ℓ-valued automata and associated ℓ-valued topologies

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Abstract: The study is to show a nice interplay among ℓ-valued approximation operator (ℓ is a complete orthomodular lattice) on an ℓ-valued approximation space, ℓ-valued topology and ℓ-valued automaton. We begin by noting that each ℓ-valued approximation space is associated with an ℓ-valued approximation operator, which turns out to be Kuratowski saturated ℓ-valued closure operator on X, if the ℓ-valued relation associated with ℓ-valued approximation space is ℓ-valued reflexive and ℓ-valued transitive. The ℓ-valued approximation operator gives rise to a saturated ℓ-valued topology on X. It is shown that the existence of dual ℓ-valued topology depends on the distributivity of associated lattice. The observations made so far are applied to ℓ-valued automata.

Keywords: ℓ-valued automaton; ℓ-valued approximation space; ℓ-valued successor; ℓ-valued source; ℓ-valued topology.

Reference to this paper should be made as follows: Tiwari, S.P., Sharan, S. and Davvaz, B. (2013) 'ℓ-valued automata and associated ℓ-valued topologies', *Int. J. Granular Computing, Rough Sets and Intelligent Systems*, Vol. 3, No. 1, pp.85–94.

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1 Introduction

The concept of ℓ-valued automata (automata theory based on quantum logic) is introduced in Ying (2000). The quantum logic can be understand as a logic whose truth value set is an orthomodular lattice, and an element of orthomodular lattice is assigned to each transition of an automaton (Ying, 2000). In Qiu (2004), it has been shown that the concepts of ℓ-valued source and ℓ-valued successor operators associated with ℓ-valued automata, induced ℓ-valued topologies on the state-set of an ℓ-valued automaton. The relationship between ℓ-valued topologies depends on the distributivity of the associated lattice. In this note, we consider an ℓ-valued approximation operator on an ℓ-valued approximation space X [i.e., a set X with an ℓ-valued reflexive and ℓ-valued transitive relation R on it, as introduced in Srivastava and Tiwari (2003)], which turns out to be a Kuratowski saturated ℓ-valued closure operator, and hence gives rise to a saturated ℓ-valued topology on X. Another ℓ-valued topology induced by an ℓ-valued approximation operator on X is defined in a natural way with the help of ℓ-valued relation R, which turns out to the dual ℓ-valued topology, when the associated lattice is distributive. Lastly, these observations are apply to ℓ-valued automata.

2 Preliminaries

In this section, we recall some basic concepts related to complete orthomodular lattices and ℓ-valued topologies. For the notations on the semantic aspect of quantum logic (cf. Chiara, 1986).

Definition 2.1 (Ying, 2000): A *complete orthomodular lattice* is a 7-tuple $\ell = (L, \leq, \wedge, \vee, \perp, 0, 1)$, where, $(L, \leq, \wedge, \vee, \perp, 0, 1)$ is complete lattice, in which 0 and 1 are respectively the least and greatest elements of L, $\leq$ is the partial ordering in L, $\forall A \subseteq L$, $\wedge A$ and $\vee A$ are respectively the greatest lower bound and the least upper bound of A, $\perp$ is an unary operator (called *orthocomplement*) on L, such that $\forall a, b \in L$,

1 $a \wedge a^{\perp} = 0, a \vee a^{\perp} = 1$

2 $a^{\perp\perp} = a$

3 $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$

4 $a \wedge (a^{\perp} \vee (a \wedge b)) \leq b$.

Definition 2.2 (Ying, 2000): Let X be a non-empty set. A mapping $A: X \to L$ is called an *ℓ-valued subset* of X.

For a non-empty set X, L^X will denote the set of all ℓ-valued subsets of X.

