
Transient and steady-state rotation centre of vehicle dynamics

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Abstract: The effect of acceleration – variable forward velocity, have been investigated on the steady-state and transient responses of turning vehicles and the resulted path of motion of the vehicle. Comparing the two mentioned responses of the vehicle could be used to prove that there is a negligible difference between the steady-state and transient centre of rotation of the vehicle in engineering applications. Dynamics of a vehicle with a constant steer angle and variable forward velocity have been analysed and presented as the proof. It has been shown that it is possible to predict the dynamics of vehicles using their steady state responses within acceptable engineering approximations. More specifically, we determine the dynamic rotation centre of vehicles and compare them with steady-state values. The result would be essential to design autodriver algorithm for autonomous vehicles.

Keywords: vehicle dynamics; steady-state vehicle behaviour; rotation centre of vehicles; variable forward velocity.

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1 Introduction

The equations of motion of the planar bicycle car model, expressed in the principal body coordinate frame B , is governed by the following set of nonlinear coupled ordinary differential equations. In the equations, the steering angle δ acts as the input and, forward

velocity of the centre of the mass of the vehicle v_x , lateral velocity v_y , and yaw rate r , are the outputs (Jazar, 2013).

$$\dot{v}_x = \frac{1}{m} F_x + r v_y \quad (1)$$

$$\dot{v}_y = \frac{C_\beta}{m v_x} v_y - \left(v_x - \frac{C_r}{m} \right) r + \frac{C_\delta}{m} \delta(t) \quad (2)$$

$$\dot{r} = \frac{D_\beta}{I_z v_x} v_y + \frac{D_r}{I_z} r + \frac{D_\delta}{I_z} \delta(t) \quad (3)$$

where

$$C_r = \frac{\partial F_y}{\partial r} = -\frac{a_1}{v_x} C_{\alpha f} + \frac{a_2}{v_x} C_{\alpha r} \quad (4)$$

$$C_\beta = \frac{\partial F_y}{\partial \beta} = -C_{\alpha f} - C_{\alpha r} \quad (5)$$

$$C_\delta = \frac{\partial F_y}{\partial \delta} = C_{\alpha f} \quad (6)$$

$$D_r = \frac{\partial M_z}{\partial r} = -\frac{a_1^2}{v_x} C_{\alpha f} - \frac{a_2^2}{v_x} C_{\alpha r} \quad (7)$$

$$D_\beta = \frac{\partial M_z}{\partial \beta} = a_2 C_{\alpha r} - a_1 C_{\alpha f} \quad (8)$$

$$D_\delta = \frac{\partial M_z}{\partial \delta} = a_1 C_{\alpha f} \quad (9)$$

$$\beta = \frac{v_y}{v_x} \quad (10)$$

The orthogonal body coordinate frame B is fixed to the vehicle at its mass centre C and is set such that the x -axis is longitudinal, y -axis is lateral, and z -axis is vertical upwards. The body coordinate and kinematics of the bicycle car model in a forward motion on a positive turn are illustrated in Figure 1.

The coefficients C_r , C_β , C_δ , D_r , D_β , D_δ in the equations of motion are slopes of the curves for lateral force F_y and yaw moment M_z as a function of r , β , and δ respectively. The coefficients C_r and D_r are functions of v_x , and the coefficients C_β , C_δ , D_β , D_δ are constant for a given vehicle. In this case it is been assumed that the vehicle is moving with a constant steer angle equal to:

$$\delta(t) = 0.1 \text{ rad} = 5.37 \text{ deg} \quad (11)$$

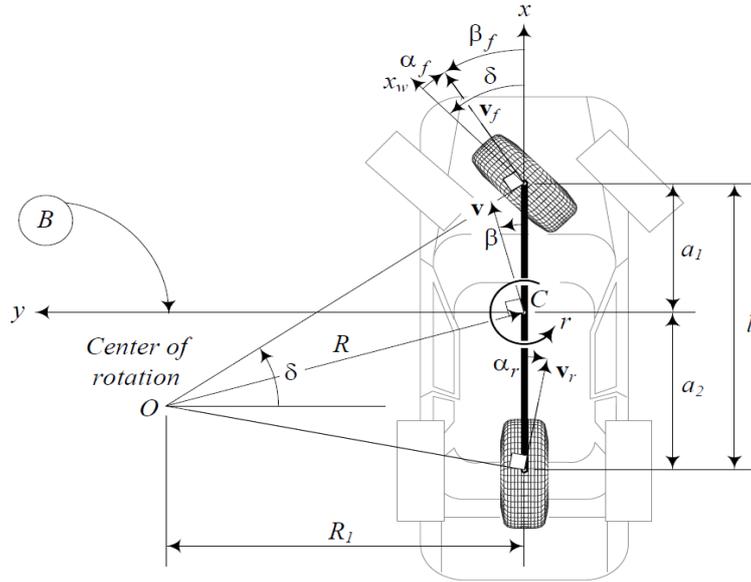
The forward velocity of the vehicle changes with time according to the following function:

