduce the total profit. As is seen, the more capital is at hand of the manager, the more he is free to align optimal decisions so as to generate maximum profit.

Table 5 Effect of capital constraint on optimal decisions

R	n	p	t_i	ξ	s_0	Total profit
20,000	2	42.96	2.75; 7.119	77	0.223	19,547
25,000	3	39	1.981; 5.171; 8.533	25	0.146	25,157
30,000	3	35.64	1.841; 4.972; 8.239	23	0.192	28,794
35,000	3	34	1.565; 4.728; 7.997	33.97	0.31	30,032
40,000	4	34.69	1.113; 3.506; 5.948; 8.479	46.871	0.424	30,356

6 Conclusions

A multi-period inventory model is developed and analysed considering various aspects of demand, servicing, replenishment schedules, preservation investment, and learning effect. The manager is allowed to choose different stock-in periods in different periods, changing pricing, service, and preservation investment accordingly. This is done by taking into account the time dependency of demand within a set business horizon. The unconstrained model is further extended by consideration of

- a warehouse space limitation
- b capital constraint.

The paper analytically establishes the existence of optimal price, and numerical results guarantee the acceptance and stability of the model. The results reveal that the parameter values related to price and time dependency, business horizon, etc. has significant impact on optimal decisions, and provide valuable insight as it suggests how the manager should adjust and realign his optimal pricing and other strategies to sustain in competition under changed business scenario.

The model may be extended in various directions. Firstly, this paper considers constant rate of deterioration which may be assumed to be of time-dependent in nature. In contrast to static pricing, one can simply put it in the dynamic form. The learning curve is taken to be of exponential type, one can try various types of learning. Finally, the current inventory model may be extended in a supply chain framework and new coordination contract may be designed to coordinate the chain successfully.

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Appendix A

Solving the first equation, we get

$$I(t) = -\frac{e^{-\eta(t_i-t)}}{\eta} \left((a + bt - ct^2 - \beta p + \delta s_i) - \frac{(b - 2ct)}{\eta} - \frac{2c}{\eta^2} \right) + k_1,$$

where the value of k_1 may be derived from the condition $I((i-1)T) = 0$ as

$$k_1 = \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \left((a + b(i-1)T - c((i-1)T)^2 - \beta p + \delta s_i) - \frac{(b - 2c(i-1)T)}{\eta} - \frac{2c}{\eta^2} \right).$$

So the inventory level can be written as

$$I(t) = -\frac{e^{-\eta(t_i-t)}}{\eta} \left\{ (a + bt - ct^2 - \beta p + \delta s_i) - \frac{(b - 2ct)}{\eta} - \frac{2c}{\eta^2} \right\} + \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \times \left\{ a + b(i-1)T - c((i-1)T)^2 - \beta p + \delta s_i - \frac{b - 2c(i-1)T}{\eta} - \frac{2c}{\eta^2} \right\}.$$

Putting $I(t_i) = -S_i$, the total backlogged amount is obtained as given in equation (3).

The replenishment occurs at time $t = t_i$. So the inventory level at time t_i would be $Q_i - S_i$.

The second equation is now solved using $Q_i - S_i$ as the initial inventory level to obtain

$$I(t) = Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i + \delta s_i t_i \\ - at - \frac{bt^2}{2} + \frac{ct^3}{3} + \beta pt - \delta s_i t.$$

Using this, the inventory level at $t_i + t_d$ is obtained as

$$I(t_i + t_d) = Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i + \delta s_i t_i - a(t_i + t_d) \\ - \frac{b(t_i + t_d)^2}{2} + \frac{c(t_i + t_d)^3}{3} + \beta p(t_i + t_d) - \delta s_i(t_i + t_d),$$

which is further used as initial condition for the third differential equation, from which direct calculation yields the inventory level

$$I(t) = -\frac{(a + bt - ct^2 - \beta p + \delta s_i)}{(1 - m)\theta} + \frac{(b - 2ct)}{(1 - m)^2\theta^2} + \frac{2c}{(1 - m)^3\theta^3} + c_3 e^{-(1-m)\theta t},$$

and the initial condition $I(t_i + t_d)$ specifies the value of c_3 as

$$c_3 = \left(Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i + \delta s_i t_i - a(t_i + t_d) - \frac{b(t_i + t_d)^2}{2} \right. \\ + \frac{c(t_i + t_d)^3}{3} + \beta p(t_i + t_d) - \delta s_i(t_i + t_d) \\ + \frac{(a + b(t_i + t_d) - c(t_i + t_d)^2 - \beta p + \delta s_i)}{(1 - m)\theta} \\ \left. - \frac{(b - 2c(t_i + t_d))}{(1 - m)^2\theta^2} - \frac{2c}{(1 - m)^3\theta^3} \right) e^{(1-m)\theta(t_i + t_d)}.$$

