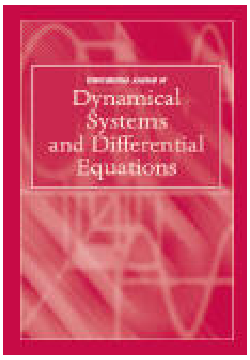

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Existence of positive solutions for a fourth-order differential equation with p-Laplacian and Riemann-Stieltjes integral boundary conditions

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Abstract: This paper investigates a problem concerning the positive solutions for a fourth-order differential equation with p-Laplacian and Riemann-Stieltjes integral boundary conditions. Krasnosel'skii proposed a theorem in 1960, which has been known as Krasnosel'skii's fixed-point theorem. This paper will use this theorem to derive two significant conclusions regarding the problem to be discussed. It is worth noting that, unlike other problems, the equation studied in this paper has a nonlinear term f that includes the first-order derivative of the unknown function.

Keywords: p-Laplacian; Riemann-Stieltjes integral boundary conditions; cone; positive solution; fixed point theorem.

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1 Introduction

Introducing the p-Laplacian operator into fourth-order differential equations adds complexity but also enhances the ability to capture nonlinear phenomena observed in real-world applications. The operator's nonlinearity presents significant mathematical challenges, making the analysis and solution of such equations an essential aspect of modern mathematical research. Introducing the p-Laplacian operator into fourth-order differential equations adds complexity to the problem, but it also enhances the ability to capture certain nonlinear phenomena observed in reality. For BVP with p-Laplacian, see (Feng et al., 2008, 2015; Ji and Ge, 2008; Ji et al., 2009, 2008; Sun, 2016b; Sun et al., 2008; Sun, 2011; Yang and Yan, 2010).

In differential equations, Riemann-Stieltjes integral boundary conditions involve using the Riemann-Stieltjes integral to represent boundary conditions. This method is particularly suitable for handling functions with variable weights, allowing the upper or lower limit of integration to be another function rather than a constant. In differential equations, using Riemann-Stieltjes integral boundary conditions can effectively represent the cumulative or distributed effects of a system. Discussions on problems featuring integral boundaries can be found in Hao et al. (2011), Jankowski (2012), Sun (2016b) and Wang and Zhang (2006).

In the work cited as Sun (2016b), Sun explored the following problem:

$$\begin{cases} v^{(3)}(x) + q(x)f(x, v(\alpha(x))) = 0, & x \in (0,1), \\ v(0) = \beta v(\eta) + \lambda[v], & v^{(2)}(0) = 0, \\ v(1) = \gamma v(\eta) + \lambda[v]. \end{cases}$$

Given that η is a real number between 0 and 1, $0 \le \gamma^2 \le \beta < \gamma < 1$, furthermore, for all x in the interval from 0 to 1, $\lambda[v] = \int_0^1 v(x)d\Lambda(x)$ illustrating the integration process via Riemann-Stieltjes.

In 2016, Sun in Wang and Zhang (2006) investigated the following BVP.

$$\begin{cases} \left(\phi_p\left(v^{(2)}(x)\right)\right)^{(2)}(x) = \lambda q(x) f \\ \left(x, v(x), v'(x), v^{(2)}(x), v^{(3)}(x)\right), 0 < x < 1, \\ v(0) = v(1) = \int_0^1 v(s)dg(s), \\ \phi_p\left(v^{(2)}(0)\right) = \phi_p\left(v^{(2)}(1)\right) \\ = \int_0^1 \phi_p\left(v^{(2)}(s)\right) dh(s), \end{cases}$$

where λ is a positive number, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $\frac{1}{q} + \frac{1}{p} = 1$.

Motivated by these studies, in this paper we attempt to investigate the solutions of

$$\begin{cases} (\phi_p(y'''(x)))' + f(x, y, y') = 0, x \in [0,1], \\ y'(0) = \alpha y(\eta) + \lambda[y], & y''(0) = 0, \\ y(1) = \delta y(\eta) + \lambda[y], & y'''(0) = 0. \end{cases} \tag{1.1}$$

and try to prove that its solutions are greater than 0. For equation (1.1), we have the following notes and explanations, $\phi_p(s) = |s|^{p-2}s$, with $p > 1$ being a p-Laplacian operator, and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, η is a real number between 0 and 1, $0 < \alpha < \delta < 1$, $\lambda[y] = \int_0^1 y(x)d\Lambda(x)$ is a Riemann-Stieltjes integral function, and Λ is an appropriate bounded variation function.

The positive solution we mentioned above refers to: y is a continuous function on [0, 1]. If x belongs to [0, 1] and is a solution of the BVP (1.1), and $y(x) > 0$, then x is a positive solution of equation (1.1).

The study is motivated by the need to establish positive solutions for fourth-order differential equations involving p-Laplacian operators, which find applications in stability theory and elastic beams. For a comprehensive understanding of fourth-order BVP, readers are referred to Wang et al. (2019), Yao (2008), Yao (2004), Yao (2008) and related literature.

This paper exhibits distinctive features. In contrast to the methodology employed in Ji et al. (2009), we use the fixed-point theorem instead of the monotone iterative technique. Diverging from the discussions in Hao et al. (2011), we extend our analysis to encompass problem (1.1) with integral boundary conditions. Furthermore, in comparison to Wang and Zhang (2006), we tackle the inclusion of the nonlinear p-Laplacian operator. Notably, this paper presents, for the first time, the properties of the Green's function of BVP (1.1), as outlined in Lemma 2.3.