$$v_x = \frac{20}{t_0} t H(t_0 - t) + 20 H(t - t_0) \text{ m/s} \quad (12)$$

where v_x changes from $v_x = 0$ m/s at $t = 0$ to the maximum speed $v_x = 20$ m/s at $t = t_0$, and remains constant afterwards. $H(t - t_0)$ is the Heaviside function:

$$H(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \quad (13)$$

Figure 1 Kinematics of a moving vehicle at steady-state conditions



The equations of motion of a two-wheeled rigid vehicle with no roll are as the following:

$$\dot{v}_x = \frac{1}{m} F_x + r v_y \quad (14)$$

$$\begin{bmatrix} \dot{v}_y \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{C_\beta}{m v_x} & \frac{C_r}{m} - v_x \\ \frac{D_\beta}{I_z v_x} & \frac{D_r}{I_z} \end{bmatrix} \begin{bmatrix} v_y \\ r \end{bmatrix} + \begin{bmatrix} \frac{C_\beta}{m} \\ \frac{D_\delta}{I_z} \end{bmatrix} \delta(t) \quad (15)$$

When v_x is known, either is constant or varying with time, equation (15) becomes independent from (14). The set of equation (15) can then be written in the form

$$\dot{\mathbf{q}} = [A] \mathbf{q} + \mathbf{u} \quad (16)$$

in which $[A]$ is a constant coefficient matrix, \mathbf{q} is the vector of control variables, and \mathbf{u} is the vector of inputs.

$$\begin{aligned}
 [A] &= \begin{bmatrix} \frac{C_\beta}{mv_x} & \frac{C_r}{m} - v_x \\ \frac{D_\beta}{I_z v_x} & \frac{D_r}{I_z} \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{C_{\alpha_f} + C_{\alpha_r}}{mv_x} & \frac{-a_1 C_{\alpha_f} + a_2 C_{\alpha_r}}{mv_x} - v_x \\ -\frac{a_1 C_{\alpha_f} - a_2 C_{\alpha_r}}{I_z v_x} & -\frac{a_1^2 C_{\alpha_f} + a_2^2 C_{\alpha_r}}{I_z v_x} \end{bmatrix}
 \end{aligned} \tag{17}$$

$$\mathbf{q} = \begin{bmatrix} v_y \\ r \end{bmatrix} \tag{18}$$

$$\mathbf{u} = \begin{bmatrix} \frac{C_{\alpha_f}}{m} \\ \frac{a_1 C_{\alpha_f}}{I_z} \end{bmatrix} \delta(t) = \begin{bmatrix} \frac{C_\beta}{m} \\ \frac{D_\delta}{I_z} \end{bmatrix} \delta(t) \tag{19}$$

The following characteristics for a sample car will be used to develop the equation of motion.

$$\begin{aligned}
 C_{\alpha_f} &= 50,000 \text{ N/rad} & C_{\alpha_r} &= 50,000 \text{ N/rad} \\
 m &= 1,000 \text{ kg} & I_z &= 1,650 \text{ kgm}^2 \\
 a_1 &= 1.0 \text{ m} & a_2 &= 1.5 \text{ m}
 \end{aligned} \tag{20}$$

In the following sections, we calculate the transient response of an understeer passenger car that is moving with a constant steering angle equal to equation (11) and a variable forward velocity which changes with time according to equation (12). The transition behaviour of the car will be determined using steady-state responses to examine the proximity of the two analysis. It will be shown that steady-state response equations are good enough to predict the transition behaviour of the car.

The results of this investigation are expected to be useful in speeding up the online calculation of autodriver algorithm (Jazar, 2010), as well as obtaining data to determine the steady-state behaviour of vehicles (Tahami et al., 2003).

