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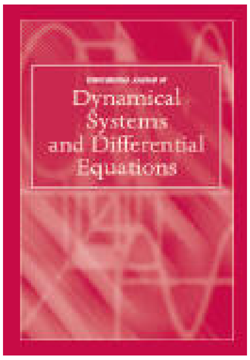

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The appropriate expression and non-uniqueness of solution for impulsive Katugampola fractional order system

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Abstract: For two impulsive generalised fractional order systems (IGFrOSs), we consider their conditions of fractional derivative and fractional integral by two new fractional order properties of piecewise function to find that the equivalent integral equations (EIEs) of the IGFrOSs are a combination of two integral equations ($\phi(t)$ and $\Phi_j(t)$) with an arbitrary constant to reveal the non-uniqueness of the IGFrOSs' solution, and moreover, we give the appropriate expressions of the EIEs to easily verify that the EIEs satisfy the conditions of fractional derivative and fractional integral in the two IGFrOSs. Finally, we apply two numerical models to illustrate the EIEs and the non-uniqueness of solution of two IGFrOSs.

Keywords: impulsive fractional differential equations; equivalent integral equations; EIEs; initial value problems; non-uniqueness of solution.

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1 Introduction

Fractional calculus was widely used in the modelling to characterise various materials and processes with the hereditary properties in many fields of science and engineering (see Bachraoui et al., 2022; Liu, 2023; Khalid et al., 2023; Yalcinkaya, 2023; Oliva-Gonzalez et al., 2023; Selvam et al., 2023; Tayebi, 2024; Raja et al., 2024; etc.). And recently, the subject of impulsive fractional order system (IFrOS) has been gaining much attention and several hundreds articles are found by searching the topic of IFrOS from the Web of Science. For the IFrOS, its equivalent integral equation (EIE) is an important tool to discuss numerical solution (Zhou et al., 2020; Cao et al., 2020), existence of solution (Gou and Li, 2020a, 2020b; Kucche et al., 2020; Heidarkhani and Salari, 2020; You and Sun, 2020; Ravichandran et al., 2020; Min and Chen, 2020; Agarwal et al., 2020), oscillation behaviour (Feng et al., 2020; Feng and Han, 2020), periodic motion (Zhang and Xiong, 2020), solvability (Xu et al., 2020), asymptotic behaviour of solution (Cheng et al., 2021), stability (Liu and Xu, 2021; Kucche et al., 2020) and non-uniqueness of solution (Zhang, 2022; Zhang et al., 2023), etc.

However, in the mainstream research regarding the IFrOS with the Caputo fractional derivative, the fractional derivative in the IFrOS was respectively considered from three aspects (including the whole interval, each subinterval and the combination of the whole interval and subintervals), which caused that there appeared three conflicting EIEs for the same IFrOS (for more details see Agarwal et al., 2016; Wang et al., 2014; Feckan et al., 2012, 2014; Wang et al., 2012; Liu, 2016, 2019; Wang et al., 2016; Zhang et al., 2014).

The above case of some controversial EIEs also appears in the study regarding the impulsive non-Caputo type fractional order system. Therefore, we will re-explore the EIE of two impulsive generalised fractional order systems (IGFrOSs):

$$\begin{cases} {}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} w(t) = g(t, w(t)), t \in (t_0, T]/\{t_1, t_2, ..., t_B\}, \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^+} \qquad\qquad j = 1, 2, ..., B, \\ = Q_j(w(t_j^-)), \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t\to t_0+} \\ = w_0, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} w(t) = g(t, w(t)), t \in (t_0, T]/\{t_1, t_2, ..., t_B\}, \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^+} \\ - {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^-} \\ = P_j(w(t_j^-)), \qquad\qquad j = 1, 2, ..., B, \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t\to t_0+} \\ = w_0, \end{cases} \tag{2}$$

where ${}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t}$ ($t_0, \kappa > 0$ and $b \in (0, 1)$) and ${}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t}$ are respectively the left Katugampola fractional derivative and fractional integral, $+\infty > T = t_{B+1} > t_B > ... > t_1 > t_0 > 0$, $g : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, and $P_j, Q_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, 2, ..., B$).

To show the connection between equations (2) and (1), we transform equation (1) into

$$\text{system (1)} \\ = \begin{cases} {}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} w(t) = g(t, w(t)), t \in (t_0, T]/\{t_1, t_2, ..., t_B\}, \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^+} \\ - {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^-} \\ = Q_j(w(t_j^-)) \\ - {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^-}, \quad j = 1, 2, ..., B, \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t\to t_0+} \\ = w_0. \end{cases} \tag{3}$$

We arrange the rest of this paper as follows. We present some definitions and conclusions of the generalised fractional calculus and two fractional order properties of piecewise function in Section 2. In Section 3, we will use the fractional order properties of piecewise function to explore the EIE of equation (2) and combine the EIE of equation (2) with the relation (3) to seek the EIE of equation (1). In Section 4, we use two numerical examples to show two IGFrOSs' EIEs and discuss the non-uniqueness of solution for the two IGFrOSs.

2 Preliminaries

For the basic definitions and conclusions of fractional calculus, we can refer to two monographs (Kilbas et al., 2006; Baleanu et al., 2012), and we mainly recall several definitions and properties of the Katugampola fractional derivative and integral in this section.

Let $+\infty > T > t_0 > 0$ and $q \geq 1$, and two notations $C[t_0, T]$ and $L^q(t_0, T)$ denote respectively the continuous function space on $[t_0, T]$ and the Lebesgue integrable function spaces on (t_0, T). And let

$$W_a^q(t_0, T) = \left\{w : [t_0, T] \to \mathbb{R} : \|w\|_{W_a^q} < \infty\right\} \ (a \in \mathbb{R}),$$

here $\|w\|_{W_a^\infty} = ess \sup\limits_{t \in [t_0, T]} |t^a w(t)|$ and $\|w\|_{W_a^q} = \left(\int_{t_0}^T |t^a w(t)|^q \frac{dt}{t}\right)^{1/q}$ $(1 \leq q < \infty)$.

Definition 2.1 (Katugampola, 2011): Let $c > 0$, $\kappa > 0$ and $w \in W_a^q(t_0, T)$. The integral

$$ {}_{t_0}^{K}\mathcal{I}_t^{c,\kappa} w(t) = \int_{t_0}^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{c-1}}{\Gamma(c)} \frac{w(r)dr}{r^{1-\kappa}} \ (t > t_0),$$

is the definition of the left Katugampola fractional integral of order c.

Definition 2.2 (Katugampola, 2014): Let $b \in (0, 1)$, $\kappa > 0$ and $w \in W_a^q(t_0, T)$. The derivative

$$\begin{aligned} {}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w(t) &= \frac{t^{1-\kappa} d}{dt} \left({}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\right) \\ &= \frac{t^{1-\kappa} d}{dt} \int_{t_0}^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{w(r)dr}{r^{1-\kappa}} \ (t > t_0), \end{aligned}$$

is the definition of the left Katugampola fractional derivative of order b.