To our knowledge, no current literature covers fourth-order BVP involving Stieltjes integral boundary conditions and p-Laplacian through the application of fixed-point theorem. Consequently, this paper contributes to and extends the current body of knowledge by refining and generalising certain results in the field.

2 Preliminaries

Definition 2.1: A mapping or operator is called a compact operator if it maps bounded sets from a topological space to relatively compact sets in the target space.

Definition 2.2: In a real Banach space, it is asserted that a functional π remains concave, continuous, and non-negative over a cone P, provided that for every y_1, y_2 within P and each k spanning $[0,1]$, $\pi : P \to [0,\infty)$ satisfies the subsequent inequality:

$$\pi(ky_1 + (1-k)y_2) \geq k\pi(y_1) + (1-t)\pi(y_2).$$

Assuming that $(H_1) - (H_4)$ are satisfied through out the paper.

- (H_1): $f\left(x, y(x), y'(x)\right) \in C([0,1] \times [0,+\infty) \times R \to [0,+\infty))$.
- (H_2): $\int_0^1 d\Lambda(x)$ and $\int_0^1 x d\Lambda(x)$ are between 0 and 1, $\int_0^1 H(x,s)d\Lambda(x) \geq 0$.
- (H_3): $\frac{(\delta-\alpha)\eta}{1-\delta+\alpha} + \theta < 1$, $\frac{\delta-\alpha+\alpha\theta}{\theta} < 1$.
- (H_4): $1 - (\delta - \alpha + \delta\eta + \alpha\eta) > 0$.

The Banach space $E = C[0,1]$ is considered, where E is equipped with the norm

$$\left\{ \|y\| = \max\left\{ \max_{x\in[0,1]} |y(x)|, \max_{x\in[0,1]} |y'(x)| \right\}, \right. \tag{2.1}$$

and define a cone $P \subset E$ by

$$P = \left\{ y \in E | y(x) \geq 0, \lambda[y] \geq 0, \text{ and } \min_{x\in\left[\theta,\frac{1}{2}-\theta\right]} y(x) \geq \xi\|y\| \right\},$$

where

$$\xi = \min\left\{ \min_{x\in\left[\theta,\frac{1}{2}-\theta\right]} h(x), \xi_1, \xi_2, \xi_3, \xi_4 \right\}.$$

Here $h(x) = 1 - \frac{x^2}{(1-x)^2}, \theta \in \left(0, \frac{1}{4}\right)$is a constant, then $\underset{x\in\left[\theta,\frac{1}{2}-\theta\right]}{h(x)} \in (0,1)$.

$\xi_1 = \frac{(\delta-\alpha)\eta+\theta(1-\delta+\alpha)}{\delta\eta-\alpha\eta+1-\delta+\alpha} = \frac{\delta\eta-\alpha\eta+\theta(1-\delta+\alpha)}{\delta\eta-\alpha\eta+1-\delta+\alpha}$, since $\theta \in \left(0, \frac{1}{4}\right)$, so $\xi_1 \in (0,1)$.

$\xi_2 = \frac{\delta+\alpha(\theta-1)}{\delta}$, since $0 < \alpha < \delta < 1, \theta \in \left(0, \frac{1}{4}\right)$, then $0 < \delta + \alpha(\theta - 1) < \delta$, so $\xi_2 \in (0,1)$.

$\xi_3 = \frac{(\delta-\alpha)\eta+\theta(1-\delta+\alpha)}{1-\delta+\alpha}$, since $0 < \alpha < \delta < 1, 0 < \eta < 1, \theta \in \left(0, \frac{1}{4}\right)$, so $\xi_3 > 0$.

From (H_3), $\frac{(\delta-\alpha)\eta}{1-\delta+\alpha} + \theta < 1$, it can be seen that $\xi_3 < 1$, therefore, $\xi_3 \in (0,1)$.

$\xi_4 = \frac{\delta-\alpha+\alpha\theta}{\alpha} = \frac{\delta+\alpha(\theta-1)}{\alpha}$, since $0 < \alpha < \delta < 1, \theta \in \left(0, \frac{1}{4}\right), \delta + \alpha(\theta - 1) > 0$. From (H_3), $\frac{\delta-\alpha+\alpha\theta}{\alpha} < 1$, so $\xi_4 \in (0,1)$.

Theorem 2.1 (see Yao, 2004): Let E be a Banach space and $P \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous, if it is satisfied:

- $\|Tx\| \le \|x\|$, for all $x \in P \cap \partial\Omega_1$, $\|Tx\| \ge \|x\|$, for all $x \in P \cap \partial\Omega_2$
- $\|Tx\| \le \|x\|$, for all $x \in P \cap \partial\Omega_2$, $\|Tx\| \ge \|x\|$, for all $x \in P \cap \partial\Omega_1$.