2 Steady-state responses

When the vehicle is turning at a steady-state condition, its behaviour is governed by the following equations:

$$F_x = -\frac{m}{R} v_x v_y \tag{21}$$

$$C_\beta \beta + (C_r v_x - m v_x^2) \frac{1}{R} = -C_\delta \delta \tag{22}$$

$$D_\beta \beta + D_r v_x \frac{1}{R} = -D_\delta \delta \quad (23)$$

from which, we can define a set of steady-state responses (Jazar, 2013):

1 curvature response, S_κ

$$S_\kappa = \frac{\kappa}{\delta} = \frac{1}{R\delta} = \frac{C_\delta D_\beta - C_\beta D\delta}{v_x (D_r C_\beta - C_r D_\beta + m v_x D_\beta)} \quad (24)$$

2 sideslip response, S_β

$$S_\beta = \frac{\beta}{\delta} = \frac{D\delta (C_r - m v_x) - D_r C_\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \quad (25)$$

3 yaw rate response, S_r

$$S_r = \frac{r}{\delta} = \frac{\kappa}{\delta} v_x = S_\kappa v_x = \frac{C_\delta D_\beta - C_\beta D\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \quad (26)$$

4 centripetal acceleration response, S_a

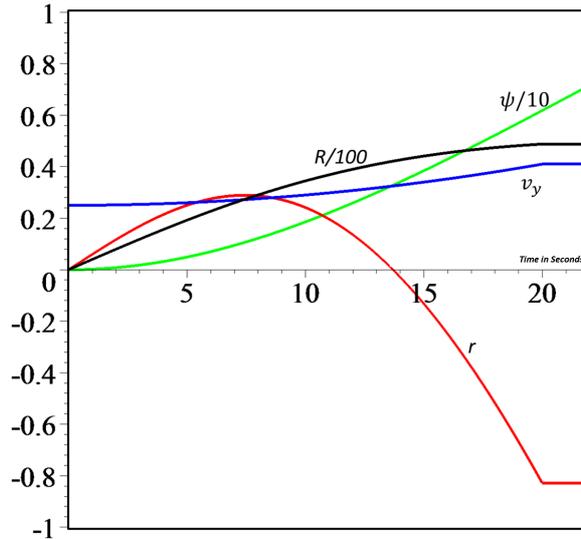
$$S_a = \frac{v_x^2 / R}{\delta} = \frac{\kappa}{\delta} v_x^2 = S_\kappa v_x^2 = \frac{(C_\delta D_\beta - C_\beta D\delta) v_x}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \quad (27)$$

5 lateral velocity response, S_y

$$S_y = \frac{v_y}{\delta} = S_\beta v_x = \frac{D\delta (C_r - m v_x) - D_r C_\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} v_x \quad (28)$$

Figure 2 illustrates steady-state responses for the sample car mentioned above.

Figure 2 Steady-state responses of a sample vehicle, as functions of forward velocity v_x (see online version for colours)



The vehicle is understeer because of a positive stability parameter K (Jazar, 2013)

$$K = \frac{m}{l^2} \left(\frac{a_2}{C_{\alpha f}} - \frac{a_1}{C_{\alpha r}} \right) = 0.0016 > 0 \quad (29)$$

And the steady-state responses for the case of variable speed will be:

1 curvature response, S_κ

$$S_\kappa = \frac{0.625 \times 10^{10}}{Z \left(\frac{0.15625 \times 10^{11}}{Z} + 0.25 \times 10^8 tH(-t+20) + 0.5 \times 10^9 H(t-20) \right)} \quad (30)$$

2 sideslip response, S_β

$$S_\beta = \frac{\frac{0.9375 \times 10^{10}}{Z} - 0.5 \times 10^8 tH(-t+20) - 0.1 \times 10^{10} H(t-10)}{\frac{0.15625 \times 10^{11}}{Z} + 0.25 \times 10^8 tH(-t+20) + 0.5 \times 10^9 H(t-10)} \quad (31)$$

3 yaw rate response, S_r

$$S_r = \frac{0.625 \times 10^{10}}{\frac{0.15625 \times 10^{11}}{Z} + 0.25 \times 10^8 tH(-t+20) + 0.5 \times 10^9 H(t-20)} \quad (32)$$

4 centripetal acceleration response, S_a

$$S_a = \frac{0.625 \times 10^{10} (Z)}{\frac{0.15625 \times 10^{11}}{Z} + 0.25 \times 10^8 tH(-t+20) + 0.5 \times 10^9 H(t-20)} \quad (33)$$

5 lateral velocity response, S_y

$$S_y = \frac{\left(\frac{0.9375 \times 10^{10}}{Z} - 0.5 \times 10^8 tH(-t+20) - 0.1 \times 10^{10} H(t-20) \right) Z}{\frac{0.15625 \times 10^{11}}{Z} + 0.25 \times 10^8 tH(-t+20) + 0.5 \times 10^9 H(t-20)} \quad (34)$$

where $Z = tH(-t+20) + 20H(t-20)$.

Employing the steady-state responses, we can determine the variation of the kinematics of motion as functions of time t .

