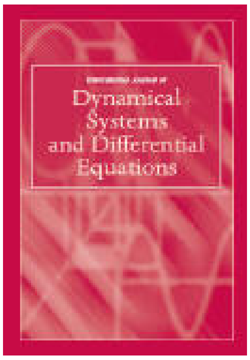

International Journal of Dynamical Systems and Differential Equations

ISSN online: 1752-3591 - ISSN print: 1752-3583
https://www.inderscience.com/ijdsde

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DOI: 10.1504/IJDSDE.2024.10069251

Article History:

Received:	16 July 2024
Last revised:	19 November 2024
Accepted:	13 December 2024
Published online:	06 March 2025

Null controllability of semilinear delay control systems

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Abstract: In this paper, the null controllability results for a class of semilinear delay control systems have been established. The fundamental semigroup is generated by using the perturbation due to the linear delay operator, which defines the fundamental solution of the system. Then, the null controllability of the associated linear delay control system has been deduced assuming that the linear non-delay control system is null controllable. The sufficient conditions have been introduced to establish the null controllability of semilinear delay control systems. The main result is proved by applying the Krasnoselskii fixed point theorem. The application of the derived result is demonstrated by a parabolic partial differential equation of diffusion process.

Keywords: null controllability; delay differential systems; fundamental solution; semilinear control systems.

Reference to this paper should be made as follows: Kumar, S. (2024) 'Null controllability of semilinear delay control systems', *Int. J. Dynamical Systems and Differential Equations*, Vol. 13, No. 4, pp.412–425.

Biographical notes: Suman Kumar is an Assistant Professor of Mathematics in the Department of Mathematics, IGNTU Amarkantak, India. He has served as member with many departmental committees. He obtained MSc in Mathematics from University of Hyderabad and PhD in Mathematics from IIT Patna, India. His research field is mathematical control theory which includes control theory of delay differential systems, abstract linear and nonlinear systems.

1 Introduction

Many real life processes observe delay in their evolution. The mathematical models of such phenomena are better represented by delay differential equations (DDEs) in which the present dynamics depends upon the historical information that incorporate delayed state function. The delayed state function acts as a feedback controller which helps in further evolution process. Therefore, the theory of DDEs has achieved diverse field of applications including biology, physics, engineering, finance, etc.

However, the modes of application of DDEs in practical systems depend upon the objectives and requirements of researchers. There are different ways to incorporate DDEs depending upon the feedback effect of the historical information, such as constant delay, state-dependent delay, time-varying delay and neutral delay. The importance of delay is observed in differential equations of integer orders and fractional orders both. Also, DDEs have significant applications in deterministic and stochastic both types of modelling of real life phenomena. Looking at the wide range of applications, several mathematicians have contributed on the existence and uniqueness of solutions of DDEs, see Driver (2012), Hale and Lunel (2013), Azbelev and Rakhmatullina (2007) and Hino et al. (1991) and references therein. The birth of control theory of DDEs was a mile stone which broadened the applications of DDEs covering physical-chemical-biological sciences, medical science, population models and network dynamics (Erneux, 2009; Smith, 2010; Piazzera, 2004; Rihan, 2021; Guo and Luo, 2002). The results presented in this paper generalises many previous controllability properties of delay systems.

DDEs are infinite-dimensional problems and there are many forms of controllability in infinite dimension control systems, viz., exact controllability (Bashirov, 2021), approximate controllability (Kumar et al., 2022a, 2022b; Raja et al., 2022; Kumar, 2023; Johnson et al., 2024), null controllability, trajectory controllability (Chalishajar et al., 2010), complete controllability (Shukla et al., 2015), constrained controllability (Sikora and Klamka, 2017), relative controllability (Wang et al., 2017), etc. This work will present the null controllability of linear and semilinear delay control systems. The null controllability is a special case of exact controllability in which the trajectory of control systems traverses from the given initial state to the zero or null state in the state space. In respect of applications to the exactly controllable systems, the arbitrary desired state can be achieved by the composition of trajectories from the given initial state to the zero state and then from the zero state to the desired state. Mathematical developments of null controllability briefly discussed in the below literature survey have motivated to explore the analysis of null controllability for delay differential systems.