Then the inventory level during the period $[t_d, T]$ can be written as

$$I(t) = \left(Q_i - S_i + at_i + \frac{bt_i^2}{2} - \frac{ct_i^3}{3} - \beta pt_i + \delta s_i t_i - a(t_i + t_d) - \frac{b(t_i + t_d)^2}{2} \right. \\ + \frac{c(t_i + t_d)^3}{3} + \beta p(t_i + t_d) - \delta s_i(t_i + t_d) \\ + \frac{(a + b(t_i + t_d) - c(t_i + t_d)^2 - \beta p + \delta s_i)}{(1 - m)\theta} - \frac{(b - 2c(t_i + t_d))}{(1 - m)^2\theta^2} \\ \left. - \frac{2c}{(1 - m)^3\theta^3} \right) e^{-(1-m)\theta(t - (t_i + t_d))} - \frac{(a + bt - ct^2 - \beta p + \delta s_i)}{(1 - m)\theta} \\ + \frac{(b - 2ct)}{(1 - m)^2\theta^2} + \frac{2c}{(1 - m)^3\theta^3}.$$

Appendix B

Noting that Q_i s and S_i s are linear in p for all i , we have

$$\begin{aligned} \frac{\partial \Pi_i}{\partial p} = & \frac{SR_i}{p} + p \left[-\frac{\beta}{\eta} + \frac{e^{-\eta(t_i - (i-1)T)}\beta}{\eta} - \beta(iT - t_i) \right] \\ & - \left(h_1 + \frac{h_2}{\alpha_2} \right) \left\{ \left(\frac{\partial Q_i}{\partial p} - \frac{\partial S_i}{\partial p} - \beta \right) t_d - \frac{\beta(2t_i + t_d)t_d}{2} \right. \\ & + \left(\frac{\partial Q_i}{\partial p} - \frac{\partial S_i}{\partial p} + \beta t_d - \frac{\beta}{(1-m)\theta} \right) \left(\frac{1 - e^{-(1-m)\theta(iT - (t_i + t_d))}}{(1-m)\theta} \right) \\ & + \left. \frac{\beta(iT - (t_i + t_d))}{(1-m)\theta} \right\} + \left(\beta(t_i - (i-1)T) + \frac{\partial S_i}{\partial p} \right) c_s \\ & - c_d(1-m)\theta \left[\left(\frac{\partial Q_i}{\partial p} - \frac{\partial S_i}{\partial p} + \beta t_d \right. \right. \\ & \left. \left. - \frac{\beta}{(1-m)\theta} \right) \left(\frac{1 - e^{-(1-m)\theta(iT - (t_i + t_d))}}{(1-m)\theta} \right) + \frac{\beta(iT - (t_i + t_d))}{(1-m)\theta} \right], \end{aligned}$$

where Π_i denotes profit in the i^{th} interval, excluding the cost of providing service which is independent of retail price. It is straightforward to deduce that

$$\begin{aligned} \frac{\partial^2 \Pi_i}{\partial p^2} = & 2 \left[-\frac{\beta}{\eta} + \frac{e^{-\eta(t_i - (i-1)T)}\beta}{\eta} - \beta(iT - t_i) \right] \\ = & -2\beta \left[iT - t_i + \frac{1 - e^{-\eta(t_i - (i-1)T)}}{\eta} \right] < 0, \end{aligned}$$

summing up over i completes the proof.

Appendix C

Proof of a is straightforward except the last part, hence omitted. For the last part, note that

$$\begin{aligned} \frac{\partial S_i}{\partial c} = & -\frac{t_i^2}{\eta} + \frac{2t_i}{\eta} - \frac{2}{\eta^2} - \frac{e^{-\eta(t_i - (i-1)T)}}{\eta} \left\{ -(i-1)^2 T^2 + \frac{2(i-1)T}{\eta} - \frac{2}{\eta^2} \right\} \\ = & -(t_i + \frac{1}{\eta})^2 - \frac{1}{\eta^2} - \frac{e^{-\eta(t_i - (i-1)T)}}{\eta} \left\{ -\left((i-1)T + \frac{1}{\eta} \right)^2 - \frac{1}{\eta^2} \right\} \\ < & -\left((i-1)T + \frac{1}{\eta} \right)^2 \left(1 - \frac{e^{-\eta(t_i - (i-1)T)}}{\eta} \right), \end{aligned}$$

since $t_i > (i-1)T \Rightarrow \left(t_i + \frac{1}{\eta} \right)^2 > \left((i-1)T + \frac{1}{\eta} \right)^2$, hence the proof is completed. Further,