Remark 1: The generalised fractional operator with $\kappa \to 0+$ and $\kappa = 1$ are respectively the Hadamard fractional operator and the Riemann-Liouville fractional operator.

Let $0 \leq b < 1$ and define

$$\begin{aligned} &C_{b,\kappa}[t_0, T] \\ &= \left\{w : (t_0, T] \to \mathbb{R}, [t^\kappa - (t_0)^\kappa]^b w(t) \in C[t_0, T]\right\}, \\ &C_{b,0}[t_0, T] \\ &= \left\{w : (t_0, T] \to \mathbb{R}, \ \left(\ln \frac{t}{t_0}\right)^b w(t) \in C[t_0, T]\right\} \end{aligned}$$

with $\|w\|_{C_{b,\kappa}} = \left\|[t^\kappa - (t_0)^\kappa]^b w(t)\right\|_C$ and $\|w\|_{C_{b,0}} = \left\|(\ln \frac{t}{t_0})^b w(t)\right\|_C$, and

$$\begin{aligned} &C_{1-b,\kappa}^b[t_0, T] \\ &= \left\{w \in C_{1-b,\kappa}[t_0, T] \text{ and } {}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w \in C_{1-b,\kappa}[t_0, T]\right\}. \end{aligned}$$

Lemma 1: Let $b \in (0,1)$ and $\kappa > 0$, and let $g : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy $g(\cdot, w(\cdot)) \in C_{1-b,\kappa}[t_0, T]$ for any $w(\cdot) \in C_{1-b,\kappa}[t_0, T]$.

Let $w(t) \in C^b_{1-b,\kappa}[t_0, T]$, then $w(t)$ satisfies

$$\begin{cases} {}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} w(t) = g(t, w(t)), \quad t \in (t_0, T], \\ {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t \to t_0+} = w_0, \end{cases} \tag{4}$$

iff $w(t)$ satisfies the following integral equation

$$\begin{aligned} w(t) &= \frac{w_0}{\Gamma(b)} \left[\frac{t^\kappa - (t_0)^\kappa}{\kappa}\right]^{b-1} \\ &+ \int_{t_0}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \frac{g(r, w(r))dr}{r^{1-\kappa}}, \ t \in (t_0, T]. \end{aligned} \tag{5}$$

For a piecewise function

$$\begin{aligned} y(t) &= \begin{cases} y_0(t), \ t \in [t_0, t_1], \\ y_1(t), \ t \in (t_1, t_2], \\ \vdots \\ y_B(t), t \in (t_B, T], \end{cases} \\ &= \begin{cases} y_0(t), t \in [t_0, t_1], \\ 0, \qquad t \in (t_1, T], \end{cases} \\ &+ \begin{cases} 0, \qquad t \in [t_0, t_1], \\ y_1(t), t \in (t_1, t_2], \\ 0, \qquad t \in (t_2, T], \end{cases} \\ &+ ... + \begin{cases} 0, \qquad t \in [t_0, t_B], \\ y_B(t), t \in (t_B, T], \end{cases} \end{aligned} \tag{6}$$

its generalised fractional derivative and integral have two different expressions respectively.

Lemma 2: Let $b \in (0,1)$, $\kappa > 0$ and $y_i(t) \in C[t_i, t_{i+1}]$ $(i = 0, 1, ..., B)$, then the left Katugampola fractional derivative of equation (6) can be computed by

$$\begin{aligned} &{}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} y(t)\Big|_{t \in [t_0, t_1]} \\ &= \frac{t^{1-\kappa} d}{dt} \int_{t_0}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{y_0(r)dr}{r^{1-\kappa}}, \ t \in [t_0, t_1], \\ &{}^{K}_{t_0}\mathcal{D}^{b,\kappa}_{t} y(t)\Big|_{t \in (t_j, t_{j+1}]} \\ &= \frac{t^{1-\kappa} d}{dt} \int_{t_0}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{y(r)dr}{r^{1-\kappa}}, \ t \in (t_j, t_{j+1}] \\ &= \frac{t^{1-\kappa} d}{dt} \left[\int_{t_0}^{t_1} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{y_0(r)dr}{r^{1-\kappa}} \right. \end{aligned}$$

$$+\int_{t_1}^{t_2}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{y_1(r)dr}{r^{1-\kappa}}$$
$$+...+\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{y_j(r)dr}{r^{1-\kappa}}\Bigg]\ \ (j=1,2,...,B); \tag{7}$$

and

$$\begin{aligned}&{}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}y(t)\\&=\begin{cases}{}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}y_0(t), & t\in[t_0,t_1],\\ \frac{t^{1-\kappa}d}{dt}\int_{t_0}^{t_1}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{y_0(r)dr}{r^{1-\kappa}}, & t\in(t_1,T],\end{cases}\\&+\begin{cases}0, & t\in[t_0,t_1],\\ {}_{t_1}^{K}\mathcal{D}_t^{b,\kappa}y_1(t), & t\in(t_1,t_2],\\ \frac{t^{1-\kappa}d}{dt}\int_{t_1}^{t_2}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{y_1(r)dr}{r^{1-\kappa}}, & t\in(t_2,T],\end{cases}\\&+...+\begin{cases}0, & t\in[t_0,t_B],\\ {}_{t_B}^{K}\mathcal{D}_t^{b,\kappa}y_B(t), & t\in(t_B,T].\end{cases}\end{aligned} \tag{8}$$

Lemma 3: Let $c>0$, $\kappa>0$ and $y_i(t)\in C[t_i,t_{i+1}]$ $(i=0,1,...,B)$, then the left Katugampola fractional integral of equation (6) can be computed by

$$\begin{aligned}&{}_{t_0}^{K}\mathcal{I}_t^{c,\kappa}y(t)\big|_{t\in[t_0,t_1]}\\&=\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_0(r)dr}{r^{1-\kappa}},\ t\in[t_0,t_1],\\&{}_{t_0}^{K}\mathcal{I}_t^{c,\kappa}y(t)\big|_{t\in(t_j,t_{j+1}]}\\&=\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y(r)dr}{r^{1-\kappa}},\ t\in(t_j,t_{j+1}]\\&=\int_{t_0}^{t_1}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_0(r)dr}{r^{1-\kappa}}+\int_{t_1}^{t_2}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_1(r)dr}{r^{1-\kappa}}\\&+...+\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_j(r)dr}{r^{1-\kappa}}\ \ (j=1,2,...,B);\end{aligned} \tag{9}$$

and

$$\begin{aligned}&{}_{t_0}^{K}\mathcal{I}_t^{c,\kappa}y(t)=\begin{cases}{}_{t_0}^{K}\mathcal{I}_t^{c,\kappa}y_0(t), & t\in[t_0,t_1],\\ \int_{t_0}^{t_1}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_0(r)dr}{r^{1-\kappa}}, & t\in(t_1,T],\end{cases}\\&+\begin{cases}0, & t\in[t_0,t_1],\\ {}_{t_1}^{K}\mathcal{I}_t^{c,\kappa}y_1(t), & t\in(t_1,t_2],\\ \int_{t_1}^{t_2}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{c-1}}{\Gamma(c)}\frac{y_1(r)dr}{r^{1-\kappa}}, & t\in(t_2,T],\end{cases}\\&+...+\begin{cases}0, & t\in[t_0,t_B],\\ {}_{t_B}^{K}\mathcal{I}_t^{c,\kappa}y_B(t), & t\in(t_B,T].\end{cases}\end{aligned} \tag{10}$$