There have been pioneering studies in the null controllability of delay systems since decades of seventies (Underwood and Young, 1979; Chukwu, 1980, 1984, 1987; Balachandran and Dauer, 1990, 1996; Dauer et al., 1998; Vieru, 2005; Chen, 2015, 2016; Davies and Haas, 2015; Du and Xu, 2018; Sathiyaraj and Balasubramaniam, 2019; Xu et al., 2020; Boujallal and Kassara, 2021; Azamov et al., 2023). Underwood and Young (1979) established null controllability for various types of linear and nonlinear functional differential equations under unlimited and square summable controls on finite intervals. They have described through an example that the null controllability of linear approximation of a nonlinear system implies the local null controllability of the main nonlinear system. Chukwu (1980, 1984) proved that if the control-free system is uniformly asymptotically stable and the linear control system is controllable with only square integrable controls in finite intervals, then the nonlinear delay system is Euclidean null controllable. Dauer with co-researchers (Balachandran and Dauer, 1996; Dauer et al., 1998) discussed the results for nonlinear neutral system with distributed and time-varying delays in control variables by using the Schauder fixed point theorem. Vieru (2005) presented characterisations for the null controllability of linear systems in Banach spaces and reflexive Banach spaces which was a problem left open by Chen and Qin (2002). Chen (2015, 2016) established null controllability for the Korteweg-de Vries equation with finite number of constraints on the state and the control variables by using an adapted Carleman inequality. Davies and Haas (2015) developed the null controllability of neutral control systems with infinite delays for controls lying in m-dimensional unit cube. Du and Xu (2018) derived the null controllability of the coupled degenerate systems with two controls, and then for one control by using the Carleman estimate and the observability inequality for the adjoint system. Sathiyaraj and Balasubramaniam (2019) applied the Schauder fixed point theorem to prove the null controllability of nonlinear fractional stochastic large-scale neutral systems in finite-dimensional space. Xu et al. (2020) established the well-posedness and the approximate null controllability of the linearised system of degenerate semilinear parabolic control system. Boujallal and Kassara (2021) presented a unified approach to investigate the asymptotic null controllability of semilinear partial differential equations with mixed input-state constraints. Azamov et al. (2023) established the stability and the null controllability for the infinite linear systems in l^2-space.

The study on controllability is influenced by the fixed point theory. There are many fixed point theorems with various applications in the nonlinear analysis. In the discussion of controllability of semilinear systems, the fixed point theorems used at large are due to Banach, Schauder, Schaefer, Brouwer and Browder. In the recent research, mathematicians have explored Bohnenblust-Karlin fixed point theorem (Raja et al., 2022) and Krasnoselskii fixed point theorem (Johnson et al., 2024) for approximate controllability results.

In this work, the null controllability of semilinear delay control systems has been established by using the Krasnoselskii fixed point theorem. The motivation of analytical discussion comes from the pioneer works of Chukwu (1980, 1984, 1987), Balachandran and Dauer (1990, 1996) and Engel and Nagel (1995). The problem formulation and solution description are presented in Section 2. The results on null controllability have been established in Section 3. The application of the obtained results is demonstrated in Section 4 by an example of parabolic control system.

2 System description and fundamental solution

Let us consider the Hilbert spaces X and U with norms $||\cdot||$ and $||\cdot||_U$, respectively. The norm $||x||_{C([-\tau,t];X)} = \sup_{s\in[-\tau,t]} ||x(s)||$, $\tau > 0$, is defined on the set $C([-\tau,t];X)$ of all continuous functions from $[-\tau,t]$ into X. Consider the semilinear delay control system as follows

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x_t + Bu(t) + f(t, x_t, u(t)),\ t \in (0,T], \\ x_0 &= \mu \text{ on } [-\tau, 0], \end{aligned} \tag{2.1}$$

where $x(t) \in X,\ u(t) \in U,\ \tau > 0$ is maximum delay, $x_t \in C([-\tau,0];X)$ defined by $x_t(\theta) = x(t+\theta)$, $A_0 : D(A_0) \subset X \to X$ is densely defined closed linear operator, $A_1 : C([-\tau,0];X) \to X$ and $B : U \to X$ are linear bounded operators, $\mu \in C([-\tau,0];X)$, and $f : [0,T] \times C([-\tau,0];X) \times U \to X$ is a nonlinear map.