Then, T has a fixed point at least in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.
Lemma 2.2: For every function y in the function space L [0, 1] on the interval [0, 1],

$$\begin{cases} y'''(x) = -z(x), & x \in [0,1], \\ y'(0) = \alpha y(\eta) + \lambda[y], & y''(0) = 0, \\ y(1) = \delta y(\eta) + \lambda[y], \end{cases} \tag{2.2}$$

there exists a unique solution:

$$\begin{aligned} y(x) = &\left(\frac{(\delta-\alpha+\alpha x)\eta}{1-(\delta-\alpha+\alpha\eta)} + x\right)\lambda[y] \\ &+ \frac{\delta-\alpha+\alpha x}{1-(\delta-\alpha+\alpha\eta)} \int_0^1 H(\eta,s)z(s)ds \\ &+ \int_0^1 H(x,s)y(s)ds, \end{aligned}$$

here

$$H(x,s) = \begin{cases} \frac{1}{2}(1-s)^2, & 0 \le x \le s \le 1, \\ \frac{1}{2}(1-x)(1+x-2s), & 0 \le s \le x \le 1. \end{cases} \tag{2.3}$$

Proof: Integrating the equation (2.2) twice from 0 to x, we get:

$$y'(x) = y'(0) - \int_0^x \left(\int_0^s z(\zeta)d\zeta\right) ds.$$

Doing the third integration, we have

$$y(x) = y(0) + y'(0)x - \frac{1}{2}\int_0^x (x-s)^2 z(s)ds.$$

Let x equals to 1, then

$$y(0) = y(1) - y'(0) + \frac{1}{2}\int_0^1 (1-s)^2 z(s)ds.$$

Substitute the boundary value conditions into the above equation

$$\begin{aligned} y(0) &= \delta y(\eta) + \lambda[y] - \alpha y(\eta) - \lambda[y] \\ &\quad + \frac{1}{2}\int_0^1 (1-s)^2 z(s)ds. \end{aligned}$$

Subsequently, we can obtain

$$\begin{aligned} y(x) &= (\delta - \alpha + \alpha x) y(\eta) + \lambda[y]x \\ &\quad + \int_0^1 H(x,s) z(s)ds. \end{aligned}$$

Substituting $x = \eta$,

$$\begin{aligned} y(\eta) &= (\delta - \alpha + \alpha\eta) y(\eta) + \lambda[y]\eta \\ &\quad + \int_0^1 H(\eta,s) z(s)ds. \end{aligned}$$

So

$$\begin{aligned} y(x) &= \left(\frac{(\delta-\alpha+\alpha x)\eta}{1-(\delta-\alpha+\alpha\eta)} + x\right)\lambda[y] \\ &\quad + \frac{\delta-\alpha+\alpha x}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(\eta,s)z(s)ds \\ &\quad + \int_0^1 H(x,s)z(s)ds, \end{aligned}$$

the proof is completed. □

Facilitated by BVP (1.1), we get

$$y^{(3)}(x) = -\varphi_q\left(\int_0^x f\left(s, y(s), y'(s)\right) ds\right).$$

Then we specify an operator $T : P \to E$ in the following manner:

$$\begin{aligned} (Ty)(x) &= \left(\frac{(\delta-\alpha+\alpha x)\eta}{1-(\delta-\alpha+\alpha\eta)} + x\right)\lambda[y] \\ &+ \int_0^1 H(x,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right) d\zeta\right) ds \\ &+ \frac{\delta-\alpha+\alpha x}{1-(\delta-\alpha+\alpha\eta)} \\ &\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right) d\zeta\right) ds. \end{aligned} \tag{2.4}$$

Therefore, when $Ty = y$, the equation (1.1) has a solution $y = y(x)$.

According to Green's function (2.3), clearly, $H(x,s) \geq 0$, $0 \leq x,\ s \leq 1$, $H(s,s) = \frac{1}{2}(1-s)^2$.

Lemma 2.3: $h(x)H(s,s) \leq H(x,s) \leq H(s,s)$.

Proof: If $s \leq x$ lies in the interval [0, 1], then

$$\begin{aligned} H(x,s) &= \frac{1}{2}\left[(1-s)^2 - (x-s)^2\right] \\ &\leq \frac{1}{2}(1-s)^2 = H(s,s), \end{aligned}$$

and

$$\frac{H(x,s)}{H(s,s)} = 1 - \frac{(x-s)^2}{(1-s)^2} \geq 1 - \frac{x^2}{(1-x)^2} = h(x).$$

If $x \leq s$ lies in the interval [0, 1], then

$$H(x,s) = \frac{1}{2}(1-s)^2 = H(s,s),$$

and

$$\frac{H(x,s)}{H(s,s)} = \frac{\frac{1}{2}(1-s)^2}{\frac{1}{2}(1-s)^2} = 1 \geq 1 - \frac{x^2}{(1-x)^2} = h(x).$$

It follows that $h(x)H(s,s) \leq H(x,s) \leq H(s,s)$, for x, s are between 0 and 1. □

Lemma 2.4: T is completely continuous.