3 The EIEs of equations (2) and (1)

To simplify some formulas, let $gdr = g(r, w(r))dr$

$$\phi(t) = \frac{w_0}{\Gamma(b)} \left[\frac{t^\kappa - (t_0)^\kappa}{\kappa}\right]^{b-1} + \int_{t_0}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \frac{gdr}{r^{1-\kappa}}, \tag{11}$$

and

$$\Phi_j(t) = \frac{w_0 + \int_{t_0}^{t_j} \frac{gdr}{r^{1-\kappa}}}{\Gamma(b)} \left[\frac{t^\kappa - (t_j)^\kappa}{\kappa}\right]^{b-1} + \int_{t_j}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \frac{gdr}{r^{1-\kappa}},\ j = 1, 2, ..., B. \tag{12}$$

And we define some function spaces:

$$\begin{aligned}
&\widetilde{C}_{1-b,\kappa}[t_0, T] \\
&= \Big\{w : (t_0, T] \to \mathbb{R} :\ [t^\kappa - (t_i)^\kappa]^{1-b}\, w(t) \in C[t_i, t_{i+1}], \\
&i = 0, 1, ..., B\}\ (\kappa \neq 0), \\
&\widetilde{C}_{1-b,0}[t_0, T] \\
&= \left\{w : (t_0, T] \to \mathbb{R} : \left[\ln \frac{t}{t_i}\right]^{1-b} w(t) \in C[t_i, t_{i+1}],\right. \\
&i = 0, 1, ..., B\}, \\
&\widetilde{C}^b_{1-b,\kappa}[t_0, T] \\
&= \Big\{w \in \widetilde{C}_{1-b,\kappa}[t_0, T],\ {}^{K}_{t_0}\mathcal{D}^{b,\kappa}_t w \in \widetilde{C}_{1-b,\kappa}[t_0, T]\Big\}, \\
&I_g([t_0, T], \mathbb{R}) \\
&= \Big\{w \in \widetilde{C}^b_{1-b,\kappa}[t_0, T] \text{ and } {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_t w(t) \\
&\in C^1\left([t_0, t_1] \cup \cup_{j=1}^{B} (t_j, t_{j+1}]\right), \\
&\lim_{t \to t_j^-} \left[\frac{d}{dt} {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_t w(t)\right] = \frac{d}{dt} {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_t w(t)\Big|_{t=t_j} < \infty \\
&\left.\text{and } \lim_{t \to t_j^+} \left[\frac{d}{dt} {}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_t w(t)\right] < \infty, \text{here } j = 1, 2, ..., B\right\}.
\end{aligned}$$

3.1 Three limit properties of equation (1) and (2)

Some basic limit properties of equations (1) and (2) are the necessary conditions that an integral equation become the integral solution of equations (1) and (2).

$$\lim_{P_i(w(t_i^-))\to 0 \text{ here } i\in\{1,2,\dots,B\}} \{\text{system (2)}\}$$

$$= \begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w(t) = g(t, w(t)), t \in (t_0, T] \text{ and } t \neq t_j \\ \qquad (j \in \{1, ..., B\}/\{i\}), \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^+} \\ - {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^-} \\ = P_j(w(t_j^-)), \qquad j \in \{1, 2, ..., B\}/\{i\}, \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t\to t_0+} \\ = w_0; \end{cases} \tag{13}$$

$$\lim_{t_j\to t_l \text{ for } \forall j\in\{1,2,\dots,B\} \text{ and } l\in\{1,2,\dots,B\}} \{\text{system (2)}\}$$

$$= \begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w(t) = g(t, w(t)), t \in (t_0, T] \text{ and } t \neq t_l, \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_l^+} \\ - {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_l^-} \\ = \sum\limits_{j=1}^{B} P_j(w(t_l^-)), \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t\to t_0+} \\ = w_0; \end{cases} \tag{14}$$

$$\lim_{Q_j(w(t_j^-))-w_0-\int_{t_0}^{t_j}\frac{gdr}{r^{1-\kappa}}\to 0 \text{ for } \forall j=1,2,\dots,B} \{\text{system (1)}\}$$

$$= \begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w(t) = g(t, w(t)), t \in (t_0, T]/\{t_1, t_2, ..., t_B\}, \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^+} \\ = w_0 + \int_{t_0}^{t_j} \frac{gdr}{r^{1-\kappa}}, \qquad j = 1, 2, ..., B, \\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t\to t_0+} \\ = w_0, \end{cases}$$

$$\Leftrightarrow w(t) = \frac{w_0}{\Gamma(b)} \left[\frac{t^\kappa - (t_0)^\kappa}{\kappa}\right]^{b-1} + \int_{t_0}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \frac{gdr}{r^{1-\kappa}},\ t \in (t_0, T]. \tag{15}$$

Remark 2: From equations (13) and (14), it shows that the impulsive effects in equation (2) have the linear additivity.

3.2 The particular solution of equations (1) and (2)

Because $\phi(t)$ satisfies ${}_{t_0}^{K}\mathcal{D}_t^{b,\kappa} w(t) = g(t, w(t))$ on $(t_0, T]$, we use equations (8) and (10) to construct *the particular solutions* of equations (2) and (1):

$$w(t)=\begin{cases}\phi(t), t\in(t_0,t_1],\\ \vdots\\ \phi(t), t\in(t_B,T],\end{cases}+\sum_{j=1}^{B}\begin{cases}0, & t\in(t_0,t_j],\\ \frac{P_j(w(t_j^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1} & ,t\in(t_j,T],\end{cases} \tag{16}$$

and

$$\begin{aligned}w(t)=&\begin{cases}\phi(t), t\in(t_0,t_1],\\ \vdots\\ \phi(t), t\in(t_B,T],\end{cases}\\ &+\begin{cases}0, & t\in(t_0,t_1],\\ \frac{Q_1(w(t_1^-))-w_0-\int_{t_0}^{t_1}\frac{gdr}{r^{1-\kappa}}}{\Gamma(b)}\left[\frac{t^\kappa-(t_1)^\kappa}{\kappa}\right]^{b-1} & ,t\in(t_1,T],\end{cases}\\ &+\sum_{j=2}^{B}\begin{cases}0, & t\in(t_0,t_j],\\ \frac{Q_j(w(t_j^-))-Q_{j-1}(w(t_{j-1}^-))-\int_{t_{j-1}}^{t_j}\frac{gdr}{r^{1-\kappa}}}{\Gamma(b)} & \\ \left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1}, & t\in(t_j,T].\end{cases}\end{aligned} \tag{17}$$

Remark 3: Equations (16) and (17) are only *the particular solution* of equations (2) and (1) respectively, because they do not include $\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}$ $(j=1,2,...,B)$ that can be used to construct another piecewise function satisfying the conditions in equation (1) [or equation (2)] [about the constructing details, we can refer to equation (23)].