Denote the operator norm $||\cdot||_{\mathscr{L}(X,Y)}$ in the space $\mathscr{L}(X,Y)$ of all bounded linear operators from space X into an space Y, and $||\cdot||_{\mathscr{L}(X)}$ for $X = Y$. Let us impose the following assumptions:

A1 A_0 generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, and there exists $M_0 \geq 1$ such that $||S(t)||_{\mathscr{L}(X)} \leq M_0$.

A2 There exists $M_1 > 0$ such that $||A_1||_{\mathscr{L}(C([-\tau,0];X),X)} \leq M_1$.

A3 There exists $L_B > 0$ such that $||Bu(t)|| \leq L_B||u(t)||_U$ for all $u \in L^2([0,T];U)$.

A4 f is continuous in $[0,T]$ and Lipschitz in $C([-\tau,0];X) \times U$ with constant $L_f > 0$ satisfying

$$||f(t,x_t,u(t)) - f(t,y_t,v(t))|| \leq L_f(||x_t - y_t||_{C([-\tau,0];X)} + ||u(t) - v(t)||_U)$$

and $f(t,0,0) = 0$ for all t.

A5 There exists $C_\mu > 0$ such that $||\mu||_{C([-\tau,0];X)} \leq C_\mu$.

Let us consider the control-free linear delay system associated to equation (2.1) given by

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_1x_t, \\ &x_0 = \mu \text{ on } [-\tau,0]. \end{aligned} \tag{2.2}$$

Define an operator $A : C([-\tau,0];X) \to X$ by $Ax = \dot{x}$ with domain

$$D(A) := \{x \in C^1([-\tau,0];X) : x(0) \in D(A_0) \text{ and } \dot{x}(0) = A_0x(0) + A_1x_0\},$$

where A_0 and A_1 satisfy assumptions A1 and A2, respectively.

By using the Desch-Schappacher perturbation theorem [Engel and Nagel (1995), Theorem 6.1], A generates a C_0-semigroup $\{P(t)\}_{t\geq 0}$ on $C([-\tau,0];X)$ given by

$$[P(t)\mu](0) = \begin{cases} \mu(t), & \text{if } t \leq 0, \\ S(t)\mu(0) + \int_0^t S(t-s)A_1P(s)\mu ds, & \text{if } t > 0. \end{cases} \tag{2.3}$$

It is called the fundamental semigroup. Clearly, $P(0) = I$, and it satisfies the translation property as follows

$$[P(t)\mu](s) = \begin{cases} \mu(t+s), & \text{for } t+s \leq 0, \\ [P(t+s)\mu](0), & \text{for } t+s > 0. \end{cases} \tag{2.4}$$

We get from equations (2.3) and (2.4) that

$$||P(t)\mu||_{C([-\tau,0];X)} \leq M_0||\mu||_{C([-\tau,0];X)} + M_0M_1\int_0^t ||P(r)\mu||_{C([-\tau,0];X)}dr.$$

By Gronwall's inequality, this implies

$$||P(t)\mu||_{C([-\tau,0];X)} \leq e^{M_0M_1}M_0||\mu||_{C([-\tau,0];X)} \ \forall \ \mu \in C([-\tau,0];X).$$

Thus,

$$||P(t)||_{\mathscr{L}(C([-\tau,0];X))} \leq e^{M_0M_1}M_0 = M_P. \tag{2.5}$$

The last equality $e^{M_0M_1}M_0 = M_P$ implies that $M_0 \leq M_P$.

Definition 2.1: A continuous function $x \in C([-\tau, T]; X)$ given by

$$x(t) = \begin{cases} \mu(t), & \text{if } t \leq 0, \\ S(t)\mu(0) + \int_0^t S(t-s)\big[A_1 x_s + Bu(s)\big]ds & \\ + \int_0^t S(t-s) f(s, x_s, u(s))ds, & \text{if } t > 0 \end{cases} \tag{2.6}$$

is called a mild solution of equation (2.1).

From assumptions A1–A4, the unique mild solution of equation (2.1) exists (Hale and Lunel, 2013; Engel and Nagel, 1995; Sukavanam and Tafesse, 2011) and is given by equation (2.6). The mild solution is also defined in terms of the fundamental semigroup as below (Sukavanam and Tafesse, 2011; Wang, 2005).

