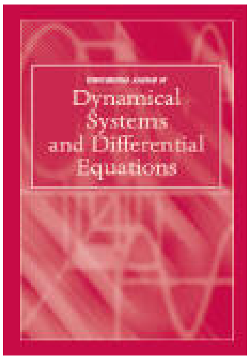

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The limit cycles of a class of discontinuous piecewise differential systems

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Abstract: The determination of the maximum number of limit cycles and their possible positions in the plane is one of the most difficult problems in the qualitative theory of planar differential systems. This problem is related to the second part of the unsolved 16th Hilbert's problem. Due to their applications in modelling many natural phenomena, piecewise differential systems have recently attracted big attention. The upper bound number of limit cycles that a class of differential systems may exhibit is typically very difficult to determine. In this work we extend the second part of the 16th Hilbert's problem to the planar discontinuous piecewise differential systems separated by a straight line and formed by an arbitrary linear centre and an arbitrary cubic uniform isochronous centre. We provide for this class of piecewise differential systems an upper bound on its maximal number of limit cycles, and we prove that such an upper bound is reached.

Keywords: cubic uniform isochronous centre; linear centre; limit cycle; discontinuous piecewise differential system.

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1 Introduction and statement of the main results

Limit cycles are one of the main remarkable and important solutions of differential equations. The notion of a limit cycle appeared first at the end of the 19th century with Poincaré (1891). Later on Hilbert stated a list of 23 problems for the advancement of mathematical science, and from then it started intensive research on these problems throughout the 20th century. From the 23 problems only the so-called 16th Hilbert's problem and the Riemann conjecture remain open until now. The second part of the 16th Hilbert problem, which has two parts, asks for an upper bound on the number of possible limit cycles, but only for planar polynomial differential systems of a given degree. For a differential system a *limit cycle* is an isolated periodic orbit in the set of all periodic orbits of this differential system.

Recently the second part of the 16th Hilbert's problem has become an interesting topic of research for many scientists because of the main role of limit cycles in understanding and explaining the dynamics of many natural phenomena, for example, the Sel'kov model of glycolysis (Sel'kov, 1968), that showed the existence of a stable limit cycle which represent the normal physiological behaviour in the human body, also some nonlinear electrical circuits exhibit limit cycle oscillations, which inspired the original Van der Pol model (van der Pol, 1920, 1926), or one of the Belousov Zhavotinskii model (Belousov, 1959), etc.

Numerous domains of applied mathematics including electronics, mechanics, neuroscience, economics, etc., commonly use the dynamics of piecewise differential systems, see for instance Bernardo et al. (2008), Makarenkov and Lamb (2012) and Simpson (2010), these systems become a very interesting topic. In 1920, Andronov et al. published their first research on piecewise linear discontinuous differential systems (Andronov et al., 1996). Many studies on piecewise linear differential systems come from applications, for instance control theory and electric circuit design. We can distinguish between two kinds of limit cycles for discontinuous piecewise differential systems: sliding and crossing. A *sliding limit cycle* is a limit cycle that contains some arc of the lines of discontinuity that separate the different differential systems that form the piecewise differential system. The *crossing limit cycle* is the one that contains only isolated points of the discontinuity lines. Here we focus only on the crossing limit cycles (see Pi and Zhang, 2013) for a more precise definition.

In recent years many publications appeared where the authors provided examples with at most three limit cycles concerning the simplest family of planar discontinuous piecewise differential systems formed by two linear differential systems separated by a straight line, see Euzébio and Llibre (2015), Freire et al. (2012, 2014, 2015), Llibre et al. (2013, 2015), Llibre and Teixeira (2017) and Llibre and Zhang (2018). Until now the solution of the extension of the second part of the 16th Hilbert's problem for this class of differential systems remains open.

Nowadays many papers consider piecewise differential systems where there is a nonlinear differential system in some pieces. However keep the straight line as the separation curve and study the maximum number of limit cycles of such piecewise differential systems. In Esteban et al. (2021), the authors solved the extension of the second part of the 16th Hilbert problem for discontinuous piecewise isochronous polynomial centres of degrees one and two separated by a straight line. Next Benterki and Llibre (2020) studied the same problem but for some classes of discontinuous piecewise isochronous polynomial centres of degrees one and three. In Benabdallah et al. (2023), the authors studied the second part of the 16th Hilbert problem for a class of discontinuous piecewise differential systems separated by a straight line and formed by linear and quadratic centres where they proved that the maximum number of limit cycles of this class of systems is at most four. In Buzzi et al. (2022), the authors study the maximum number of limit cycles of some classes of planar discontinuous piecewise differential systems separated by a straight line and formed by combinations of linear centres (consequently isochronous) and cubic isochronous centres with homogeneous nonlinearities.

Studying the maximum number of limit cycles for discontinuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = \begin{cases} \mathbf{F}^-(\mathbf{x}, \mathbf{y}) = \left(F_1^-(x,y), F_2^-(x,y)\right)^T \\ \quad \text{if } (x,y) \in \Gamma^-, \\ \mathbf{F}^+(\mathbf{x}, \mathbf{y}) = \left(F_1^+(x,y), F_2^+(x,y)\right)^T \\ \quad \text{if } (x,y) \in \Gamma^+, \end{cases} \tag{1}$$

is the main objective of this paper, where the straight line $\Gamma = \{(x, y) : x = 0\}$ is the separation curve of the plane, that separates it on the two half-planes

$$\Gamma^- = \{(x,y) : x \leq 0\}, \qquad \Gamma^+ = \{(x,y) : x \geq 0\},$$

and formed by a linear centre and a cubic uniform isochronous centre. On the straight line of discontinuity the flow of the piecewise differential system is defined following the rules of Filippov (1988).