3.3 The EIEs of equations (2) and (1)

We will first consider the EIE of equation (2) by applying equation (8), equation (10) and the particular solution (16) in this subsection.

Theorem 4: Let η be an arbitrary constant, and let $g:(t_0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy $g(\cdot,w(\cdot))\in\widetilde{C}_{1-b,\kappa}[t_0,T]$ for any $w(\cdot)\in\widetilde{C}_{1-b,\kappa}[t_0,T]$.

Let $w(t)\in I_g([t_0,T],\mathbb{R})$, then $w(t)$ satisfies (2) iff $w(t)$ satisfies

$$w(t)=\begin{cases}\frac{w_0}{\Gamma(b)}\left[\frac{t^\kappa-(t_0)^\kappa}{\kappa}\right]^{b-1}+\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}},\\ t\in(t_0,t_1],\\ \frac{w_0}{\Gamma(b)}\left[\frac{t^\kappa-(t_0)^\kappa}{\kappa}\right]^{b-1}+\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}\\ +\sum_{i=1}^{j}\frac{P_i(w(t_i^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_i)^\kappa}{\kappa}\right]^{b-1}\\ +\eta\sum_{i=1}^{j}P_i(w(t_i^-))\left\{\frac{w_0+\int_{t_0}^{t_i}\frac{gdr}{r^{1-\kappa}}}{\Gamma(b)}\left[\frac{t^\kappa-(t_i)^\kappa}{\kappa}\right]^{b-1}\right.\\ +\int_{t_i}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}-\frac{w_0}{\Gamma(b)}\left[\frac{t^\kappa-(t_0)^\kappa}{\kappa}\right]^{b-1}\\ -\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}-\frac{[\frac{t^\kappa-(t_i)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)}\int_{t_0}^{t_i}\frac{[\frac{(t_i)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\\ \left.-\int_{t_i}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_i}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}}\right\},\\ t\in(t_j,t_{j+1}],\ j=1,2,...,B.\end{cases} \tag{18}$$

Remark 4: For easy verification that equation (18) satisfies the conditions of the fractional derivative and integral in equation (2), we give another expression of equation (18):

$$x(t)=\begin{cases}\phi(t), t\in(t_0,t_1],\\ \vdots\\ \phi(t), t\in(t_B,T],\end{cases}$$
$$+\sum_{j=1}^{B}\begin{cases}0, & t\in(t_0,t_j],\\ \frac{P_j(w(t_j^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1} & ,t\in(t_j,T],\end{cases}$$
$$+\sum_{j=1}^{B}\begin{cases}0,\\ t\in(t_0,t_j],\\ \eta P_j(w(t_j^-))\{\Phi_j(t)-\phi(t)\\ -\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}}\\ -\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)}\int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{-\kappa}}\Big\},\\ t\in(t_j,T].\end{cases} \tag{19}$$

Proof: We first prove the necessity. Consider equation (2) with an impulse: For $\forall j\in\{1,2,...,B\}$,

$$\begin{cases}{}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}w(t)=g(t,w(t)), \quad t\in(t_0,T] \text{ and } t\neq t_j,\\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa}w(t)\Big|_{t=t_j^+}\\ -{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa}w(t)\Big|_{t=t_j^-}\\ =P_j(w(t_j^-)),\\ {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa}w(t)\Big|_{t\to t_0+}=w_0.\end{cases} \tag{20}$$

By equation (13) and *Lemma 1*, the solution of equation (20) satisfies

$$\lim_{P_j(w(t_j^-))\to 0}\{w(t)\}=\phi(t)=\frac{w_0}{\Gamma(b)}\left[\frac{t^\kappa-(t_0)^\kappa}{\kappa}\right]^{b-1}$$
$$+\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}},\ t\in(t_0,T], \tag{21}$$

and

$$w(t)=\phi(t)=\frac{w_0}{\Gamma(b)}\left[\frac{t^\kappa-(t_0)^\kappa}{\kappa}\right]^{b-1}$$
$$+\int_{t_0}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}},\quad t\in(t_0,t_j], \tag{22}$$

with ${}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^-} = w_0 + \int_{t_0}^{t_j} \frac{gdr}{r^{1-\kappa}}$. And then, we use Lemmas 1–3 to construct an approximate EIE of equation (20):

$$
\begin{aligned}
\widetilde{w}(t) = &\begin{cases} \phi(t), & t\in(t_0,t_j],\\ \frac{{}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\big|_{t=t_j^+}}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1} \\ +\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}, & t\in(t_j,T],\end{cases}\\
&-\begin{cases} 0, & t\in(t_0,t_j],\\ \int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}}, \\ +\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)}\int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(t_j,T].\end{cases}
\end{aligned}
\tag{23}
$$

Equation (23) meets the conditions of the fractional derivative, the impulsive condition and the initial value in equation (20) to be only the approximate EIE of equation (20) because equation (23) dissatisfies equation (21).

Substituting ${}^{K}_{t_0}\mathcal{I}^{1-b,\kappa}_{t} w(t)\Big|_{t=t_j^+} = w_0 + \int_{t_0}^{t_j} \frac{gdr}{r^{1-\kappa}} + P_j(w(t_j^-))$ into (23), we have

$$
\begin{aligned}
\widetilde{w}(t) = &\begin{cases} \phi(t), t\in(t_0,t_j],\\ 0, \quad t\in(t_j,T],\end{cases}\\
&+\begin{cases} 0, & t\in(t_0,t_j],\\ \frac{P_j(w(t_j^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1}, & t\in(t_j,T],\end{cases}\\
&+\begin{cases} 0, & t\in(t_0,t_j],\\ \Phi_j(t)-\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} \\ -\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)}\int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(t_j,T].\end{cases}
\end{aligned}
\tag{24}
$$

Let $e(t)=w(t)-\widetilde{w}(t)$ for $t\in(t_0,T]$ denote the error between $\widetilde{w}(t)$ and the EIE of equation (20).