Definition 2.2: The mild solution (2.6) with the fundamental semigroup $\{P(t)\}_{t \geq 0}$ written as

$$x(t) = \begin{cases} \mu(t), & \text{if } t \leq 0, \\ [P(t)\mu](0) + \int_0^t S(t-s)Bu(s)ds & \\ + \int_0^t S(t-s) f(s, x_s, u(s))ds, & \text{if } t > 0 \end{cases} \tag{2.7}$$

is called the fundamental solution of equation (2.1).

The linear delay control system corresponding to equation (2.1) is

$$\begin{aligned} &\dot{y}(t) = A_0 y(t) + A_1 y_t + Bu(t), \\ &y_0 = \mu \text{ on } [-\tau, 0]. \end{aligned} \tag{2.8}$$

The mild solution of equation (2.8) is

$$y(t; u) = \begin{cases} \mu(t), & \text{if } t \leq 0, \\ S(t)\mu(0) + \int_0^t S(t-s)A_1 y_s ds & \\ + \int_0^t S(t-s)Bu(s)ds, & \text{if } t > 0, \end{cases} \tag{2.9}$$

and its fundamental solution is given by

$$y(t; u) = [P(t)\mu](0) + \int_0^t S(t-s)Bu(s)ds. \tag{2.10}$$

The linear non-delay control system is

$$\begin{aligned} &\dot{z}(t) = A_0 z(t) + Bu(t), \ t > 0, \\ &z(0) = \mu(0), \end{aligned} \tag{2.11}$$

and its mild solution is

$$z(t; u) = S(t)\mu(0) + \int_0^t S(t-s)Bu(s)ds. \tag{2.12}$$

3 Null controllability

Let us denote the solution of the semilinear system (2.1) as $x(t, \mu, u, f)$, the solution of equation (2.8) as $x(t, \mu, u, 0) = y(t; u)$ and the solution of equation (2.2) as $x(t, \mu, 0, 0) = y(t; 0)$.

Definition 3.1 (null controllability): The semilinear control system (2.1) is said to be null controllable on $[0, T]$ if for each $\mu \in C([-\tau, 0]; X)$ there is a control $u \in L^2([0, T]; U)$ such that $x(0, \mu, u, f) = \mu(0)$ and $x(T, \mu, u, f) = 0$.

Define the controllability operator $\mathcal{B}^T : L^2([0, T]; U) \to X$ as

$$\mathcal{B}^T u = \int_0^T S(T-s)Bu(s)ds \tag{3.1}$$

and denote the nonlinear part of $x(T, \mu, u, f)$ as

$$\mathcal{F}^T(f) = \int_0^T S(T-s)f(s, x_s, u(s))ds. \tag{3.2}$$

Definition 3.2 (reachable set): The reachable set of equation (2.8) is defined as

$$\mathscr{R}_T = \left\{ \mathcal{B}^T u \mid u \in L^2([0, T]; U) \right\}.$$

Theorem 3.3: If the non-delay linear control system (2.11) is null controllable on $[0, T]$, then the linear delay control system (2.8) is null controllable.

Proof: From the fundamental solution (2.10), we have $y(t; 0) = [P(t)\mu](0)$, where $P(t)$ is bounded linear operator from $C([-\tau, 0]; X)$ into itself. Since equation (2.11) is null controllable on $[0, T]$, we get $u \in L^2([0, T]; U)$ satisfying

$$z(T; u) = S(T)\mu(0) + \mathcal{B}^T u = 0.$$

Then, $S(T)\mu(0) = -\mathcal{B}^T u$ and it implies $S(T)X \subset \mathcal{B}^T(L^2([0, T]; U))$.

Now, for each $\mu \in C([-\tau, 0]; X)$, we want to find a bounded linear operator $H : C([-\tau, 0]; X) \to L^2([0, T]; U)$ such that $u = H\mu$ satisfies $y(0, H\mu) = \mu(0)$ and $y(T, H\mu) = 0$.

Let the null space of $\mathcal{B}^T$ be N and its orthogonal complement in $L^2([0, T]; U)$ be $N^\perp$. Then, the operator $\mathcal{B}_0 : N^\perp \to X_\mathcal{B}$ is bijective linear operator, where $\mathcal{B}_0 \equiv \mathcal{B}^T|_{N^\perp}$ and $X_\mathcal{B} = \mathcal{B}^T(L^2([0, T]; U))$. So, $\mathcal{B}_0^{-1}$ exists.