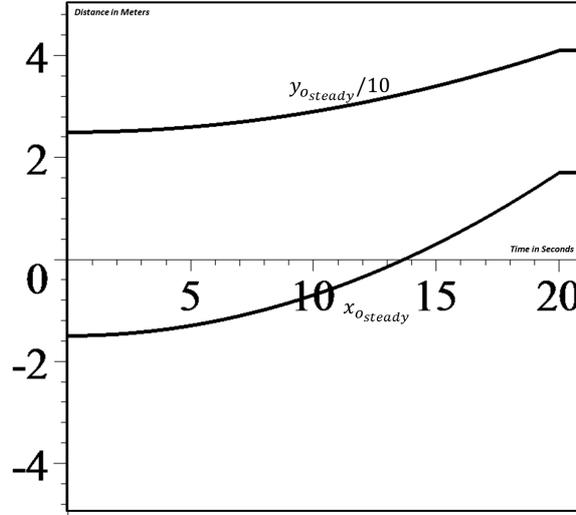
3 Steady-state rotation centre and path of motion

Having the steady-state responses $S_\kappa = 1/R/\delta$ and $S_\beta = \beta/\delta$, we are able to determine the steady-state position of the rotation centre (x_O, y_O) of a vehicle in the vehicle body coordinate frame as is illustrated in Figure 3.

$$x_O = -R \sin \beta = -\frac{1}{S_k \delta} \sin(S_\beta \delta) \quad (35)$$

$$y_O = R \cos \beta = \frac{1}{S_k \delta} \cos(S_\beta \delta) \quad (36)$$

Figure 3 The coordinates $(x_{O_{steady}}, y_{O_{steady}})$ of the steady-state rotation centre of the vehicle in the body frame for $0 \leq t \leq 20$ s



At steady-state conditions the radius of rotation R can be found from the curvature response S_k , equation (30), and the vehicle sideslip angle β from the sideslip response S_β [equation (31)].

$$R = \frac{1}{\delta S_k} = \frac{v_x (D_r C_\beta - C_r D_\beta + m v_x D_\beta)}{(C_\delta D_\beta - C_\beta D_\delta) \delta} \quad (37)$$

$$\beta = \delta S_\beta = \frac{D_\delta (C_r - m v_x) - D_r C_\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \delta \quad (38)$$

Therefore, the steady-state position of the dynamic centre of rotation O in the body coordinate frame B , about which the vehicle will actually turn, is at

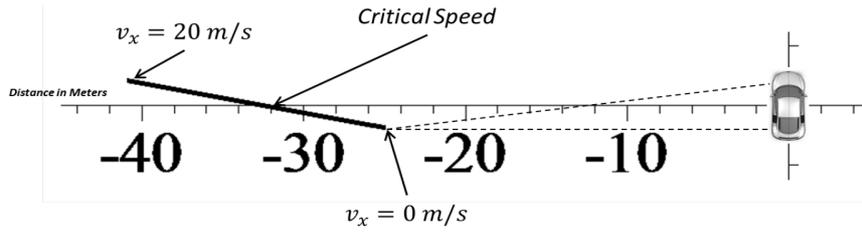
$$x_O = -\frac{v_x (D_r C_\beta - C_r D_\beta + m v_x D_\beta)}{(C_\delta D_\beta - C_\beta D_\delta) \delta} \sin \left(\frac{D_\delta (C_r - m v_x) - D_r C_\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \delta \right) \quad (39)$$

$$y_O = \frac{v_x (D_r C_\beta - C_r D_\beta + m v_x D_\beta)}{(C_\delta D_\beta - C_\beta D_\delta) \delta} \cos \left(\frac{D_\delta (C_r - m v_x) - D_r C_\delta}{D_r C_\beta - C_r D_\beta + m v_x D_\beta} \delta \right) \quad (40)$$

Figure 4 illustrates the coordinates $x_{O_{steady}}$ and $y_{O_{steady}}$ of the vehicle for $0 \leq t \leq 20$ after substituting equation (12) in to equations (39) and (40) At a constant steer angle, the

dynamic rotation centre of the understeer vehicle moves away and forward by increasing the forward velocity. The rate of the displacement of the rotation centre increases by speed. Figure 4 depicts how location of the centre of rotation with respect to the body frame of the vehicle changes with speed. It indicates the critical speed at which the velocity vector of the vehicle at its mass centre is in the x direction only. At the critical speed, we have $\beta = 0$ and the dynamics centre of rotation is located on the y -axis.