By equations (21) and (24), we get

$$
\begin{aligned}
\lim_{P_j(w(t_j^-))\to0} e(t) &= \lim_{P_j(w(t_j^-))\to0}\{w(t)\}\\
&- \lim_{P_j(w(t_j^-))\to0}\{\widetilde{w}(t)\}\\
&=\begin{cases} 0, & t\in(t_0,t_j],\\ \phi(t)-\Phi_j(t)+\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}} \\ +\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}}, & t\in(t_j,T].\end{cases}
\end{aligned}
\tag{25}
$$

From equation (25), we suppose

$$e(t) = f(P_j(w(t_j^-))) \lim_{P_j(w(t_j^-))\to 0} e(t),$$

here f is an undetermined function

$$= -\begin{cases} 0, & t \in (t_0, t_j], \\ f(P_j(w(t_j^-)))\,\{\Phi_j(t) - \phi(t) \\ -\int_{t_j}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j} \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} \\ -\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)}\int_{t_0}^{t_j} \\ \frac{[\frac{(t_j)^\kappa - s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\Big\}, & t \in (t_j, T]. \end{cases} \tag{26}$$

By plugging equations (24) and (26) into $w(t) = \widetilde{w}(t) + e(t)$, the EIE of equation (20) is

$$w(t) = \begin{cases} \phi(t), t \in (t_0, t_j], \\ \phi(t), t \in (t_j, T], \end{cases}$$
$$+\begin{cases} 0, & t \in (t_0, t_j], \\ \frac{P_j(w(t_j^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1}, & t \in (t_j, T], \end{cases}$$
$$+\begin{cases} 0, & t \in (t_0, t_j], \\ \left[1 - f(P_j(w(t_j^-)))\right]\{\Phi_j(t) - \phi(t) \\ -\int_{t_j}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j} \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} \\ -\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa - s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\Big\}, & t \in (t_j, T]. \end{cases} \tag{27}$$

Because equation (20) is a special case of equation (2), equation (27) is a special case of the EIE of equation (2). And moreover, the impulsive effects in equation (2) have the linear additivity by equations (13) and (14). Thus, by combining equation (27) and the linear additivity of impulsive effects with equation (16), we give the EIE of equation (2):

$$x(t) = \begin{cases} \phi(t), t \in (t_0, t_1], \\ \vdots \\ \phi(t), t \in (t_B, T], \end{cases}$$
$$+\sum_{j=1}^{B}\begin{cases} 0, & t \in (t_0, t_j], \\ \frac{P_j(w(t_j^-))}{\Gamma(b)}\left[\frac{t^\kappa-(t_j)^\kappa}{\kappa}\right]^{b-1}, & t \in (t_j, T], \end{cases}$$

$$
+\sum_{j=1}^{B}\begin{cases} 0, & t\in(t_0,t_j],\\ \left[1-f(P_j(w(t_j^-)))\right]\left[\Phi_j(t)-\phi(t)\right. & \\ -\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} & \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} & \\ -\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} & \\ \left.\int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right], & t\in(t_j,T]. \end{cases} \tag{28}
$$

Because equation (28) must meet equation (14), we have

$$
\begin{aligned}
&\left[1-f(P_i(w(t_i^-)))\right]+\left[1-f(P_j(w(t_j^-)))\right]\\
&=1-f\left(P_i(w(t_i^-))+P_j(w(t_j^-))\right)\\
&\text{for } \forall P_i(w(t_i^-)),P_j(w(t_j^-))\in\mathbb{R},
\end{aligned} \tag{29}
$$

then $1-f(P_j(w(t_j^-)))=\eta P_j(w(t_j^-))$ where η is an arbitrary real, and equation (28) is equation (19).

‘Sufficiency’. We use two equations (8) and (10) to compute the fractional integral and derivative for a part of equation (19):

$$
\begin{aligned}
&{}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}\begin{cases} 0, & t\in(t_0,t_j],\\ \Phi_j(t)-\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} & \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} & \\ -\phi(t)-\frac{[\frac{t^\kappa-(t_i)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} & \\ \int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(t_j,T], \end{cases}\\
&=\begin{cases} 0, & t\in(t_0,t_j],\\ g(t,w(t))-\frac{t^{1-\kappa}d}{dt} & \\ \left[\int_{t_0}^{t_j}\frac{(\frac{t^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right. & \\ \left.+\int_{t_j}^{t}\frac{(\frac{t^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right], & \\ t\in(t_j,T], & \end{cases}\\
&=\begin{cases} 0, & t\in(t_0,t_j],\\ g(t,w(t))-{}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}\phi(t), & t\in(t_j,T], \end{cases}\\
&\ \text{(by using Lemma 1)}\\
&=0,\ \text{ here } j=1,2,...,B,
\end{aligned} \tag{30}
$$

and

$$
{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa}\begin{cases} 0, & t\in(t_0,t_j],\\ \Phi_j(t)-\int_{t_j}^{t}\frac{(\frac{t^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right] \\ \frac{dr}{r^{1-\kappa}}-\phi(t)-\frac{[\frac{t^\kappa-(t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(t_j,T],\end{cases}
$$

$$
=\begin{cases} 0, & t\in(t_0,t_j],\\ w_0+\int_{t_0}^{t}\frac{gdr}{r^{1-\kappa}}-\int_{t_j}^{t} \\ \left[\frac{r^{1-\kappa}d}{dr}\int_{t_0}^{t_j}\frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} \\ -\int_{t_j}^{t}\frac{(\frac{t^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}} \\ -\int_{t_0}^{t_j}\frac{[\frac{(t_j)^\kappa-s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(t_j,T],\end{cases}
$$

$$
=\begin{cases} 0, & t\in(t_0,t_j],\\ w_0+\int_{t_0}^{t}\frac{gdr}{r^{1-\kappa}}-\int_{t_0}^{t}\frac{(\frac{t^\kappa-s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \\ \left[\frac{w_0}{\Gamma(b)}\left(\frac{s^\kappa-(t_0)^\kappa}{\kappa}\right)^{b-1}\right. \\ \left.+\int_{t_0}^{s}\frac{(\frac{s^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(b)}\frac{gdr}{r^{1-\kappa}}\right]\frac{ds}{s^{1-\kappa}}, & t\in(t_j,T],\end{cases}
$$

$$
=\begin{cases} 0, & t\in(t_0,t_j],\\ \int_{t_0}^{t}\frac{gdr}{r^{1-\kappa}} \\ -\int_{t_0}^{t}\left[\int_{r}^{t}\frac{(\frac{t^\kappa-s^\kappa}{\kappa})^{-b}(\frac{s^\kappa-r^\kappa}{\kappa})^{b-1}}{\Gamma(1-b)\Gamma(b)}\frac{ds}{s^{1-\kappa}}\right] \\ \frac{gdr}{r^{1-\kappa}}, & t\in(t_j,T],\end{cases}
$$

$$
=0,\ \text{here } j=1,2,...,B. \tag{31}
$$

By equations (30) and (31), equation (19) satisfies ${}_{t_0}^{K}\mathcal{D}_t^{b,\kappa}w(t)=g(t,w(t))$ for $t\in\cup_{i=0}^{B}(t_i,t_{i+1}]$ [that is, equation (19) satisfies the fractional derivative] and

$$
{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa}w(t)=\begin{cases} w_0+\int_{t_0}^{t}\frac{gdr}{r^{1-\kappa}}, t\in(t_0,t_1],\\ \vdots \\ w_0+\int_{t_0}^{t}\frac{gdr}{r^{1-\kappa}}, t\in(t_B,T],\end{cases}
$$

$$
+\sum_{j=1}^{B}\begin{cases} 0, & t\in(t_0,t_j],\\ P_j(w(t_j^-)), t\in(t_j,T].\end{cases} \tag{32}
$$

By equation (32), we deduce that equation (19) satisfies the impulsive conditions and initial value in (2). And it is easily verified that equation (19) also satisfies equations (13) and (14). The proof is completed. □

Theorem 5: Let η be an arbitrary constant, and let $g:(t_0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy $g(\cdot,w(\cdot))\in\widetilde{C}_{1-b,\kappa}[t_0,T]$ for any $w(\cdot)\in\widetilde{C}_{1-b,\kappa}[t_0,T]$.