Let us define an operator $H : C([-\tau, 0]; X) \to L^2([0, T]; U)$ by

$$H\mu = -\mathcal{B}_0^{-1}([P(T)\mu](0)).$$

Then,

$$\begin{aligned} y(T, H\mu) &= [P(T)\mu](0) + \mathcal{B}^T(H\mu) \\ &= [P(T)\mu](0) + \mathcal{B}^T(-\mathcal{B}_0^{-1}[P(T)\mu](0)) = 0. \end{aligned}$$

Next, we prove that H is bounded. Suppose that $\{\mu_n\}$ is a convergent sequence in $C([-\tau,0];X)$ such that $\{H\mu_n\}$ converges in $L^2([0,T];U)$, and let $\mu=\lim_{n\to\infty}\mu_n$, $u=\lim_{n\to\infty}H\mu_n$, $u_n=H\mu_n$.

Since $N^{\perp}$ is closed in $L^2([0,T];U)$, $u\in N^{\perp}$ and

$$[P(T)\mu](0)+\mathcal{B}^T u=[P(T)\mu](0)+\lim_{n\to\infty}\mathcal{B}^T(H\mu_n)$$
$$=\lim_{n\to\infty}\left([P(T)\mu_n](0)+\mathcal{B}^T(H\mu_n)\right)=0,$$

we get, $u=-\mathcal{B}_0^{-1}([P(t)\mu](0))=H\mu$. Hence, by the closed graph theorem, H is bounded. This completes the proof. □

The adjoint operator $(\mathcal{B}^T)^*:X\to L^2([0,T];U)$ of $\mathcal{B}^T$ is given by

$$((\mathcal{B}^T)^*x)(t)=B^*S^*(T-t)x.$$

Let us define the controllability grammian $G^T:X\to X$ by

$$G^Tx=\mathcal{B}^T(\mathcal{B}^T)^*x=\int_0^T S(T-s)BB^*S^*(T-s)xds. \tag{3.3}$$

Definition 3.4 (complete): The linear delay control system is called complete on $[0,T]$ if $\exists\ \delta>0$ such that $\mathcal{N}_\delta(0)\subset\mathscr{R}_T$, where $\mathcal{N}_\delta(0)$ is open ball of radius δ and centre 0. In other words, $0\in\text{Int}\ \mathscr{R}_T$.

Krasnoselskii fixed point theorem (Smart, 1974): Let Ω be a closed convex nonempty subset of a Banach space X. Suppose that Q_1 and Q_2 map Ω into X, $Q_1x+Q_2y\in\Omega$ for all $x,y\in\Omega$, Q_1 is contraction map, and Q_2 is compact continuous map. Then there exists $x\in\Omega$ such that $Q_1x+Q_2x=x$.

Theorem 3.5: Let us put the following assumptions:

a The linear delay control system is complete, i.e., $0\in\text{Int}\mathscr{R}_T$.

b $||f(t,x_t,u(t))||\le e^{-\beta t},\ t\ge 0$ for some $\beta>0$.

c $e^{M_P^2L_BM_GL_f}M_P^3L_B^2M_GL_fT^2<1$.

Then the semilinear delay control system (2.1) is null controllable.

Proof: From assumption a, the linear delay control system is complete. So, $0\in\text{Int}\ \mathscr{R}_T$ and $G^Tz=0\Rightarrow z=0$. Then, G^T is invertible and the inverse operator $(G^T)^{-1}$ is bounded. So, there exists $M_G>0$ such that $||(G^T)^{-1}||\le M_G$.

The adjoint operator of $\mathcal{B}^T$ is the operator $(\mathcal{B}^T)^*:X\to L^2([0,T];U)$ given by $(\mathcal{B}^T)^*=B^*S^*(T-\cdot)$ with $((\mathcal{B}^T)^*x)(s)=B^*S^*(T-s)x$ on $[0,T]$ and $x\in X$.