Figure 4 The location of the steady-state rotation centre in the body frame for $0 \leq v_x \leq 20$



The global frame G is fixed to the ground, and B coincides G at the beginning of manoeuvres. The B -frame moves with the vehicle while the axes z and Z are always parallel. Therefore, the velocity vector of the vehicle in the global frame is

$$\begin{aligned} {}^G \mathbf{v} &= {}^G R_B {}^B \mathbf{v} = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ &= \begin{bmatrix} v_x \cos \psi - v_y \sin \psi \\ v_y \cos \psi + v_x \sin \psi \end{bmatrix} = \begin{bmatrix} v_X \\ v_Y \end{bmatrix} \end{aligned} \quad (41)$$

where ${}^G R_B$ is the transformation between B and G and the velocity vector of the vehicle in the body frame is

$${}^B \mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad (42)$$

Therefore, the global coordinates of the mass centre of the vehicle would be

$$X = \int_0^t v_X dt = \int_0^t (v_x \cos \psi - v_y \sin \psi) dt \quad (43)$$

$$Y = \int_0^t v_Y dt = \int_0^t (v_y \cos \psi + v_x \sin \psi) dt \quad (44)$$

When the steer angle is kept constant, and after reaching the top speed of 20 m/s the vehicle will eventually be turning on a constant circular path. The position of the steady-state rotation centre of the vehicle in the B -frame is at

$$x_{Osteady} = -R \sin \beta = 1.6994 \text{ m} \quad (45)$$

$$y_{Osteady} = R \cos \beta = 40.965 \text{ m} \quad (46)$$

because

$$\lim_{t \rightarrow \infty} R = 41 \text{ m} \quad (47)$$

$$\lim_{t \rightarrow \infty} \beta = -0.04146 \text{ rad} \quad (48)$$

The global coordinates of the steady-state rotation centre is

$$\begin{bmatrix} X_{O_{steady}} \\ Y_{O_{steady}} \end{bmatrix} = {}^G \mathbf{r}_B + {}^G R_B \begin{bmatrix} x_{O_{steady}} \\ y_{O_{steady}} \end{bmatrix} = \begin{bmatrix} -5.8 \text{ m} \\ -23.9 \text{ m} \end{bmatrix} \quad (49)$$

where ${}^G \mathbf{r}_O$ is the G -expression of the position vector of the origin of the B -frame with respect to the origin of the G -frame at any point on the steady-state conditions (Jazar, 2011). In this example, we used the calculated data at $t = 2$ s when we have

$$\begin{bmatrix} x_{O_{steady}} \\ y_{O_{steady}} \end{bmatrix} = \begin{bmatrix} 1.6994 \text{ m} \\ 40.965 \text{ m} \end{bmatrix} \quad (50)$$

$$\psi = 3.84 \text{ rad} \quad (51)$$

$${}^G \mathbf{r}_O = \begin{bmatrix} -27.345 \text{ m} \\ -50.182 \text{ m} \end{bmatrix} \quad (52)$$

Figure 5 The steady-state path of motion of the vehicle and its rotation centre (see online version for colours)

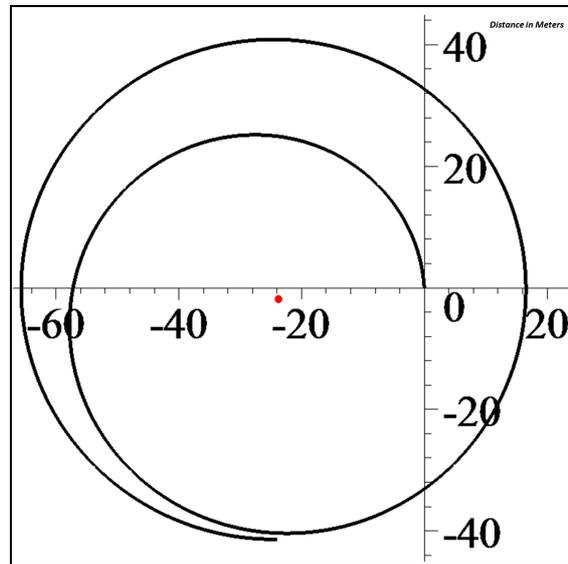


Figure 5 illustrates the instantaneous path of motion of the vehicle in global frame and its final steady-state rotation centre. The magnification of the transient stage depicts how the vehicle approaches its steady state circular path.

4 Dynamic solution rotation centre and path of motion

In order to be able to compare the dynamic solution results which will be called the ‘exact solution’ from now on with the steady-state solutions called ‘approximate solution’ let’s solve equations (15), (16) and (19). The equations of motion will turn to the following after substituting all the variables and nominal values from the sample car:

$$\dot{v}_y(t) = \left(\frac{25}{tH(-t+20) + 20H(t-20)} - tH(-t+20) - 20H(t-20) \right) r(t) - 5 \quad (53)$$

$$\dot{r}(t) = \frac{15.1515v_y(t)}{tH(-t+20) + 20H(t-20)} + \frac{98.4848r(t)}{tH(-t+20) + 20H(t-20)} - 3.03 \quad (54)$$

These are exactly equal to the same values for the approximate solutions in equations (47) and (48). This means that the path of motion for both solutions is exactly the same after they get to the steady-state condition. So, every deviation in the final path of motion of the two solutions has happened during the transition time.