Let $w(t) \in I_g([t_0, T], \mathbb{R})$, then $w(t)$ satisfies (1) iff $w(t)$ satisfies

$$
\begin{aligned}
w(t) &= \overline{w}(t) + \eta \left[Q_1(w(t_1^-)) - w_0 - \int_{t_0}^{t_1} \frac{g dr}{r^{1-\kappa}} \right] \\
&\times \begin{cases} 0, & t \in (t_0, t_1], \\ \Phi_1(t) - \phi(t) - \frac{[\frac{t^\kappa - (t_1)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_1} \frac{[\frac{(t_1)^\kappa - s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \\ - \int_{t_1}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa} d}{dr} \int_{t_0}^{t_1} \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \right] \frac{dr}{r^{1-\kappa}}, & t \in (t_1, T], \end{cases} \\
&+ \eta \sum_{j=2}^{B} \left[Q_j(w(t_j^-)) - Q_{j-1}(w(t_{j-1}^-)) - \int_{t_{j-1}}^{t_j} \frac{g dr}{r^{1-\kappa}} \right] \\
&\times \begin{cases} 0, & t \in (t_0, t_j], \\ \Phi_j(t) - \phi(t) - \frac{[\frac{t^\kappa - (t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_j} \frac{[\frac{(t_j)^\kappa - s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \\ - \int_{t_j}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa} d}{dr} \int_{t_0}^{t_j} \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \right] \frac{dr}{r^{1-\kappa}}, & t \in (t_j, T], \end{cases}
\end{aligned} \tag{33}
$$

where $\overline{w}(t)$ is the particular solution (17).

Proof: 'Necessity'. By (3) and the particular solution (17) and Theorem 4, the solution of equation (1) satisfies

$$
\begin{aligned}
w(t) &= \overline{w}(t) \\
&+ \eta \sum_{j=1}^{B} \left[Q_j(w(t_j^-)) - {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t) \Big|_{t=t_j^-} \right] \\
&\times \begin{cases} 0, & t \in (t_0, t_j], \\ \Phi_j(t) - \phi(t) - \frac{[\frac{t^\kappa - (t_j)^\kappa}{\kappa}]^{b-1}}{\Gamma(b)} \\ \int_{t_0}^{t_j} \frac{[\frac{(t_j)^\kappa - s^\kappa}{\kappa}]^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \\ - \int_{t_j}^{t} \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{b-1}}{\Gamma(b)} \\ \left[\frac{r^{1-\kappa} d}{dr} \int_{t_0}^{t_j} \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-b}}{\Gamma(1-b)} \frac{\phi(s) ds}{s^{1-\kappa}} \right] \frac{dr}{r^{1-\kappa}}, & t \in (t_j, T], \end{cases}
\end{aligned} \tag{34}
$$

with ${}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^+} = Q_j(w(t_j^-))$ $(j = 1, 2, ..., B)$.

By using equation (31), the fractional integral of equation (34) is

$$
{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t) = {}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} \overline{w}(t)
$$

$$
= \begin{cases} w_0 + \int_{t_0}^{t} \frac{gdr}{r^{1-\kappa}}, t \in (t_0, t_1], \\ \vdots \\ w_0 + \int_{t_0}^{t} \frac{gdr}{r^{1-\kappa}}, t \in (t_B, T], \end{cases}
$$

$$
+ \begin{cases} 0, & t \in (t_0, t_1], \\ Q_1(w(t_1^-)) - w_0 - \int_{t_0}^{t_1} \frac{gdr}{r^{1-\kappa}}, & t \in (t_1, T], \end{cases}
$$

$$
+ \sum_{k=2}^{N} \begin{cases} 0, & t \in (t_0, t_j], \\ Q_j(w(t_j^-)) - Q_{j-1}(w(t_{j-1}^-)) & \\ - \int_{t_{j-1}}^{t_j} \frac{gdr}{r^{1-\kappa}}, & t \in (t_j, T]. \end{cases} \tag{35}
$$

By equation (35) and the conditions ${}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^+} = Q_j(w(t_j^-))$ $(j = 1, 2, ..., B)$, we have

$$
{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_1^-} = w_0 + \int_{t_0}^{t_1} \frac{gdr}{r^{1-\kappa}},
$$

$$
{}_{t_0}^{K}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^-} = Q_{j-1}(w(t_{j-1}^-))
$$

$$
+ \int_{t_{j-1}}^{t_j} \frac{gdr}{r^{1-\kappa}}, \ j = 2, ..., B. \tag{36}
$$

Then we plug equation (36) into equation (34) to obtain that the EIE of equation (1) is equation (33).

'Sufficiency'. By using equations (30) and (31), we easily verify that equation (33) satisfies the conditions of fractional derivative and integral, impulsive conditions and initial value in equation (1). And we easily verify that equation (33) satisfies equation (15). The proof is completed. □

4 Applications

For two IGFrOSs, two numerical models are given to show their EIEs and the non-uniqueness of solution of two IGFrOSs in this section.

Example: Consider two IGFrOSs:

$$
\begin{cases} {}_{1}^{K}\mathcal{D}_t^{\frac{1}{2},\kappa} x(t) = t, & \kappa > 0, t \in (1, 5] \text{ and } t \neq 3, \\ {}_{1}^{K}\mathcal{I}_t^{\frac{1}{2},\kappa} x(t)\Big|_{t=3^+} & \\ - {}_{1}^{K}\mathcal{I}_t^{\frac{1}{2},\kappa} x(t)\Big|_{t=3^-} = 1, & \\ {}_{1}^{K}\mathcal{I}_t^{\frac{1}{2},\kappa} x(t)\Big|_{t\to 1^+} = 1, & \end{cases} \tag{37}
$$

and

$$
\begin{cases} {}_{1}^{K}\mathcal{D}_t^{\frac{1}{2},\kappa} x(t) = t, & \kappa > 0, t \in (1, 5] \text{ and } t \neq 3, \\ {}_{1}^{K}\mathcal{I}_t^{\frac{1}{2},\kappa} x(t)\Big|_{t=3^+} = 1, & \\ {}_{1}^{K}\mathcal{I}_t^{\frac{1}{2},\kappa} x(t)\Big|_{t\to 1^+} = 1. & \end{cases} \tag{38}
$$