Definition 2.3 (Ying, 2000): For given two ℓ-valued subsets A and B of $X \models^{\ell} A \subseteq B$ if $A(x) \leq B(x)$, $\forall x \in X$. For given ℓ-valued sets $(A_i)_{i\in I}$, the ℓ-valued sets $(\bigcup_{i\in I} A_i)$ and $(\bigcap_{i\in I} A_i)$ are respectively given by

$$(\bigcup_{i\in I} A_i)(x) \overset{def}{=} \bigvee_{i\in I} A_i(x), \forall x \in X,$$

$$(\bigcap_{i\in I} A_i)(x) \overset{def}{=} \bigwedge_{i\in I} A_i(x), \forall x \in X.$$

Definition 2.4 (Qiu, 2004): Let X be a non-empty set and $\tau \subseteq L^X$. Then τ is an *ℓ-valued topology* on X if

1 $\phi, X \in \tau$

2 if $A, B \in \tau$, then $A \cap B \in \tau$

3 if $A_i \in \tau$, $i \in I$, then $\bigcup_{i\in I} A_i \in \tau$.

The pair (X, τ) is called an ℓ-valued topological space and ℓ-valued subset of X in τ are called *ℓ-valued open sets*. The complement of an ℓ-valued open set is called *ℓ-valued closed set*.

Definition 2.5 (Qiu, 2004): A mapping c: $L^X \to L^X$ is called an *ℓ-valued closure operator* if, $\forall A, B \in L^X$,

1 $\models^{\ell} c(\phi) \equiv \phi$

2 $\models^{\ell} A \subseteq c(A)$

3 $\models^{\ell} c(A \cup B) \equiv c(A) \cup c(B)$

4 $\models^{\ell} c(c(A)) \equiv c(A)$.

An ℓ-valued interior operator can be introduced similarly.

3 ℓ-valued approximation operators and associated ℓ-valued topologies

In this section, we indicate that study of an ℓ-valued topology can be carried out much on the same lines as done in Srivastava and Tiwari (2003), with the fact that the existence of dual topology depends on the distributivity of lattice. The concept of ℓ-valued approximation operators given in this section is similar to the concept given in She and Wang (2009), with the difference that instead of using residuated lattice, we use complete orthomodular lattice. We shall begin with the following definition.

Definition 3.1: An *ℓ-valued relation* R on a set X is a map R: $X \times X \to L$. The ℓ-valued relation R is called

1 *ℓ-valued reflexive* if $R(x, x) = 1$, $\forall x \in X$

2 *ℓ-valued symmetric* if $R(x, y) \equiv R(y, x)$, $\forall x, y \in X$

3 *ℓ-valued transitive* if $R(x, z) \geq \bigvee\{R(x, y) \wedge R(y, z) : y \in X\}$, $\forall x, z \in X$.

Definition 3.2:

1 A pair (X, R) is called an *ℓ-valued approximation space* if X is a set and R is an *ℓ-valued relation* on X.

2 For an approximation space (X, R), $\overline{c} : L^X \to L^X$, defined as,

$$\overline{c}(A)(x) \overset{def}{=} \bigvee\{R(x, y) \wedge A(y) : y \in X\},\ A \in L^X,\ x \in X$$

is called an *ℓ-valued approximation operator* on X (induced by R).

Remark 3.1: What we have named above as an ℓ-valued approximation operator on X, is a natural generalisation of an upper approximation operator (Yao, 1996). A natural generalisation of lower approximation operator to ℓ-valued lower approximation operator can also be given. However, our interest is in only the ℓ-valued topology which ℓ-valued upper approximation operator induces on X. We have chosen the name ℓ-valued approximation operator rather than an ℓ-valued upper approximation operator.

Proposition 3.1: An ℓ-valued relation R on a non-empty set X is ℓ-valued reflexive and ℓ-valued transitive if and only if (the associated) ℓ-valued approximation operator $\overline{c}$ is a Kuratowski saturated[1] ℓ-valued closure operator on X.

Proof: Let R be an ℓ-valued reflexive and ℓ-valued transitive relation on X. To show that $\overline{c}$ is an ℓ-valued closure operator on X, we need to verify the following conditions: $\forall A$, $A_j \in L^X, j \in J$:

1 $|=^\ell c(\phi) \equiv \phi$

2 $|=^\ell A \subseteq \overline{c}(A)$

3 $|=^\ell \overline{c}(\bigcup\{A_j : j \in J\}) \equiv \bigcup\{\overline{c}(A_j) : j \in J\}$

4 $|=^\ell \overline{c}(\overline{c}(A)) \equiv \overline{c}(A)$.