The centre of rotation of the vehicle in the body frame (X_o, Y_o) are found as the following:

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} 1.6994 \text{ m} \\ 40.965 \text{ m} \end{bmatrix} \quad (55)$$

which are again the same, which indicates that the path of motion taken by the vehicle in both situations will look the same from the eye of the passengers of the car.

Figure 6 The coordinate (x_o, y_o) of the steady-state rotation centre of the vehicle in the body frame for $0 \leq t \leq 20$ m

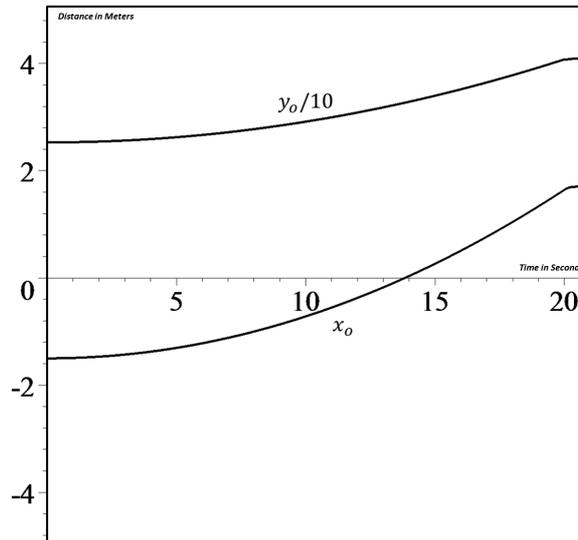
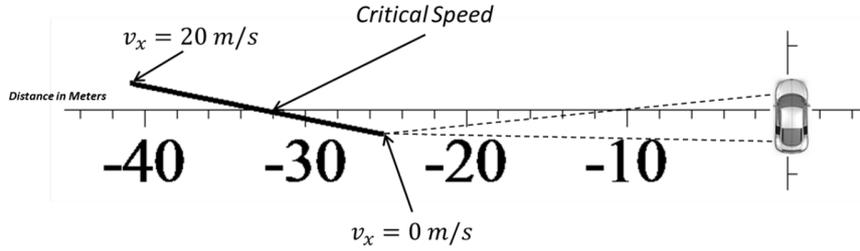


Figure 7 The location of the dynamic rotation centre in the body frame for $0 \leq v_x \leq 20$ m/s



But in order to find the centre of rotation in the global frame we will need to repeat the equations (43) and (44). Following the same steps as before we will get:

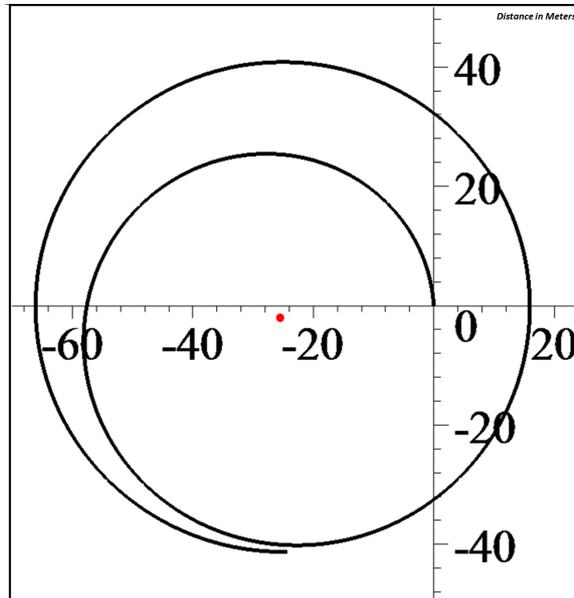
$$\psi = 3.82 \text{ rad} \tag{56}$$

$${}^G \mathbf{r}_O = \begin{bmatrix} -27.345 \text{ m} \\ -50.182 \text{ m} \end{bmatrix} \tag{57}$$

We will get,

$$\begin{bmatrix} X_O \\ Y_O \end{bmatrix} = {}^G \mathbf{r}_O + {}^G R_B \begin{bmatrix} x_O \\ y_O \end{bmatrix} = \begin{bmatrix} -5.61 \text{ m} \\ -24.22 \text{ m} \end{bmatrix} \tag{58}$$