By Theorems 4–5, the EIEs of equations (37) and (38) are respectively given by

$$x(t) = \begin{cases} \phi(t), t \in (1,3], \\ \phi(t), t \in (3,5], \end{cases}$$
$$+ \begin{cases} 0, & t \in (1,3], \\ \frac{1}{\Gamma(\frac{1}{2})} \left[\frac{t^\kappa - 3^\kappa}{\kappa}\right]^{-\frac{1}{2}}, & t \in (3,5], \end{cases}$$
$$+\eta \begin{cases} 0, & t \in (1,3], \\ \Phi(t) - \int_3^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \left[\frac{r^{1-\kappa} d}{dr} \int_1^3 \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s)ds}{s^{1-\kappa}}\right] \frac{dr}{r^{1-\kappa}} & \\ -\phi(t) - \frac{[\frac{t^\kappa - 3^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \int_1^3 \frac{[\frac{3^\kappa - s^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s)ds}{s^{1-\kappa}}, & t \in (3,5], \end{cases} \tag{39}$$

and

$$x(t) = \begin{cases} \phi(t), t \in (1,3], \\ \phi(t), t \in (3,5], \end{cases}$$
$$- \begin{cases} 0, & t \in (1,3], \\ \frac{3^{\kappa+1} - 1}{(\kappa+1)\Gamma(\frac{1}{2})} \left[\frac{t^\kappa - 3^\kappa}{\kappa}\right]^{-\frac{1}{2}}, & t \in (3,5], \end{cases}$$
$$+\eta \frac{3^{\kappa+1} - 1}{\kappa + 1} \begin{cases} 0, & t \in (1,3], \\ \int_3^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \left[\frac{r^{1-\kappa} d}{dr} \int_1^3 \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s)ds}{s^{1-\kappa}}\right] & \\ \frac{dr}{r^{1-\kappa}} & \\ -\Phi(t) + \phi(t) & \\ + \frac{[\frac{t^\kappa - 3^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \int_1^3 \frac{[\frac{3^\kappa - s^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s)ds}{s^{1-\kappa}}, & t \in (3,5], \end{cases} \tag{40}$$

where η is an arbitrary constant,

$$\phi(t) = \frac{1}{\Gamma(\frac{1}{2})} \left[\frac{t^\kappa - 1}{\kappa}\right]^{-\frac{1}{2}} + \int_1^t \frac{(\frac{t^\kappa - s^\kappa}{\kappa})^{-\frac{1}{2}} s^\kappa}{\Gamma(\frac{1}{2})} ds, \tag{41}$$

and

$$\Phi(t) = \frac{3^{\kappa+1} + \kappa}{(\kappa + 1)\Gamma(\frac{1}{2})} \left[\frac{t^\kappa - 3^\kappa}{\kappa}\right]^{-\frac{1}{2}}$$
$$+ \int_3^t \frac{(\frac{t^\kappa - s^\kappa}{\kappa})^{-\frac{1}{2}} s^\kappa}{\Gamma(\frac{1}{2})} ds. \tag{42}$$

Figure 1 Three solution trajectories of equation (37) with $\kappa = 0.1$ (see online version for colours)

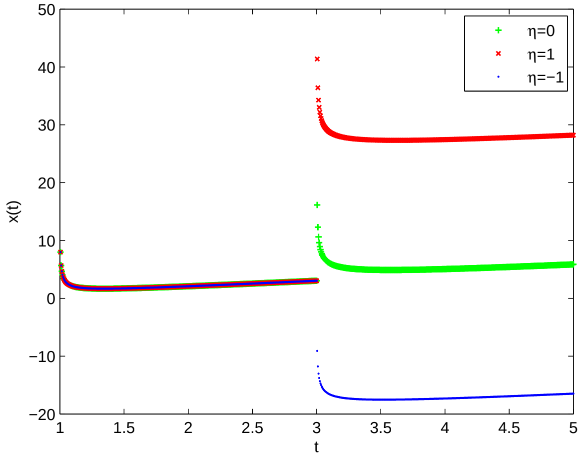
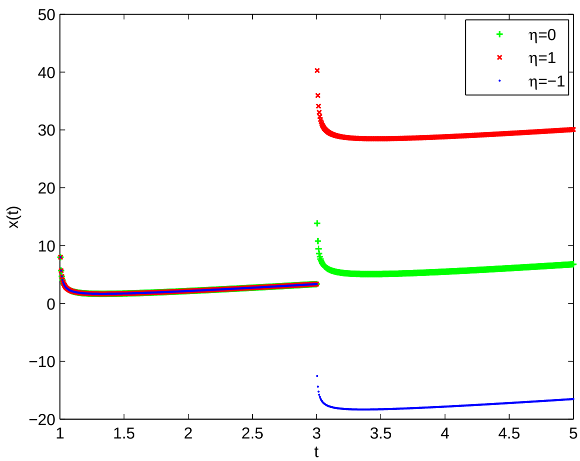


Figure 2 Three solution trajectories of equation (37) with $\kappa = 0.5$ (see online version for colours)



Plugging $b = \frac{1}{2}$ and $g(t, w(t)) = t$ into equations (30) and (31) respectively, we have

$$
{}_1^K\mathcal{D}_t^{\frac{1}{2},\kappa} \begin{cases} 0, & t \in (1, 3], \\ \Phi(t) - \int_3^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \\ \left[\frac{r^{1-\kappa} d}{dr} \int_1^3 \frac{(\frac{r^\kappa - s^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s) ds}{s^{1-\kappa}} \right] \frac{dr}{r^{1-\kappa}} \\ -\phi(t) - \frac{[\frac{t^\kappa - 3^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \\ \int_1^3 \frac{[\frac{3^\kappa - s^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\phi(s) ds}{s^{1-\kappa}}, & t \in (3, 5], \end{cases}
$$

$$
= 0, \tag{43}
$$

and

$$
{}_1^K\mathcal{I}_t^{\frac{1}{2},\kappa}\begin{cases} 0, & t\in(1,3],\\ \Phi(t) - \int_3^t \frac{(\frac{t^\kappa - r^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \left[\frac{r^{1-\kappa}d}{dr}\int_1^3 \frac{(\frac{r^\kappa-s^\kappa}{\kappa})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\frac{\phi(s)ds}{s^{1-\kappa}}\right]\frac{dr}{r^{1-\kappa}} & \\ -\phi(t) - \frac{[\frac{t^\kappa-3^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & \\ \int_1^3 \frac{[\frac{3^\kappa - s^\kappa}{\kappa}]^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\frac{\phi(s)ds}{s^{1-\kappa}}, & t\in(3,5], \end{cases}
$$

$$
= 0. \tag{44}
$$

Thus, by equations (43) and (44), we easily verify that equations (39) and (40) satisfy all conditions in equations (37) and (38) respectively.