Proof: $(Ty)(x)$ as shown in equation (2.4),

$$\begin{aligned} (Ty)'(x) = &\left(\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)} + 1\right)\lambda[y] \\ &+ \int_0^x (s-x)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right) d\zeta\right) ds \\ &+ \frac{\alpha}{1-(\delta-\alpha+\alpha\eta)} \\ &\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right) d\zeta\right) ds, \end{aligned}$$

$$\begin{aligned} (Ty)''(x) = &-\int_0^x \varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right) d\zeta\right) ds \\ &\leq 0. \end{aligned}$$

Evidently, $(Ty)(x)$ exhibits concavity over the interval [0, 1]. By equation (2.4),

$$(Ty)(0) = \frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\lambda[y]$$
$$+\frac{1}{2}\int_0^1 (1-s)^2\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds$$
$$+\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \geq 0,$$

$$(Ty)(1) = \left(\frac{\delta\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y]$$
$$+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \geq 0.$$

This plus $(Ty)''(x) \leq 0$, one has $(Ty)(x) \geq 0$. Considering $\lambda[x] = \int_0^1 x(t)d\Lambda(t)$, we have

$$\lambda[Ty] = \int_0^1 \Bigg[\left(\frac{(\delta-\alpha+\alpha x)\eta}{1-(\delta-\alpha+\alpha\eta)}+x\right)\lambda[y]$$
$$+\int_0^1 H(x,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds$$
$$+\frac{\delta-\alpha+\alpha x}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds\Bigg]d\Lambda(x) \geq 0.$$

Subsequently, it must be demonstrated that $\min\limits_{x\in\left[\theta,\frac{1}{2}-\theta\right]} (Ty)(x) \geq \xi|(Ty)(x)|$.

Compared $\min\limits_{x\in\left[\theta,\frac{1}{2}-\theta\right]}(Ty)(x)$ and $\max\limits_{x\in[0,1]}|(Ty)(x)|$. When $\xi^* = \min\left\{\min\limits_{x\in\left[\theta,\frac{1}{2}-\theta\right]} h(x), \xi_1, \xi_2\right\}$, where

$$\xi_1 = \frac{\delta\eta-\alpha\eta+\theta(1+\alpha-\delta)}{1+\alpha+\delta\eta-\delta-\alpha\eta},$$
$$\xi_2 = \frac{\delta+\alpha(\theta-1)}{\delta}, \xi_1, \xi_2 \in (0,1).$$

We can obtain

$$\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}(Ty)(x) \geq \xi^* \max_{x\in[0,1]}|(Ty)(x)|. \tag{2.5}$$

Compared $\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}(Ty)(x)$ and $\max_{x\in[0,1]}|(Ty)'(x)|$. When $\xi^{**}=\min\{\xi_3,\xi_4\}$ and (H_3) hold, where

$$\begin{aligned}\xi_3&=\frac{\theta(1+\alpha-\delta)+\delta\eta-\alpha\eta}{1+\alpha-\delta},\\ \xi_4&=\frac{\delta+\alpha\theta-\alpha}{\alpha},\xi_3,\xi_4\in(0,1)\,.\end{aligned}$$

Then

$$\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}(Ty)(x)\geq\xi^{**}\max_{x\in[0,1]}|(Ty)'(x)|. \tag{2.6}$$

Given equations (2.5), (2.6) and

$$\xi=\min\left\{\min_{x\in\left[\theta,\frac{1}{2}-\theta\right].}h(x),\xi_1,\xi_2,\xi_3,\xi_4\right\},$$

then

$$\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}Ty(x)\geq\xi\|Ty(x)\|. \tag{2.7}$$

It is apparent that T exhibits continuity. Considering $\Omega\subset P$ as a bounded domain, it becomes clear that $T\Omega$ possesses boundedness and equicontinuity. Utilising the Arzela-Ascoli theorem leads to the deduction that $T\Omega$ has relative compactness. Thus, T demonstrates compactness. To encapsulate, $T:P\to P$ can be regarded as a fully continuous operator. □

3 The main results

For ease of reference, we introduce the subsequent notations:

$$\begin{aligned}f_1&=\lim_{|y|+|y'|\to0}\inf\min_{x\in[0,1]}\frac{f\left(x,y,y'\right)}{\left(|y|+|y'|\right)^{p-1}},\\ f_2&=\lim_{|y|+|y'|\to\infty}\inf\min_{x\in[0,1]}\frac{f\left(x,y,y'\right)}{\left(|y|+|y'|\right)^{p-1}},\\ f_3&=\lim_{|y|+|y'|\to0}\sup\max_{x\in[0,1]}\frac{f\left(x,y,y'\right)}{\left(|y|+|y'|\right)^{p-1}},\\ f_4&=\lim_{|y|+|y'|\to\infty}\sup\max_{x\in[0,1]}\frac{f\left(x,y,y'\right)}{\left(|y|+|y'|\right)^{p-1}}.\end{aligned}$$

Theorem 3.1: Assume that (H_1)–(H_4) hold, moreover

$$f_2\in\left[B^{p-1},+\infty\right),f_3\in\left(0,A^{p-1}\right].$$

Then BVP (1.1) has a non-negaitve solution, where

$$A = \min\left\{\frac{1-m}{M}, \frac{1-m'}{M'}\right\},$$

$$B = \max\left\{\frac{1-n}{N}, \frac{1-n'}{N'}\right\},$$

$$m = \frac{1-(\delta-\alpha+\delta\eta+\alpha\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 d\Lambda(x),$$

$$m' = \frac{1+\alpha-\delta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 d\Lambda(x),$$

$$M = \frac{2(1+\alpha-\alpha\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds,$$

$$M' = \frac{2\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds,$$

$$n = \frac{\delta\eta-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \xi d\Lambda(x),$$

$$n' = \frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \xi d\Lambda(x),$$

$$N = \left(\frac{1-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\right)$$
$$\int_\theta^{\frac{1}{2}-\theta} \min_{x\in\left[\theta,\frac{1}{2}-\theta\right]} h(x)H(s,s)\xi\left(\frac{1}{2}-\theta\right)^{q-1} ds,$$

$$N' = \left(\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\right)$$
$$\int_\theta^{\frac{1}{2}-\theta} \min_{x\in\left[\theta,\frac{1}{2}-\theta\right]} h(x)H(s,s)\xi\left(\frac{1}{2}-2\theta\right)^{q-1} ds.$$

Proof: From (H_2) and (H_4), it can be seen that $m, m', n, n' \in (0,1)$. From $(H_2), (H_4)$ and $H(s,s) = \frac{1}{2}(1-s)^2$, it can be seen that $M, M', N, N' \in (0,1)$.