(1) is obviously satisfied and (2) follows using the reflexivity of R. (3) is satisfied, since for $x \in X$ and $A_j \in L^X$, $j \in J$, $\overline{c}(\bigcup\{A_j : j \in J\})(x) = \bigvee\{R(x, y) \wedge (\bigvee\{A_j(y) : j \in J\}) : y \in X\} = \bigvee\{\bigvee\{R(x, y) \wedge A_j(y) : y \in X\} : j \in J\} = \bigcup\{\overline{c}(A_j) : j \in J\}(x)$. Finally, (4) is also satisfied by using the transitivity of R as: $\forall A \in L^X$, and $\forall x \in X$, $\overline{c}(\overline{c}(A))(x) = \bigvee\{R(x, y) \wedge \overline{c}(A)y : y \in X\} = \bigvee\{R(x, y) \wedge (\bigvee\{R(y, z) \wedge A(z) : z \in X\}) : y \in X\} = \bigvee\{(\bigvee\{R(x, y) \wedge R(y, z) : y \in X\}) \wedge A(z) : z \in X\} \leq \bigvee\{R(x, z) \wedge A(z) : z \in X\} = \overline{c}(A)(x)$.

Conversely, let $\overline{c}$ be an ℓ-valued closure operator on X and $x \in X$. Then $1 = 1_x(x) \leq \overline{c}(1_x)(x)$. Thus, $\overline{c}(1_x)(x) = 1$, hence $\bigvee\{R(x, y) \wedge 1_x(y) : y \in X\} = 1 = R(x, x)$. Hence, R is an ℓ-valued reflexive. Next, let $x, z \in X$. Then $\overline{c}(\overline{c}(1_z))(x) \leq \overline{c}(1_z)(x)$, i.e., $\bigvee\{R(x, y) \wedge \overline{c}(1_z)(y) : y \in X\} \leq \overline{c}(1_z)(x)$, or that $\bigvee\{R(x, y) \wedge (\bigvee\{R(y, u) \wedge 1_z(u) : u \in X\}) : y \in X\} \leq \bigvee\{R(x, y) \wedge 1_z(y) : y \in X\}$, or that $\bigvee\{R(x, y) \wedge R(y, z) : y \in X\} \leq \bigvee\{R(x, y) \wedge 1_z(y) : y \in X\}$. Thus, $\bigvee\{R(x, y) \wedge R(y, z) : y \in X\} \leq R(x, z)$. Hence, R is an ℓ-valued transitive also.

As a consequence, the ℓ-valued approximation operator, say $\overline{c}$ on X associated with an ℓ-valued approximation space (X, R), induces a saturated ℓ-valued topology on X, which we shall denote as $\tau(X)$.

Proposition 3.2: Let R be an ℓ-valued reflexive and ℓ-valued transitive relation on X and R^* be another ℓ-valued relation on X such that $R^*(x, y) \equiv R(y, x)$. Then R^* is also an ℓ-valued reflexive and ℓ-valued transitive relation on X.

Proof: Let R^* be ℓ-valued relation on X such that $R^*(x, y) \equiv R(y, x)$, $\forall x, y \in X$. We have to show that R^* is an ℓ-valued reflexive and ℓ-valued transitive relation on X, i.e., $R^*(x, x) = 1$ and $R^*(x, z) \geq R(x, y) \wedge R(y, z)$, $\forall x, y, z \in X$. Now, $R^*(x, x) \equiv R(x, x) = 1$. Again, $R^*(x, y) \equiv R(y, x)$ and $R^*(y, z) \equiv R(z, y)$. Thus, $R^*(x, z) \geq R(z, x) \geq R(z, y) \wedge R(y, x) = R^*(y,; z) \wedge R^*(x, y) = R^*(x, y) \wedge R^*(y, z)$. Hence, R^* is an ℓ-valued reflexive and ℓ-valued transitive relation on X.