For the solution pair (x,u), let us take a control

$$u(t)=-B^*S^*(T-t)(G^T)^{-1}g(t),\ t\in[0,T], \tag{3.4}$$

and

$$x_t(0,\mu,u,f) = [P(t)\mu](0) + \int_0^t S(t-s)Bu(s)ds \\ + \int_0^t S(t-s)f(s,x_s,u(s))ds, \tag{3.5}$$

where

$$g(t) = [P(t)\mu](0) + \int_0^t S(t-s)f(s,x_s,u(s))ds.$$

Then, equation (3.5) is a solution of equation (2.1) corresponding to u and satisfies $x(T) = 0$. We wish to verify that u is admissible control. First, we have

$$\begin{aligned} ||g(t)|| &\le ||P(t)\mu||_{C([-\tau,0];X)} + \int_0^t ||S(t-s)f(s,x_s,u(s))||ds \\ &\le M_P||\mu||_{C([-\tau,0];X)} + M_0 \int_0^t ||f(s,x_s,u(s))||ds \\ &\le M_P C_\mu + M_P \int_0^t e^{-\beta s} ds \\ &\le M_P\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big). \end{aligned}$$

Then, from equation (3.4), we get

$$||u||_2 \le L_B M_G M_P^2 \sqrt{T}\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big).$$

Let us define an operator $Q : C([-\tau,T];X) \to C([-\tau,T];X)$ by

$$(Qx)(t) = [P(t)\mu](0) + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x_s,u(s))ds.$$

Take operators $Q_1, Q_2 : C([-\tau,T];X) \to C([-\tau,T];X)$ as follows

$$(Q_1x)(t) = [P(t)\mu](0) + \int_0^t S(t-s)Bu(s)ds$$

and

$$(Q_2x)(t) = \int_0^t S(t-s)f(s,x_s,u(s))ds$$

so that $Qx = Q_1x + Q_2x$. Consider a ball $\Omega = \{x \in C([-\tau,T];X) : ||x||_{C([-\tau,T];X)} \le R\} \subset C([-\tau,T];X)$. Now, we claim that $Q : \Omega \to C([-\tau,T];X)$ has a fixed point in Ω. For the Krasnoselskii fixed point theorem, we verify that: $Q_1x + Q_2y \in \Omega$ for $x, y \in \Omega$, Q_1 is contraction map, and Q_2 is compact continuous map.

1 Let $x, y \in \Omega$ be two trajectories corresponding to controls $u_1, u_2 \in L^2([0,T];U)$. Then

$$\begin{aligned}
||Q_1x + Q_2y|| &\le M_P C_\mu + M_P L_B \sqrt{T} L_B M_G M_P^2 \sqrt{T}\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big) \\
&+ M_P \frac{1-e^{-\beta T}}{\beta} \le M_P\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big) \\
&+ M_P^3 L_B^2 M_G T\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big) \\
&\le M_P\Big(C_\mu + \frac{1-e^{-\beta T}}{\beta}\Big)(1 + M_P^2 L_B^2 M_G T) \le R.
\end{aligned}$$

2 For $x, y \in \Omega$ as in 1, we have

$$\begin{aligned}
||Q_1x - Q_1y|| &\le \int_0^t ||S(t-s)B(u_1(s) - u_2(s))||ds \\
&\le M_P L_B \int_0^t ||u_1(s) - u_2(s)||ds. \qquad (3.6)
\end{aligned}$$

Since $u_1(t) = -B^*S^*(T-t)(G^T)^{-1}g_1(t)$ and $u_2(t) = -B^*S^*(T-t)(G^T)^{-1}g_2(t)$, where

$$g_1(t) = [P(t)\mu](0) + \int_0^t S(t-s)f(s, x_s, u_1(s))ds$$

and

$$g_2(t) = [P(t)\mu](0) + \int_0^t S(t-s)f(s, y_s, u_2(s))ds,$$

therefore

$$\begin{aligned}
u_1(t) - u_2(t) &= -B^*S^*(T-t)(G^T)^{-1}(g_1(t) - g_2(t)) \\
&= -B^*S^*(T-t)(G^T)^{-1}\int_0^t S(t-s)\Big(f(s, x_s, u_1(s)) \\
&- f(s, y_s, u_2(s))\Big)ds.
\end{aligned}$$