On one side,

$$f_3 \in \left(0, A^{p-1}\right],$$

then there exists $r_1 > 0$, and when $|y| + |y'| \le 2r_1$,

$$f(x, y, y') \le A^{p-1} \cdot (|y| + |y'|)^{p-1}.$$

Given equation (2.1), it entails that $|y| + |y'| \le 2\|y\|$.

Define

$$\Omega_1 = \{y \in P : \|y(x)\| < r_1\},$$

then when y belongs to the set $P \cap \partial\Omega_1$, $y(x) \le \|y\| = r_1$, $y'(x) \le \|y\| = r_1$, that is $|y| + |y'| \le 2r_1$. Sequentially,

$$\begin{aligned}
&\max_{x\in[0,1]} |(Ty)(x)| \\
&\le \left(\frac{\delta\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y] \\
&+\int_0^1 \frac{1}{2}(1-s)^2\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \\
&+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)} \\
&\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \\
&\le \frac{1-(\delta-\alpha+\delta\eta+\alpha\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 y(x)d\Lambda(x) \\
&+\int_0^1 H(s,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \\
&+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)} \\
&\int_0^1 H(s,s)\varphi_q\left(\int_0^s f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \\
&\le \frac{1-(\delta-\alpha+\delta\eta+\alpha\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 y(x)d\Lambda(x) \\
&+\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)} \\
&\int_0^1 H(s,s)\varphi_q\left(\int_0^1 f\left(\zeta, y(\zeta), y'(\zeta)\right)d\zeta\right)ds \\
&\le \frac{1-(\delta-\alpha+\delta\eta+\alpha\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 y(x)d\Lambda(x) \\
&+\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)} \\
&\int_0^1 H(s,s)\varphi_q\left(\int_0^1 A^{p-1}\cdot\left(|y|+|y'|\right)^{p-1}\right)ds \\
&\le \left[m+\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds\cdot A\right]\|y\| \\
&\le (m+AM)\|y\| \le \|y\|.
\end{aligned}$$

$$\begin{aligned}
&\max_{x\in[0,1]} |(Ty)'(x)| \\
&\le \left(\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y] \\
&+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}
\end{aligned}$$

$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^1 f\left(\zeta,y(\zeta),y'(\zeta)\right)d\zeta\right)ds$$
$$\le \left(\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\right)\int_0^1 y(x)d\Lambda(x)$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(s,s)\varphi_q\left(\int_0^1 A^{p-1}\cdot\left(|y|+|y'|\right)^{p-1}d\zeta\right)ds$$
$$\le \left[m'+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds\cdot A\right]\|y\|$$
$$\le (m'+AM')\|y\|\le\|y\|.$$

Thus,

$$\|Ty\|\le\|y\|,\ \forall y\in P\cap\partial\Omega_1. \tag{3.1}$$

On the other side, by condition

$$f_2 =\in \left[B^{p-1},+\infty\right),$$

then there is a $R_1>0$, when $|y|+|y'|\ge R_1$,

$$f\left(x,y,y'\right)\ge B^{p-1}\cdot(|y|+|y'|)^{p-1}.$$

Let $r_2=\max\left\{\frac{R_1}{\xi},\frac{r_1}{\xi}\right\},\Omega_2=\{y\in E\mid \|y\|<r_2\}$, when $y\in P\cap\partial\Omega_2$, we get $\xi\|y\|\le |y|\le\|y\|$, then

$$|y|+|y'|\ge\xi\|y\|=\xi r_2\ge R_1, x\in\left[\theta,\frac{1}{2}-\theta\right].$$

Hence

$$\max_{x\in[0,1]}|(Ty)(x)|$$
$$\ge\left(\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\right)\lambda[y]$$
$$+\int_0^1 H(x,s)\varphi_q\left(\int_0^s f\left(\zeta,y(\zeta),y'(\zeta)\right)d\zeta\right)ds$$
$$+\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta,y(\zeta),y'(\zeta)\right)d\zeta\right)ds$$
$$\ge\left(\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\right)\lambda[y]$$

$$+\left(\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}+1\right)$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_{\theta}^{\frac{1}{2}-\theta}f\left(\zeta,y(\zeta),y'(\zeta)\right)d\zeta\right)ds$$
$$\geq\left(\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\right)\lambda[y]+\left(\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}+1\right)$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_{\theta}^{\frac{1}{2}-\theta}B^{p-1}\cdot(|y|+|y'|)^{p-1}\,d\zeta\right)ds$$
$$\geq\left[\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1\xi d\Lambda(x)\right.$$
$$+\left(\frac{1-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\right)$$
$$\left.\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)ds\cdot B\xi\left(\frac{1}{2}-2\theta\right)^{q-1}\right]\|y\|$$
$$\geq(n+NB)\|y\|\geq\|y\|.$$