Remark 3.2: R^* being ℓ-valued reflexive and ℓ-valued transitive relation, it will induce another ℓ-valued approximation operator, say $\overline{c}^*$, on X. We shall denote by $\tau^*(X)$, the ℓ-valued topology on X induced by $\overline{c}^*$.

Recall the following from Qiu (2004).

Proposition 3.3: The following statements are equivalent:

1 L satisfies the distributive law: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $\forall a, b, c \in L$.

2 $\forall a, b \in L$, $b^\perp \vee (b \wedge a) \geq a$.

3 $\forall a, b \in L$, $b \wedge (b^\perp \vee a) \leq a$.

The relationship between the ℓ-valued topologies $\tau(X)$ and $\tau^*(X)$ are given by the following proposition.

Proposition 3.4: If L is a distributive lattice then the ℓ-valued topologies $\tau(X)$ and $\tau^*(X)$ are dual, i.e., $A \in L^X$ is $\tau(X)$-open if and only if A is $\tau^*(X)$-closed.

Proof: Let A be $\tau(X)$-open. Then $\models^\ell \overline{c}(A^\perp) \equiv A^\perp$. We show that A is $\tau^*(X)$-closed, for which it suffices to show that $\models^\ell \overline{c}^*(A) \subseteq A$. For $y \in X$, $R(y, x) \wedge A^\perp(x) \leq A^\perp(y)$. Now, $[R^*(x, y)]^\perp \vee A^\perp(y) \geq [R^*(x, y)]^\perp \vee \{R(y, x) \wedge A^\perp(x)\} = [R(y, x)]^\perp \vee \{R(y, x) \wedge A^\perp(x)\} \geq A^\perp(x) \Rightarrow [R^*(x, y)] \wedge A(y) \leq A(x)$. Thus, $\models^\ell \overline{c}^*(A) \subseteq A$. It is obvious that $\models^\ell A \subseteq \overline{c}^*(A)$. This shows that A is $\tau^*(X)$-closed.

Conversely, let A be $\tau^*(X)$-closed. Then $\models^\ell \overline{c}(A^\perp) \subseteq A^\perp$. We show that A is $\tau(X)$-open, for which we only need to show that $\models^\ell \overline{c}(A^\perp) \subseteq A^\perp$. It is obvious that $\models^\ell A \subseteq \overline{c}(A^\perp)$. Also, for all $y \in X$, $R^*(y, x) \wedge A(x) \leq A(y)$. Now, $[R(x, y)]^\perp \vee A(y) \geq [R(x, y)]^\perp \vee \{R^*(y, x) \wedge A(x)\} = [R(x, y)]^\perp \vee \{R(x, y) \wedge A(x)\} \geq A(x) \Rightarrow [R(x, y)] \wedge A^\perp(y) \leq A^\perp(x)$. Thus, $\models^\ell \overline{c}(A^\perp) \subseteq A^\perp$. Hence, A is $\tau(X)$-open.

4 ℓ-valued topologies for ℓ-valued automata

In this section, we apply the observations made so for to ℓ-valued automata. We introduce ℓ-valued topologies on the state-set of an ℓ-valued automaton, which turn out the same as introduced by Qiu (2004). Lastly, we introduce the concept of ℓ-valued core of an ℓ-valued automaton, on the lines of Bavel (1968), and Srivastava and Tiwari (2002).

Throughout this section, L stands for a distributive complete orthomodular lattice.

Definition 4.1 (Qiu, 2004): An *ℓ-valued automaton* is a triple $M = (Q, X, \delta)$, where Q is a non-empty set (of states of M), X is a monoid (the input monoid of M) whose identity shall be denoted as e, and δ is an ℓ-valued subset of $Q \times X \times Q$, i.e., a map δ: $Q \times X \times Q \to L$ such that $\forall p, q \in Q$,

$$\delta(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

and, $\delta(q, xy, p) = \vee\{\delta(q, x, r) \wedge \delta(r, y, p): r \in Q\}$, $\forall x, y \in X$.