This gives

$$\begin{aligned}
||u_1(t) - u_2(t)|| &\le M_P L_B M_G \int_0^t ||S(t-s)\big(f(s, x_s, u_1(s)) - f(s, y_s, u_2(s))\big)||ds \\
&\le M_P^2 L_B M_G \int_0^t ||f(s, x_s, u_1(s)) - f(s, y_s, u_2(s))||ds \\
&\le M_P^2 L_B M_G L_f \int_0^t (||x_s - y_s||_{C([-\tau,0];X)} + ||u_1(s) - u_2(s)||)ds \\
&\le M_P^2 L_B M_G L_f \int_0^t ||x_s - y_s||_{C([-\tau,0];X)}ds
\end{aligned}$$

$$
\begin{aligned}
&+ M_P^2 L_B M_G L_f \int_0^t ||u_1(s) - u_2(s)||ds \\
&\leq M_P^2 L_B M_G L_f T ||x - y||_{C([-\tau,T];X)} \\
&+ M_P^2 L_B M_G L_f \int_0^t ||u_1(s) - u_2(s)||ds.
\end{aligned}
$$

By Gronwall's inequality,

$$
||u_1(t) - u_2(t)||_U \leq e^{M_P^2 L_B M_G L_f} M_P^2 L_B M_G L_f T ||x - y||_{C([-\tau,T];X)}. \tag{3.7}
$$

Then, from equation (3.6), we get

$$
||Q_1 x - Q_1 y|| \leq e^{M_P^2 L_B M_G L_f} M_P^3 L_B^2 M_G L_f T^2 ||x - y||_{C([-\tau,T];X)}.
$$

Since $e^{M_P^2 L_B M_G L_f} M_P^3 L_B^2 M_G L_f T^2 < 1$, therefore Q_1 is a contraction map.

3 For $x, y \in \Omega$ as in 1, we have

$$
\begin{aligned}
||Q_2 x - Q_2 y|| &\leq \int_0^t ||S(t-s)(f(s, x_s, u_1(s)) - f(s, y_s, u_2(s)))||ds \\
&\leq M_P L_f \int_0^t (||x_s - y_s||_{C([-\tau,0];X)} + ||u_1(s) - u_2(s)||_U) ds \\
&\leq M_P L_f \int_0^t \Big(||x - y||_{C([-\tau,T];X)} \\
&+ e^{M_P^2 L_B M_G L_f} M_P^2 L_B M_G L_f T ||x - y||_{C([-\tau,T];X)} \Big) ds \\
&\leq M_P L_f T (1 + e^{M_P^2 L_B M_G L_f} M_P^2 L_B M_G L_f T) ||x - y||_{C([-\tau,T];X)}.
\end{aligned}
$$

This shows that Q_2 is Lipschitz, and hence continuous. To show the compactness, we shall verify that $Q_2(\Omega)$ is uniformly bounded and equicontinuous. For any $x \in \Omega$ corresponding to some control $u \in L^2([0,T];U)$, we get

$$
\begin{aligned}
||(Q_2 x)(t)|| &\leq M_P \int_0^t ||f(s, x_s, u(s))||ds \\
&\leq M_P \frac{1 - e^{-\beta t}}{\beta} \text{for all } t > 0.
\end{aligned}
$$

This implies that $||Q_2 x||_{C([-\tau,T];X)} \leq M_P \frac{1 - e^{-\beta T}}{\beta}$. Hence, $Q_2(\Omega)$ is uniformly bounded. Next, for $t_1, t_2 \in [0,T]$, we have

$$
\begin{aligned}
||(Q_2 x)(t_2) - (Q_2 x)(t_1)|| &\leq \int_{t_1}^{t_2} ||S(t-s) f(s, x_s, u(s))||ds \\
&\leq M_P \left| \int_{t_1}^{t_2} ||f(s, x_s, u(s))||ds \right| \\
&\leq M_P \left| \frac{e^{-\beta t_1} - e^{-\beta t_2}}{\beta} \right|
\end{aligned}
$$

$$\leq \frac{M_P}{\beta} \frac{|e^{\beta t_2} - e^{\beta t_1}|}{e^{\beta(t_1+t_2)}}$$
$$\leq \frac{M_P}{\beta}|\beta t_2 - \beta t_1|$$
$$\leq M_P|t_2 - t_1|.$$

This shows that $Q_2(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem Nair (2001), $Q_2(\Omega)$ is relatively compact in $C([-\tau, T]; X)$, which renders Q_2 is a compact operator. Hence, the operator $Q = Q_1 + Q_2$ has a fixed point in Ω by the Krasnoselskii fixed point theorem. It is the solution of equation (2.1) under control (3.4) satisfying $x(T, \mu, u, f) = 0$. This completes the proof of theorem. □