Figure 8 The dynamics response (exact) path of motion of the vehicle and its rotation centre (see online version for colours)



5 Conclusions

The steady-state responses of a vehicle which starts moving with a constant steering angle, while the forward velocity is changing with time has been calculated for a sample car. The same sample has been used to solve the problem with the same condition dynamically. The final path of motion has been calculated for both of the above conditions. By comparing the exact and approximate paths of motion we will be able to decide whether the approximate method is accurate enough for engineering purposes. The reason behind all these is to make the calculation expenses for autonomous algorithms for autodrivers shorter and less complicated. There will be two methods for comparing the two resulted paths of motion. The first method is to compare the position of the vehicle at any moment of time on the two paths and check the distance between the positions of the two. The difference resulted in this case will include the transition time lag for the exact path which will cause the vehicle to fall behind. The reason is that for the approximate method no time is taken by the system to achieve the steady-state condition. For this case the values used for calculating the global position of the centre of rotation can be used for calculating the global position of the centre of rotation can be used from equations (52) and (59). These are the actual positions of the vehicle on the paths of motion at time $t = 14.985$ s.

These values indicate that the two vehicles are just 0.624 m apart in the x direction and 0.668 m apart in the y direction. The total distance of the two paths of motion at this moment of time is only 0.91 m which is only 2.2% of the radius of rotation. Figures 9 and 10 indicate the deviation in the x and y direction during the time of motion which makes it possible to find out the maximum deviation of the two resulted paths.

Figure 9 Deviation in the x direction – time lag included

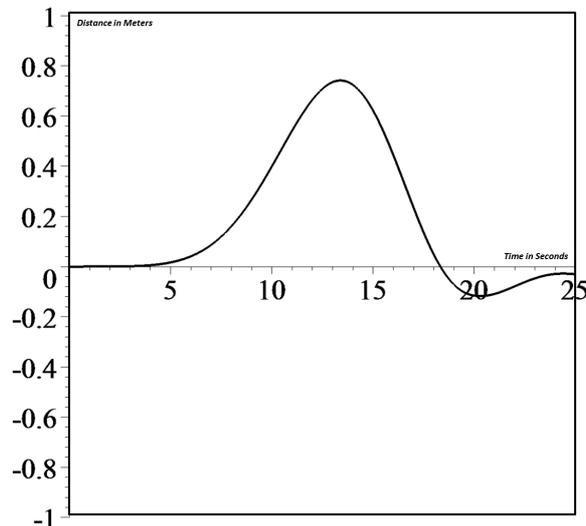
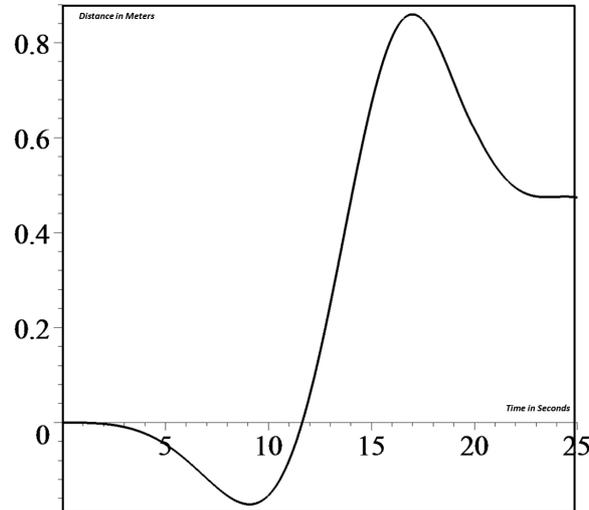


Figure 10 Deviation in the y direction – time lag included

The only problem here is that there is no way to make sure at what time during the motion we are getting the biggest deviation of the two paths. In order to get a more accurate result on the deviation of the two paths, we will have to find a second method for comparing. By comparing only the global centre of rotation of the two paths of motion we can compare the two paths only by their geometry, which is a more accurate way because, firstly what is important for the research is to find the difference in the geometry of the two generated paths and secondly we can make sure this is the biggest deviating distance between the two paths of motion, geometrically.

Doing the above mentioned we will have only 0.19 m deviation in the x direction and 0.32 m in the y direction. The total distance of the two global centres of rotation is only 0.37 m which is just 0.9% of the radius of rotation.

To conclude, in the case of moving with a constant steering angle but a varying velocity, the resulted path of motion using the steady-state responses of the vehicle or solving the dynamic equations of the motion of the vehicle are definitely close enough to make us able to neglect the difference. So, it is suggested to at least in the case of complicated calculation of equations of motion to use the steady-state responses for solving the problem.