Definition 4.2 (Qiu, 2004): Let (Q, X, δ) be an ℓ-valued automaton and $A \in L^Q$. Then *ℓ-valued source* and *ℓ-valued successor* of A are respectively defined as follows:

$$\sigma(A)(q) \overset{def}{=} \vee\left\{A(p) \wedge d(q, x, p) : p \in Q, x \in X\right\}, \text{ and}$$

$$s(A)(q) \overset{def}{=} \vee\left\{A(p) \wedge \delta(p, y, q) : p \in Q, y \in X\right\},$$

$\forall q \in Q$.

Let (Q, X, δ) be an ℓ-valued automaton. Consider an ℓ-valued relation R on Q given by

$$R(p, q) \overset{def}{=} s\left(1_{\{p\}}\right)(q), \forall p, q \in Q.$$

We can show that ℓ-valued relation R on Q is ℓ-valued reflexive and ℓ-valued transitive. To show R is ℓ-valued reflexive, we need to show that $R(p, p) = 1$. Now, $R(p, p) = s(1_{\{p\}})(p) = \vee\{1_{\{p\}}(q) \wedge \delta(q, x, p): q \in Q, x \in X\} = \delta(p, e, p) = 1$. Thus, R is ℓ-valued reflexive. To show R is ℓ-valued transitive. We need to show that $R(p, r) \geq R(p, q) \wedge R(q, r)$, $\forall p, q, r \in Q$, i.e., $s(1_{\{p\}})(r) \geq s(1_{\{p\}})(q) \wedge s(1_{\{q\}})(r)$. Now $s(1_{\{p\}})(q) \wedge s(1_{\{q\}})(r) = [\vee\{1_{\{p\}}(r) \wedge \delta(r, x, q): r \in Q, x \in X\}] \wedge [\vee\{1_{\{q\}}(p) \wedge \delta(p, y, r): p \in Q, y \in X\}] = \vee\{(1_{\{p\}}(r) \wedge \delta(r, x, q)) \wedge (\vee\{1_{\{q\}}(p) \wedge \delta(p, y, r)): r, p \in Q, x, y \in X\} = [\vee\{\delta(p, x, q)\}: x \in X] \wedge [\vee\{\delta(q, y, r)\}: y \in X] = \vee[\delta(p, x, q) \wedge \delta(q, y, r): x, y \in X] = \vee\{\delta(p, z, r): z \in X\} = \vee\{1_{\{p\}}(p) \wedge \delta(p, z, r): z \in X\} = s(1_{\{p\}})(r) = R(p, r)$. Thus, $R(p, r) = R(p, q) \wedge R(q, r)$. Hence, R is ℓ-valued transitive.

So, as in Definition 3.2, there is an ℓ-valued approximation operator on Q given by

$$\overline{c}(A)(q) \overset{def}{=} \bigvee\left\{s\left(1_{\{p\}}\right)(q) \wedge A(p) : p \in Q\right\}, \forall A \in L^Q, \forall q \in Q.$$

This operator $\overline{c}$ must be a Kuratowski saturated ℓ-valued closure operator on Q (Proposition 3.1). Thus, $\overline{c}$ induces an saturated ℓ-valued topology on Q, say $\tau(Q)$.

Remark 4.1:

1 Let (Q, X, δ) be an ℓ-valued automaton. Similar to above, if we define another ℓ-valued relation R^* on Q, given by $R^*(p, q) \overset{def}{=} \sigma(1_{\{p\}})(q),\ \forall p,\ q \in Q$. Then $R^*(p, q) \equiv R(q, p)$ and so by Proposition 3.2, R^* is also a ℓ-valued reflexive and ℓ-valued transitive relation on Q, and hence it will induce another ℓ-valued approximation operator, say $\overline{c}^*$, on Q, which will induce an ℓ-valued topology on Q, say $\tau^*(Q)$.

2 It can be seen that $|=^\ell \overline{c}(A) \equiv s(A)$ and $|=^\ell \overline{c}^*(A) \equiv \sigma(A)$. Thus, the saturated ℓ-valued topologies $\tau(Q)$ and $\tau^*(Q)$ on Q are precisely the ℓ-valued topologies $\Im_S$ and $\Im_R$ respectively, introduced in Qiu (2004).