$$\max_{x\in[0,1]}|(Ty)'(x)|$$
$$=\left[\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right]\lambda[y]$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1H(\eta,s)\varphi_q$$
$$\left(\int_0^s f\left(\zeta,y(\zeta),y'(\zeta)\right)d\zeta\right)ds$$
$$\geq\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1\xi d\Lambda(x)$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_{\theta}^{\frac{1}{2}-\theta}B^{p-1}\cdot(|y|+|y'|)^{p-1}\,d\zeta\right)ds$$
$$\geq\left[n'+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\right.$$

$$\int_{\theta}^{\frac{1}{2}-\theta} \min_{x\in\left[\theta,\frac{1}{2}-\theta\right]} h(x)H(s,s)ds \cdot B\xi\left(\frac{1}{2}-2\theta\right)^{q-1}\Bigg]\|y\|$$
$$\geq (n' + N'B)\|y\| \geq \|y\|.$$

Thus

$$\|Ty\| \geq \|y\|, \ \forall y \in P \cap \partial\Omega_2. \tag{3.2}$$

Applying Theorem 2.1 along with the inequalities (3.1) and (3.2), we can infer that the operator T has a fixed point, denoted as y^*, belonging to $P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$, satisfying $r_1 < \|y^*\| < r_2$. It is evident that y^* constitutes a non-negative solution to the BVP represented by equation (1.1). □

Theorem 3.2: Assume that (H_1)–(H_4) hold, moreover

$$f_1 \in \left[B^{p-1}, +\infty\right), f_4 \in (0, A^{p-1}],$$

then BVP (1.1) has a non-negaitve solution.

Proof: On the one hand,

$$f_1 \in \left[B^{p-1}, +\infty\right),$$

indicating the existence of $r_1 > 0$. For $|y| + |y'| \leq 2r_1$, it holds that

$$f\left(x, y, y'\right) \geq B^{p-1} \cdot \left(|y| + |y'|\right)^{p-1}.$$

Let $\Omega_1 = \{y \in E \mid \|y\| < r_1\}$, then when $y \in P \cap \Omega_1$, we obtain

$$\xi\|y\| \leq |y| \leq \|y\|, \text{ then } |y| + |y'| \geq \xi\|y\|,$$
$$x \in \left[\theta, \frac{1}{2} - \theta\right].$$

Sequentially,

$$\max_{x\in[0,1]} \mid (Ty(x) \mid \geq \left(\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\right)\lambda[y]$$
$$+\int_0^1 H(x,s)\varphi_q\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds$$
$$+\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds$$
$$\geq \frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\lambda[y]$$

$$+\left(\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}+1\right)$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_{\theta}^{\frac{1}{2}-\theta}f\left(\zeta,x(\zeta),x'(\zeta)\right)d\zeta\right)ds$$
$$\geq\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\lambda[y]$$
$$+\left(\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}+1\right)$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_{\theta}^{\frac{1}{2}-\theta}B^{p-1}\cdot\left(|y|+|y'|\right)^{p-1}d\zeta\right)ds$$
$$\geq\left[\frac{(\delta-\alpha)\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1\xi d\Lambda(x)\right.$$
$$+\left(\frac{\delta-\alpha}{1-(\delta-\alpha+\alpha\eta)}+1\right)$$
$$\left.\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)ds\cdot\xi B\left(\frac{1}{2}-2\theta\right)^{q-1}\right]\|y\|$$
$$\geq(n+NB)\|y\|\geq\|y\|.$$

$$\max_{x\in[0,1]}|(Ty)'(x)|=\left[\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right]\lambda[y]$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_0^1 H(\eta,s)\varphi_q\left(\int_0^s f\left(\zeta,x(\zeta),x'(\zeta)\right)d\zeta\right)ds$$
$$\geq\frac{1-\delta+\alpha}{1-(\delta-\beta+\beta\eta)}\lambda[y]+\frac{\beta}{1-(\delta-\alpha+\alpha\eta)}$$
$$\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)\varphi_q$$
$$\left(\int_0^s B^{p-1}\cdot\left(|y|+|y'|\right)^{p-1}d\zeta\right)ds$$
$$\geq\left[\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1\xi d\Lambda(x)\right.$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_{\theta}^{\frac{1}{2}-\theta}\min_{x\in\left[\theta,\frac{1}{2}-\theta\right]}h(x)H(s,s)ds$$

$$\cdot \xi B\left(\frac{1}{2}-2\theta\right)^{q-1}\Bigg]\|y\| \geq (n'+N'B)\|y\| \geq \|y\|.$$

Thus

$$\|Ty\| \geq \|y\|, \ \forall y \in P \cap \partial\Omega_2. \tag{3.3}$$

On the other side, since

$$f_4 \in \left(0, A^{p-1}\right],$$

then there exists $R_1 > 0$, such that $|y| + |y'| \geq R_1$,

$$f\left(x, y, y'\right) \leq A^{p-1} \cdot \left(|y| + |y'|\right)^{p-1}.$$

Then, case 1, f is bounded, case 2, f is not bounded.