$$\begin{aligned}
\frac{\partial Q_i}{\partial c} &= \frac{\partial S_i}{\partial c} + \frac{t_i^3}{3} - \frac{(t_i + t_d)^3}{3} + \frac{(t_i + t_d)^2}{(1-m)\theta} - \frac{2(t_i + t_d)}{(1-m)^2\theta^2} + \frac{2}{(1-m)^3\theta^3} \\
&\quad - \left(\frac{2}{(1-m)^3\theta^3} + \frac{i^2 T^2}{(1-m)\theta} - \frac{2iT}{(1-m)^2\theta^2} \right) e^{(1-m)\theta(iT-(t_i+t_d))} \\
&= \frac{1}{(1-m)\theta} \left(t_i + t_d - \frac{1}{(1-m)\theta} \right)^2 + \frac{1}{(1-m)^3\theta^3} \\
&\quad - e^{(1-m)\theta(iT-(t_i+t_d))} \frac{1}{(1-m)^3\theta^3} \\
&\quad - \frac{1}{(1-m)\theta} \left(iT - \frac{1}{(1-m)\theta} \right)^2 e^{(1-m)\theta(iT-(t_i+t_d))} \\
&< - \left((i-1)T + \frac{1}{\eta} \right)^2 \left(1 - \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \right) \\
&\quad + \frac{1}{(1-m)\theta} \left(iT - \frac{1}{(1-m)\theta} \right)^2 \\
&\quad - \frac{1}{(1-m)\theta} \left(iT - \frac{1}{(1-m)\theta} \right)^2 e^{(1-m)\theta(iT-(t_i+t_d))} \\
&\quad + \frac{1}{(1-m)^3\theta^3} \left(1 - e^{(1-m)\theta(iT-(t_i+t_d))} \right) \\
&= - \left((i-1)T + \frac{1}{\eta} \right)^2 \left(1 - \frac{e^{-\eta(t_i-(i-1)T)}}{\eta} \right) \\
&\quad - \frac{1}{(1-m)^3\theta^3} \left(e^{(1-m)\theta(iT-(t_i+t_d))} - 1 \right) \\
&\quad - \frac{1}{(1-m)\theta} \left(iT - \frac{1}{(1-m)\theta} \right)^2 \left(e^{(1-m)\theta(iT-(t_i+t_d))} - 1 \right) (iT \\
&\quad - \frac{1}{(1-m)\theta})^2 < 0,
\end{aligned}$$

completing the proof.

For b, it is observed that

$$\frac{\partial S_i}{\partial b} = \frac{t_i - \{1 + e^{-\eta(t_i-(i-1)T)}(i-1)T\}}{\eta}.$$

Since η is very small in magnitude, approximating $e^{-\eta(t_i-(i-1)T)}$ linearly, straightforward calculation reveals that whenever $[t_i - (i-1)T][1 + \eta(i-1)T] > 1$, or $t_i - (i-1)T > \frac{1}{1+\eta(i-1)T}$, $\frac{\partial S_i}{\partial b} > 0$. For the second part, it is observed that $\frac{\partial Q_i}{\partial b} = \frac{\partial S_i}{\partial b} + A_i$, where $A_i = \left(\frac{(t_i+t_d)^2}{2} - \frac{t_i^2}{2} \right) + \frac{1}{(1-m)\theta} (iT e^{(1-m)\theta(iT-(t_i+t_d))} - (t_i + t_d)) + \frac{1}{(1-m)^2\theta^2} (e^{(1-m)\theta(iT-(t_i+t_d))} - 1)$, so that $\frac{\partial S_i}{\partial b} > 0$ implies $\frac{\partial Q_i}{\partial b} > 0$.

For c, we may write $\frac{\partial S_i}{\partial t_i} = \frac{1}{\eta} \left(b - 2ct_i + \frac{2c}{\eta} \right) + e^{-\eta(t_i-(i-1)T)} R$, where $R = a + b(i-1)T - c((i-1)T)^2 - \beta p + \delta s_i - \frac{(b-2c(i-1)T)}{\eta} - \frac{2c}{\eta^2}$, and subsequently $\frac{\partial^2 S_i}{\partial t_i^2} =$

$-\left[\frac{2c}{\eta} + \eta R e^{-\eta(t_i - (i-1)T)}\right] < 0$, establishing unique maxima. As t_i 's are time points in the i^{th} interval on the time horizon, t_i s increase with i , so there is a value t^* to be obtained from first order condition $\frac{\partial S_i}{\partial t_i} = 0$ such that whenever t_i 's are lesser than t^* , S_i increases, and decreases when t_i 's are higher in magnitude than t^* .