Definition 4.3 (Qiu, 2004): $A \in L^Q$ is called an *ℓ-valued subautomaton* of ℓ-valued automaton (Q, X, δ) if

$$A(q) \le \wedge\left\{\delta(q, x, p)^\perp \vee \left(\delta(q, x, p) \wedge A(p)\right) : p \in Q, x \in X\right\}, \forall q \in Q.$$

The following ℓ-valued topological characterisation of an ℓ-valued subautomaton is given in Qiu (2004).

Proposition 4.1: $A \in L^Q$ is an ℓ-valued subautomaton of ℓ-valued automaton (Q, X, δ) if and only if $|=^\ell s(A) \equiv A$ (i.e., A is ℓ-valued $\tau^*(Q)$-open).

Definition 4.4: An ℓ-valued subautomaton $A \in L^Q$ is called *ℓ-valued separated subautomaton* of ℓ-valued automaton (Q, X, δ) if

$$A(p) \le \wedge\left\{A(q) \vee \delta(q, x, p)^\perp : q \in Q, x \in X\right\}, \forall p \in Q.$$

Proposition 4.2: $A \in L^Q$ is an ℓ-valued separated subautomaton of ℓ-valued automaton $M = (Q, X, \delta)$ if and only if A is $\tau^*(Q)$-clopen (i.e., $\tau^*(Q)$-open as well as $\tau^*(Q)$-closed).

Proof: Let A be ℓ-valued separated subautomaton of M, i.e., $\forall p \in Q$, $A(p) \le \wedge\{A(q) \vee \delta(q, x, p)^\perp: q \in Q, x \in X\}$. Then $A^\perp(p) \ge \vee\{A^\perp(q) \wedge \delta(q, x, p): q \in Q, x \in X\}$. Therefore, $s(A^\perp)(p) \le A^\perp(p)$, or that $|=^\ell s(A^\perp) \subseteq A^\perp$, i.e., $A^\perp$ is $\tau^*(Q)$-open. Hence, A is $\tau^*(Q)$-closed. Also, A being an ℓ-valued subautomaton, A is already $\tau^*(Q)$-open. Thus, A is $\tau^*(Q)$-clopen if and only if A is an ℓ-valued separated subautomaton of M.

Before stating next, recall from Willard (1970), that a topological space (X, τ) is called R_0 if $\forall x, y \in X, x \in cl(y) \Rightarrow y \in cl(x)$. A natural generalisation of this concept to an ℓ-valued topological space is as follows.

Definition 4.5: An ℓ-valued topological space (X, τ) is called R_0, if $\forall x, y \in X$, $c(1_{\{y\}})(x) \le c(1_{\{x\}})(y)$.

Definition 4.6 (Qiu, 2007): An ℓ-valued automaton (Q, X, δ) is called *retrievable* if $\forall p, q \in Q$ and $\forall x \in X$, $\delta(p, x, q) \le \vee\{\delta(q, y, p): y \in X\}$.

Proposition 4.3: An ℓ-valued automaton $M = (Q, X, \delta)$ is retrievable if and only if the ℓ-valued topology $\tau(Q)$ is R_0.

Proof: Let M be retrievable. Then $\forall p, q \in Q$ and $\forall x \in X$, $\delta(p, x, q) \leq \vee\{\delta(q, y, p)$: $y \in X\}$. We show that the ℓ-valued topology $\tau(Q)$ is R_0, for which it suffices to show that $s(1_{\{p\}})(q) \leq s(1_{\{q\}})(p)$, $\forall p, q \in Q$. As $s(1_{\{p\}})(q) = \vee\{\delta(p, x, q): x \in X\}$ and $s(1_{\{q\}})(p) = \vee\{\delta(q, y, p): y \in X\}$, from the retrievability of M, $s(1_{\{p\}})(q) \leq s(1_{\{q\}})(p)$, $\forall p, q \in Q$. Therefore, the ℓ-valued topology $\tau(Q)$ is R_0. Conversely, let $\tau(Q)$ be an R_0 ℓ-valued topology on Q. Then $\forall p, q \in Q$, $s(1_{\{p\}})(q) \leq s(1_{\{q\}})(p)$, i.e., $\delta(p, x, q) \leq \delta(q, y, p)$, $\forall x, y \in X$ and $\forall p, q \in Q$. Thus, M is retrievable.