Case 1

If f is bounded, then L is a positive constant such that $f(x, y, y') \leq L^{p-1}$.

Define $\overline{H}_2 = \max\left\{\frac{r_1}{\xi}, \frac{LM}{1-m}, \frac{LM'}{1-m'}\right\}$, $\Omega_2 = \{y \in E\colon \|y\| \leq \overline{H}_2\}$. When $y \in P \cap \partial\Omega_2$, we get:

$$\begin{aligned}
&\max_{x\in[0,1]} |(Ty)(x)| \\
&\leq \left(\frac{\delta\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y] \\
&+\int_0^1 \frac{1}{2}(1-s)^2\varphi_q\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds \\
&+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(\eta,s)\varphi_q \\
&\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds \\
&\leq \frac{1-(1-\eta)(\delta-\alpha)}{1-(\delta-\alpha+\alpha\eta)}\lambda[y] \\
&+\int_0^1 H(s,s)\varphi_q\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds \\
&+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q \\
&\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right)d\zeta\right)ds \\
&\leq \frac{1-(1-\eta)(\delta-\alpha)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 y(x)d\Lambda(x)
\end{aligned}$$

$$+\frac{1-\alpha\eta+\beta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q\left(\int_0^1 L^{p-1}d\zeta\right)ds$$
$$\leq \frac{1-(1-\eta)(\delta-\alpha)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \overline{H}_2 d\Lambda(x)$$
$$+L\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds$$
$$= m\overline{H}_2+\frac{1}{2}ML\leq \overline{H}_2=\|y\|.$$

$$\max_{x\in[0,1]}|(Ty)'(x)|\leq\left(\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y]$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(\eta,s)\varphi_q$$
$$\left(\int_0^s f\left(\zeta,x(\zeta),x'(\zeta)\right)d\zeta\right)ds$$
$$\leq\left(\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y]$$
$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q\left(\int_0^1 L^{p-1}d\zeta\right)ds$$
$$\leq\frac{1+\alpha-\delta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \overline{H}_2 d\Lambda(x)$$
$$+L\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds$$
$$= m'\overline{H}_2+\frac{1}{2}M'L\leq \overline{H}_2=\|y\|.$$

Case 2

Assuming that f is not bounded, choose $\overline{H}_2>\max\left\{r_1,\frac{R_1}{2}\right\}$, when x is between 0 and 1, $0<|y|+|y'|\leq 2\overline{H}_2$, we have

$$f\left(x,y,y'\right)\leq f\left(x,\overline{H}_2,\overline{H}_2\right)\leq A^{p-1}\cdot\left(2\overline{H}_2\right)^{p-1}.$$

Define $\Omega_2=\left\{y\in E\mid \|y\|<\overline{H}_2\right\}$, considering $\overline{H}_1+\overline{H}_2\geq R_1$. Hence, when $y\in P\cap\partial\Omega_2$,

$$\max_{x\in[0,1]}|(Ty)(x)|\leq\left(\frac{\delta\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y]$$
$$+\int_0^1\frac{1}{2}(1-s)^2\varphi_q\left(\int_0^1 f\left(\zeta,x(\zeta),x'(\zeta)\right)d\zeta\right)ds$$
$$+\frac{\delta}{1-(\delta-\alpha+\alpha\eta)}\int_0^s H(\eta,s)\varphi_q$$

$$\left(\int_0^s f\left(\zeta, x(\zeta), x'(\zeta)\right) d\zeta\right) ds$$

$$\leq \frac{1-(1-\eta)(\delta-\alpha)}{1-(\delta-\alpha+\alpha\eta)}\lambda[y] + \frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}$$

$$\int_0^1 H(s,s)\varphi_q\left(\int_0^1 f\left(\zeta, \overline{H}_2, \overline{H}_2\right) d\zeta\right) ds$$

$$\leq \frac{1-(1-\eta)(\delta-\alpha)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \overline{H}_2 d\Lambda(x)$$

$$+\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q$$

$$\left(\int_0^1 A^{p-1}\cdot\left(2\overline{H}_2\right)^{p-1}\right) ds$$

$$\leq \frac{1-(\delta-\alpha)(1-\eta)}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 \overline{H}_2 d\Lambda(x)$$

$$+\frac{1+\alpha-\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds\cdot 2A\overline{H}_2$$

$$\leq (m+MA)\overline{H}_2 \leq \overline{H}_2 = \|y\|.$$

$$\max_{x\in[0,1]} |(Ty)'(x)| \leq \left(\frac{\alpha\eta}{1-(\delta-\alpha+\alpha\eta)}+1\right)\lambda[y]$$

$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(\eta,s)\varphi_q$$

$$\left(\int_0^1 f(\zeta, x(\zeta), x(\zeta))d\zeta\right) ds$$

$$\leq \left(\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\right)\int_0^1 \overline{H}_2 d\lambda(x)$$

$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q$$

$$\left(\int_0^1 f\left(\zeta, \overline{H}_2, \overline{H}_2\right) d\zeta\right) ds$$

$$\leq \left(\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\right)\int_0^1 \overline{H}_2 d\lambda(x)$$

$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)\varphi_q$$

$$\left(\int_0^1 A^{p-1}\cdot\left(2\overline{H}_2\right)^{p-1}\right) ds$$

$$\leq \left(\frac{1-\delta+\alpha}{1-(\delta-\alpha+\alpha\eta)}\right)\int_0^1 \overline{H}_2 d\Lambda(x)$$

$$+\frac{\alpha}{1-(\delta-\alpha+\alpha\eta)}\int_0^1 H(s,s)ds\cdot 2A\overline{H}_2$$

$$\leq (m' + M'A)\overline{H}_2 \leq \overline{H}_2 = \|y\|.$$

In both cases, then

$$\|Ty\| \leq \|y\|, \ \forall y \in P \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.1 along with the inequalities (3.3) and (3.4), we can conclude T possesses a fixed point, denoted as y^*, belonging to $P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$, and $H_1 \leq \|y^*\| \leq \overline{H}_2$. Notably, it is evident that y^* constitutes a non-negative solution to the BVP represented by equation (1.1).