4 Application

Example 4.1: Consider a diffusion process given by the parabolic control system as follows

$$\frac{\partial y}{\partial t}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + y(x, t-\tau) + b(x)\int_0^t u(s,x)ds$$
$$+ f(t, y_t(x,\cdot), u(x,t)),\ t \in [0,T],\ x \in [0,\pi], \tag{4.1a}$$
$$y(0,t) = y(\pi,t) = 0,\ t \in [0,T], \tag{4.1b}$$
$$y_0(x,\theta) = \mu(x,\theta),\ \theta \in [-\tau, 0],\ x \in [0,\pi], \tag{4.1c}$$

where $y(x,t)$ is density at point x and time t; $b \in L^\infty([0,\pi]; \mathbb{R}^+)$ is weight function for control u.

Let $X = L^2[0,\pi] = U$ represent the state and control space both. Let $y(\cdot,t) \in X$ be state and $u(\cdot,t) \in U$ be control. Define operator A_0 by $A_0 y = \frac{d^2y}{dx^2}$ with

$$D(A_0) = \{y \in L^2[0,\pi] :\ y, \frac{dy}{dx} \text{ are absolutely continuous,}$$
$$\frac{d^2y}{dx^2} \in L^2[0,\pi] \text{ and } y(0) = 0 = y(\pi)\}.$$

Then, $\{\xi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx) : 0 \leq x \leq \pi\}$ is an orthonormal basis for $L^2[0,\pi]$ associated to the eigenspectrum $\{\lambda_n = -n^2\}$, $n \in \mathbb{N}$, of operator A_0. Further, A_0 generates the strongly continuous semigroup $\{S(t)\}_{t\geq 0}$. For $y = \sum_{n=0}^{\infty}\langle y, \xi_n\rangle\xi_n$, the semigroup is given as

$$S(t)y = \sum_{n=0}^{\infty} e^{-n^2 t}\langle y, \xi_n\rangle\xi_n,$$

satisfying $||S(t)||_{op} \leq 1 = M_0$.

Take $A_1 : C([-\tau, 0]; X) \to X$ defined by $A_1 y_t = y(t-\tau)$. Clearly A_1 is bounded linear operator with $M_1 = 1$.

Define B as $Bu(x,t) = b(x)\int_0^t u(s,x)ds$. Thus, B is bounded linear operator with $L_B = \text{ess sup}_{x\in[0,\pi]} b(x)\sqrt{T}$.

Let the nonlinear function f be given by

$$f(t, y_t(\cdot), u(t)) = \frac{1}{n^2}\frac{||y(t-\tau)||}{1+||y(t-\tau)||}\xi_n(x) + \frac{1}{(n+k)^2}\frac{||u(t)||}{1+||u(t)||}\xi_{n+k}(x)$$

for chosen $n, k \in \mathbb{N}$. For $y_t, z_t \in C([-\tau, 0]; X)$ and controls $u, v \in L^2([0, T]; U)$, we have

$$\begin{aligned} &||f(t, y_t, u(t)) - f(t, z_t, v(t))||_X \\ &\leq \frac{1}{n^2}\big(||y_t(\cdot) - z_t(\cdot)||_{C([-\tau,0];X)} + ||u(t) - v(t)||\big). \end{aligned}$$

This implies that the Lipschitz condition holds with $L_f \geq \frac{1}{n^2}$. Moreover, we have

$$||f(t, y_t, u(t))|| \leq \frac{1}{n^2}(||\xi_n(x)|| + ||\xi_{n+k}(x)||) \leq \frac{2}{n^2}\sqrt{\frac{2}{\pi}}.$$

For sufficiently large n, we get $\frac{2}{n^2}\sqrt{\frac{2}{\pi}} \leq e^{-\beta t}$ for $t \in [0, T]$. Hence, the semilinear parabolic control system (4.1) is null controllable by Theorem 3.5.

5 Conclusions

The fundamental solution for the semilinear control system with delay has been described. The null controllability of the retarded linear and the semilinear control systems have been established under natural assumptions on the operators and the nonlinear term. The delay term is considered in linear and nonlinear forms. The linear form of delay generates the perturbed semgigroup, which gives the fundamental solution of the system. By using the fundamental solution, the null controllability of the semilinear delay system has been established with the help of Krasnoselskii fixed point theorem. There is future option to extend this analysis for stochastic delay control systems.

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