The following is an easy generalisation of the characterisation of R_0-topological spaces.

Definition 4.7: An ℓ-valued topological space (X, τ) is R_0 if and only if each ℓ-valued open set contains the closure of each of its points.

Proposition 4.4: An ℓ-valued automaton $M = (Q, X, \delta)$ is retrievable if and only if for any ℓ-valued subautomaton $A \in L^Q$ of M, $\models^\ell \sigma(A) \equiv A$.

Proof: Let M be retrievable. Then $\tau(Q)$ is R_0 ℓ-valued topology. Thus, for any ℓ-valued open subset A of Q, $\models^\ell \sigma(A) \subseteq A$. Hence, $\models^\ell \sigma(A) \equiv A$. Conversely, let $A \in L^Q$ be an ℓ-valued subautomaton of M such that $\models^\ell \sigma(A) \equiv A$. Then as A is $\tau(Q)$ ℓ-valued open and contains the closure of its each point, the ℓ-valued topology $\tau(Q)$ is R_0. Thus, M is retrievable.

Corollary 4.1: An ℓ-valued automaton M is retrievable if and only if each ℓ-valued subautomaton of M is separated.

Proof: Follows from Propositions 4.1 and 4.4.

We close this section by introducing the concept of ℓ-valued core of an ℓ-valued automaton and show that it turns out to be an ℓ-valued interior operator for ℓ-valued topology $\tau^*(Q)$. It can be seen that some ℓ-valued automata theoretic results can be more conveniently expressed in terms of it, but as the arguments involved in proving theses are parallel to the ones used in Srivastava and Tiwari (2002), we have not carried out these exercises in this paper.

Definition 4.8: Let (Q, X, δ) be an ℓ-valued automaton and $A \in L^Q$. Then *ℓ-valued core* of A is defined as follows:

$$\mu(A)(q) \overset{def}{=} \wedge\{\delta(p, x, q)^\perp \vee A(p) : p \in Q, x \in X\}, \forall q \in Q.$$

Proposition 4.5: Let (Q, X, δ) be an ℓ-valued automaton and $A, B \in L^Q$. Then

1 $\models^\ell \mu(\phi) \equiv \phi$

2 $\models^\ell \mu(A) \subseteq (A)$

3 $\models^\ell \mu(A \cap B) \equiv \mu(A) \cap \mu(B)$

4 $\models^\ell \mu(\mu(A)) \equiv \mu(A)$.

Proof: (1) is obviously satisfied. Let $q \in Q$. Then $\mu(A)(q) \overset{def}{=} \wedge\{\delta(p, x, q)^\perp \vee A(p)$: $p \in Q, x \in X\} \leq \delta(q, e, q)^\perp \vee A(q) = A(q)$. Thus, $\mu(A)(q) \leq (A)(q)$, showing that (2) is satisfied. (3) is satisfied, since $\mu(A \cap B)(q) = \wedge\{\delta(p, x, q)^\perp \vee (A \wedge B)(p): p \in Q, x \in X\} = \wedge \{(\delta(p, x, q)^\perp \vee A(p)) \wedge (\delta(p, x, q)^\perp \vee B(p)): p \in Q, x \in X\} = (\wedge\{\delta(p, x, q)^\perp \vee A(p)$:

$p \in Q, x \in X\}) \wedge (\wedge\delta(p, x, q)^{\perp} \vee B(p): p \in Q, x \in X\}) = (\mu(A) \cap \mu(B))(q)$. That (4) is also satisfied, since $\mu(\mu(A))(q) = \wedge\{\delta(p, x, q)^{\perp} \vee \mu(A)(p): p \in Q, x \in X\} = \wedge\{\delta(p, x, q)^{\perp} \vee (\wedge\{\delta(r, y, p)^{\perp} \vee A(r): r \in Q, y \in X\}): p \in Q, x \in X\} = \wedge\{(\wedge\{\delta(r, y, p)^{\perp} \vee \delta(p, x, q)^{\perp}: p \in Q, y \in X, x \in X\}) \vee A(r): r \in Q\} = \wedge\{\delta(r, yx, q)^{\perp} \vee A(r): y \in X, x \in X, r \in Q\} = \wedge\{\delta(r, z, q)^{\perp} \vee A(r): z \in X, r \in Q] = \mu(A)(q)$. Thus, $|=^{\ell} \mu(\mu(A)) \equiv \mu(A)$.