Figure 11 shows the comparison between the two resulted paths of motion and Figure 12 shows the comparison between the movement of the centre of rotation of the two vehicles in their respective body frames. As mentioned earlier the movement of the two vehicles look the same for the passengers, that is why the two lines in Figure 12 are not recognisable.

Figure 11 The exact and approximate paths of motion (see online version for colours)

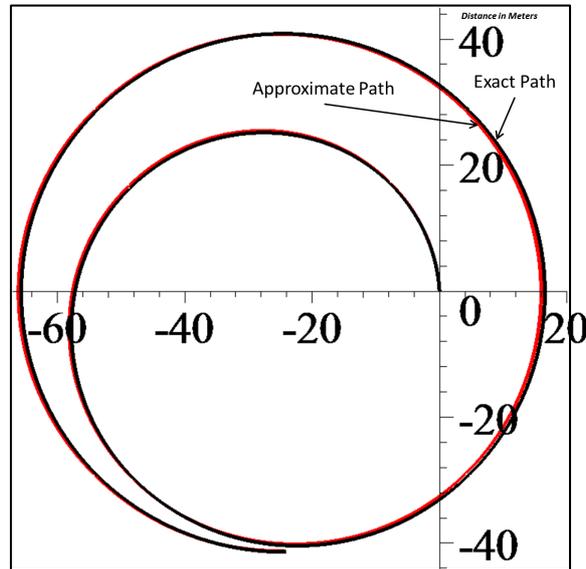
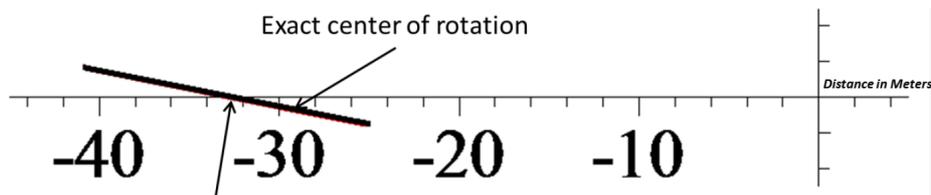


Figure 12 The exact and approximate centres of rotation in the body frame (see online version for colours)



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Nomenclature

$a \equiv \ddot{x}$	acceleration
a_i	distance of the axle number i from the mass centre
$[A]$	force coefficient matrix
$B(Cxyz)$	vehicle coordinate frame
C	mass centre
C_α	tire sideslip coefficient
$C_{\alpha f}$	front sideslip coefficient
$C_{\alpha r}$	rear sideslip coefficient
C_r, \dots, D_δ	force system coefficients
C_r	proportionality coefficient between F_y and r
C_β	proportionality coefficient between F_y and β
C_δ	proportionality coefficient between F_y and δ
D_r	proportionality coefficient between M_z and r
D_β	proportionality coefficient between M_z and β
D_δ	proportionality coefficient between M_z and δ
\mathbf{d}	frame position vector
F_x	longitudinal force, forward force, traction force
F_y	lateral force
F_{yf}	front lateral force
F_{yr}	rear lateral force
\mathbf{g}, g	gravitational acceleration
$G(OXYZ)$	global coordinate frame
I	mass moment
K	stability factor
m	mass
M_x	roll moment, bank moment, tilting torque
M_y	pitch moment
M_z	yaw moment, aligning moment
$p = \dot{\varphi}$	roll rate
\mathbf{q}	control variable vector

q_i	generalised coordinate
$r = \dot{\psi}$	yaw rate
\mathbf{r}	position vector
R	radius of rotation
R_w	tire radius
${}^G R_B$	rotation matrix to go from B frame to G frame
$S_\kappa = \kappa/\delta$	curvature response
$S_\beta = \beta/\delta$	sideslip response
$S_r = r/\delta$	yaw rate response
$S_a = v_x^2 / R / \delta$	centripetal acceleration response
S_y	lateral velocity responses
t	time
T	tire coordinate frame
T_w	wheel torque
\mathbf{u}	input vector
$\mathbf{v} \equiv \dot{\mathbf{x}}, \mathbf{v}$	velocity
w	wheelbase
x, y, z, \mathbf{x}	displacement
α	sideslip angle
β	vehicle sideslip angle, attitude angle
δ	steer angle
δ_f	front steer angle
δ_r	rear steer angle
$\kappa = 1/R$	curvature
λ	eigenvalue
φ	roll angle
$\dot{\varphi} = p$	roll rate
ψ	yaw angle
$\dot{\psi} = r$	yaw rate
ω	angular velocity
$\dot{\omega}$	angular acceleration.