Using Theorem 2.1 and inequalities (3.3) and (3.4), we can conclude $y^* \in P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ is the fixed point of T and $H_1 \leq \|y^*\| \leq \overline{H}_2$. Then y^* is a non-negative solution of equation (1.1). □

4 Conclusions

This paper demonstrates the existence of positive solutions for a fourth-order differential equation with p-Laplacian and Riemann-Stieltjes integral boundary conditions. The novelty of this work is that the nonlinear term f explicitly includes the first-order derivative of the unknown function $y(x)$.

References

Feng, H.Y., Pang, H.H. and Ge, W.G. (2008) 'Multiplicity of symmetric positive solutions for a multipoint boundary value problem with a one-dimensional p-Laplacian', *Nonlinear Analysis-theory Methods and Applications*, Vol. 69, No. 9, pp.3050–3059.

Feng, X.F., Feng, H.Y. and Tan, H.X. (2015) 'Existence and iteration of positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian', *Applied Mathematics and Computation*, Vol. 266, No. 1, pp.634–641.

Hao, X.N., Liu, L.S. and Wu, Y.H. (2011) 'Positive solutions for second order impulsive differential equations with integral boundary conditions', *Communications in Nonlinear Science and Numerical Simulation*, Vol. 16, No. 1, pp.101–111.

Jankowski, T. (2012) 'Existence of positive solutions to third order differential equations with advanced arguments and nonlocal boundary conditions', *Nonlinear Analysis, Theory, Methods and Applications*, Vol. 75, No. 2, pp.913–923.

Ji, D.H. and Ge, W.G. (2008) 'Existence of multiple positive solutions for Sturm-Liouville-like four-point boundary value problem with p-Laplacian', *Nonlinear Analysis-Theory, Methods and Applications*, Vol. 68, No. 9, pp.2638–2646.

Ji, D.H., Yang, Y.T. and Ge, W.G. (2008) 'Triple positive pseudo-symmetric solutions to a four-point boundary value problem with p-Laplacian', *Applied Mathematics Letters*, Vol. 21, No. 3, pp.268–274.

Ji, D.H., Tian, Y. and Ge, W.G. (2009) 'Positive solutions for one-dimensional p-Laplacian boundary value problems with sign changing nonlinearity', *Nonlinear Analysis-Theory, Methods and Applications*, Vol. 71, No. 11, pp.5406–5416.

Sun, B. (2016a) 'Monotonic iteration positive symmetric solutions to a boundary value problem with p-Laplacian and Stieltjes integral boundary conditions', *Boundary Value Problems*, Article No. 151, pp.1–12.

Sun, B. (2016b) 'Monotone iterative technique and positive solutions to a third-order differential equation with advanced arguments and Stieltjes integral boundary conditions', *Advances in Difference Equations*, Article No. 218, pp.1–11.

Sun, B., Qu, Y. and Ge, W.G. (2008) 'Existence and iteration of positive solutions for a multipoint one-dimensional p-Laplacian boundary value problem', *Applied Mathematics and Computation*, Vol. 197, No. 1, pp.389–398.

Sun, Y. (2011) 'Triple positive solutions for a class of third-order p-Laplacian singular boundary value problems', *Journal of Applied Mathematics and Computing*, Vol. 37, No. 1, pp.587–599.

Wang, Z.Y. and Zhang, J.H. (2006) 'Positive solutions for one-dimensional p-Laplacian boundary value problems with dependence on the first order derivative', *Journal of Mathematical Analysis and Applications*, Vol. 314, No. 2, pp.618–630.

Wang, S.L., Chai, J.L. and Zhang, G.W. (2019) 'Positive solutions of beam equations under nonlocal boundary value conditions', *Advances in Difference Equations*, Vol. 470, No. 1, pp.1–13.

Yang, C. and Yan, J.R. (2010) 'Positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian', *Computers and Mathematics with Applications*, Vol. 59, No. 6, pp.2059–2066.

Yao, Q.L. (2004) 'Positive solutions for eigenvalue problems of fourth-order elastic beam equations', *Applied Mathematics Letters*, Vol. 17, No. 2, pp.237–243.

Yao, Q.L. (2008) 'Existence and multiplicity of positive solutions to a singular elastic beam equation rigidly fixed at both ends', *Nonlinear Analysis-Theory, Methods and Applications*, Vol. 69, No. 8, pp.2683–2694.

Yao, Q.L. (2008) 'Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right', *Nonlinear Analysis-Theory, Methods and Applications*, Vol. 69, Nos. 5–6, pp.1570–1580.