Proposition 4.6: Let (Q, X, δ) be an ℓ-valued automaton. The map $\mu: L^Q \to L^Q$, which sends each $A \in \mathrm{L}^Q$ to $\mu(A)$, is an ℓ-valued interior operator on Q.

Proof: Follows from Proposition 4.5.

Proposition 4.7: Let (Q, X, δ) be an ℓ-valued automaton. Then the topology on Q given by the interior operator μ is precisely $\tau^*(Q)$.

Proof: To prove this proposition, it is enough to show that $|=^{\ell} \mu(A) \equiv (A)$ if and only if $|=^{\ell} \sigma(A) \equiv A$, $\forall A \in L^Q$. First, let $|=^{\ell} \sigma(A) \equiv A$. Then $\forall p \in Q$, $\sigma(A)(p) \leq A(p)$, i.e., $A(q) \wedge \delta(p, x, q) \leq A(p)$, $\forall q \in Q$ and $\forall x \in X$. We show that $|=\ell\ \mu(A) \equiv (A)$, for which it suffices to show that $|=^{\ell} A \subseteq \mu(A)$, i.e., $A(q) \leq \mu(A)(q)$, $\forall q \in Q$, i.e., $A(q) \leq \delta(p, x, q)^{\perp} \vee A(p)$, $\forall q \in Q$ and $\forall x \in X$. Now, $\delta(p, x, q)^{\perp} \vee A(p) \geq \delta(p, x, q)^{\perp} \vee (A(q) \wedge \delta(p, x, q)) = \delta(p, x, q)^{\perp} \vee (\delta(p, x, q) \wedge A(q)) \geq A(q)$. Thus, $|=^{\ell} \mu(A) \equiv (A)$. Conversely, let $|=^{\ell} \mu(A) \equiv (A)$. Then $\forall q \in Q$, $A(q) \leq \mu(A)(q)$, i.e., $A(q) \leq \delta(p, x, q)^{\perp} \vee A(p)$. We show that $|=^{\ell} \sigma(A) \equiv A$, for which we only need to show that $|=^{\ell} \sigma(A) \subseteq A$, i.e., $\forall p, q \in Q$ and $\forall x \in X$, $A(q) \wedge \delta(p, x, q) \leq A(p)$. Now, $A(q) \wedge \delta(p, x, q) \leq (\delta(p, x, q)^{\perp} \vee A(p)) \wedge \delta(p, x, q) = \delta(p, x, q) \wedge (\delta(p, x, q)^{\perp} \vee A(p)) \leq A(p)$. Thus, $|=^{\ell} \sigma(A) \equiv A$.

5 Conclusions

We have tried to establish the relationships among ℓ-valued approximation operator, ℓ-valued topology, and ℓ-valued automata, with the hope that this may offer some new insights in quantum computation. Possibly, much more can be done than what has been presented here. For example, it may possible to introduce an ℓ-valued topology on the state-set of product of two ℓ-valued automata. Also, as in Srivastava and Tiwari (2002), and Tiwari and Srivastava (2005), the decompositions of an ℓ-valued automaton can be proposed and it will be interesting to see that up to which extent these concepts depend on the distributivity of associated lattice.

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Notes

1 A Kuratowski ℓ-valued closure operator k: $L^X \to L^X$ on X is being called here saturated if the (usual) requirement $|=^\ell k(A \cup B) \equiv k(A) \cup k(B)$ is replaced by $|=^\ell k(\cup A_j) \equiv \cup k(A_j)$, where A, B, $A_j \in L^X, j \in J$.