Figure 3 Three solution trajectories of equation (37) with κ = 0.9 (see online version for colours)

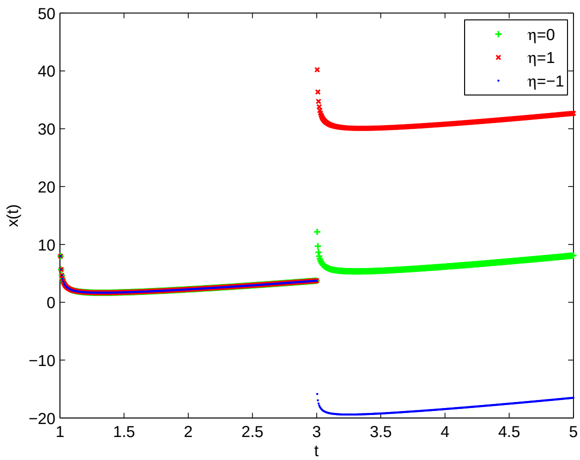


Figure 4 Three solution trajectories of equation (37) with κ = 1.5 (see online version for colours)

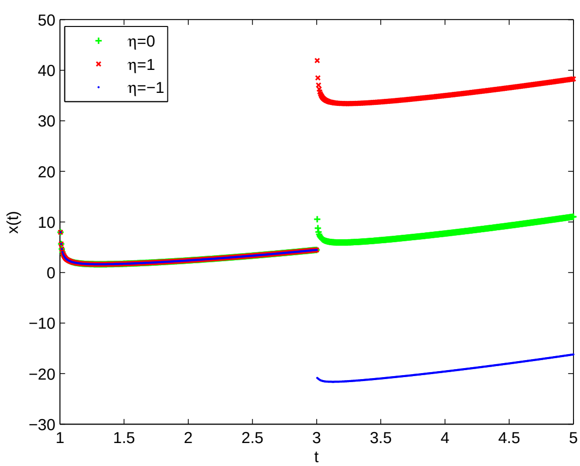


Figure 5 Three solution trajectories of equation (38) with κ = 0.1 (see online version for colours)

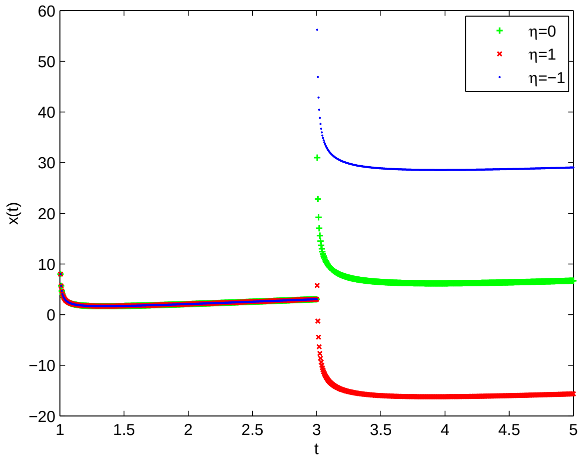
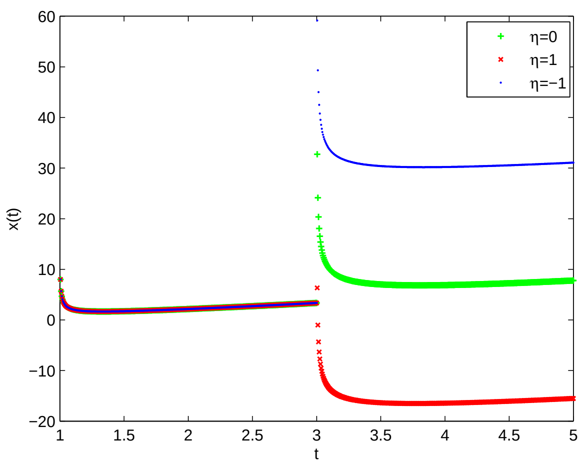


Figure 6 Three solution trajectories of equation (38) with κ = 0.5 (see online version for colours)



To show the non-uniqueness of solution of equations (37) and (38), we use the numerical simulation to draw the solution trajectories of four cases (κ = 0.1, 0.5, 0.9 and 1.5) of equations (37) and (38) in Figures 1–8 respectively, which three curves of solution in each figure represent equations (39) and (40) with the corresponding κ and η = 0, 1, –1, respectively. And the numerical simulation is based on the Euler's method of numerical approximation of definite integral with the step size l = 0.005.

5 Conclusions

The EIEs of equations (1) and (2) are two integral equations with an arbitrary constant, which reveal the non-uniqueness of solution of equations (1) and (2). And moreover, equations (1) and (2) have the relation

$$\text{system (1)} = \text{system (2) with } P_j(w(t_j^-)) = Q_j(w(t_j^-)) - {}^{K}_{t_0}\mathcal{I}_t^{1-b,\kappa} w(t)\Big|_{t=t_j^-} \quad j = 1, 2, ..., B.$$

Figure 7 Three solution trajectories of equation (38) with κ = 0.9 (see online version for colours)

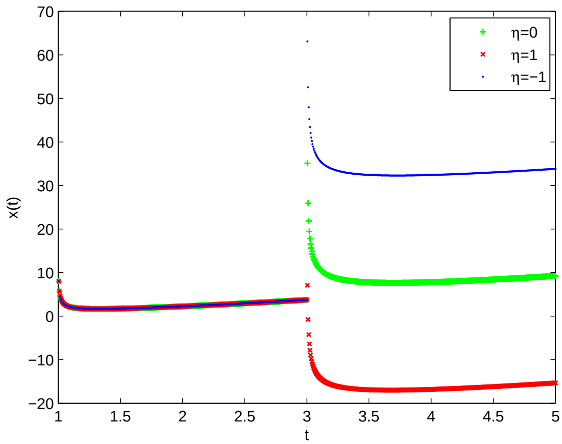
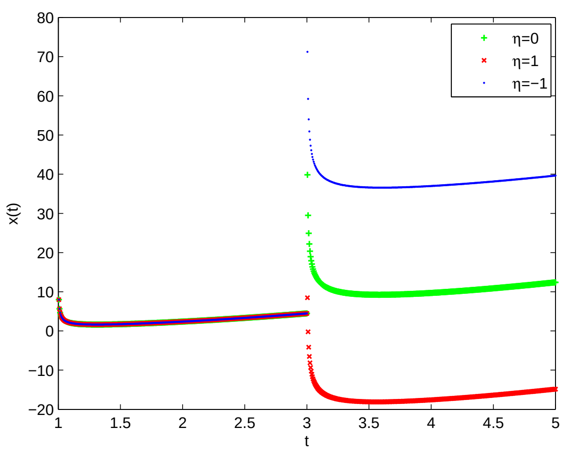


Figure 8 Three solution trajectories of equation (38) with κ = 1.5 (see online version for colours)



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