In the next lemma we give the normal form of an arbitrary linear differential centre.

Lemma 1: Any linear differential centre can be written as

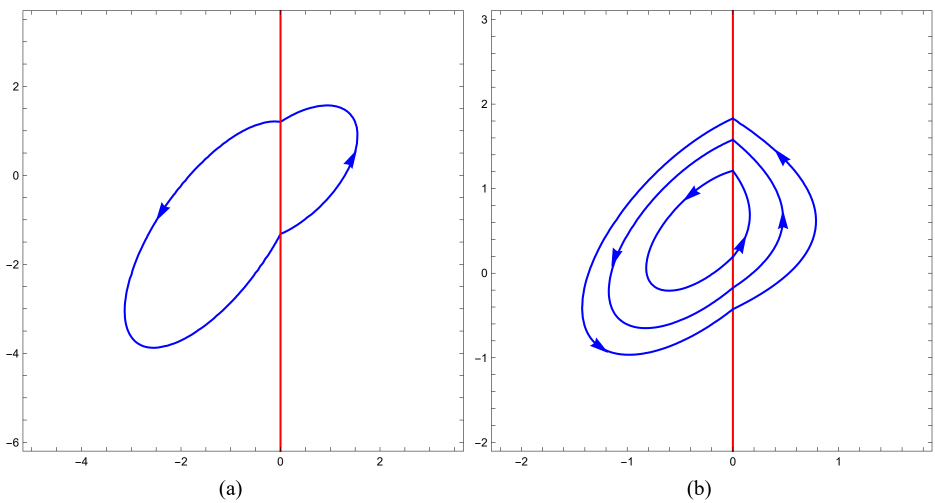
$$\dot{x} = d_1 - \beta x - \frac{y\left(4\beta^2 + \omega^2\right)}{4\alpha}, \quad \dot{y} = c_1 + \alpha x + \beta y, \\ \text{with } \alpha > 0, \ \omega > 0, \tag{2}$$

with its first integral

$$H(x, y) = 8\alpha(c_1 x - d_1 y) + 4(\alpha x + \beta y)^2 + y^2\omega^2. \tag{3}$$

For the proof of Lemma 1 see Llibre and Teixeira (2018).

Figure 1 (a) The unique limit cycle of the discontinuous piecewise differential system (21)–(22) (b) The three limit cycles of the discontinuous piecewise differential system (30)–(31) (see online version for colours)



The next result is due to Collins (1997), who classified the cubic polynomial uniform isochronous centres.

Lemma 2: A cubic polynomial differential system has a uniform isochronous centre at the origin if and only if after an affine change of variables and a rescaling of the independent variable it can be written as

$$\dot{x} = -y + xf(x, y), \quad \dot{y} = x + yf(x, y), \tag{4}$$

where $f(x, y) = a_1 x + a_2 y + a_4 xy$, and satisfies $a_1 a_2 = 0$ and $a_4 \neq 0$.

For other proof of Lemma 2 see Section 2 of Artés et al. (2017).

The normal form of the first integrals of the uniform cubic polynomial isochronous centres (4) is given in the following theorem:

Theorem 3: The first integrals of system (4) are described in what follows:

Case 1: $a_1^2 + a_2^2 = 0$, the corresponding first integral of system (4) is

$$H_1(x, y) = \frac{x^2 + y^2}{1 - a_4 x^2}. \tag{5}$$

Case 2: $a_1^2 + a_2^2 \neq 0$, the first integral of system (4)

- *Subcase 2.1:* If $4a_4 - a_1^2 < 0$ and $a_2 = 0$ is

$$H_2^{(1)}(x, y) = \frac{(a_1 + 2a_4 y + S)^{1-S/a_1} \left(x^2 + y^2\right)^{S/a_1}}{(-a_1 - 2a_4 y + S)^{S/a_1+1}}, \tag{6}$$

  with $S = \sqrt{a_1^2 - 4a_4}$.
- *Subcase 2.2:* If $4a_4 + a_2^2 > 0$ and $a_1 = 0$ is

$$H_2^{(2)}(x, y) = \frac{(a_2 + 2a_4 x + S)^{1-S/a_2} \left(x^2 + y^2\right)^{S/a_2}}{(-a_2 - 2a_4 x + S)^{S/a_2+1}}, \tag{7}$$

  with $S = \sqrt{a_2^2 + 4a_4}$.
- *Subcase 2.3:* If $4a_4 - a_1^2 > 0$ and $a_2 = 0$ is

$$H_2^{(3)}(x, y) = \left(\frac{x^2 + y^2}{a_1 y + a_4 y^2 + 1}\right)^{S/a_1} \times e^{-2 \arctan((a_1 + 2a_4 y)/S)}, \tag{8}$$

  with $S = \sqrt{4a_4 - a_1^2}$.
- *Subcase 2.4:* If $4a_4 + a_2^2 < 0$ and $a_1 = 0$ is

$$H_2^{(4)}(x, y) = \left(\frac{x^2 + y^2}{a_2 x + a_4 x^2 - 1}\right)^{S/a_2} \times e^{-2 \arctan((a_2 + 2a_4 x)/S)}. \tag{9}$$

- *Subcase 2.5:* If $4a_4 - a_1^2 = 0$ and $a_2 = 0$ is

$$H_2^{(5)}(x, y) = \frac{\left(x^2 + y^2\right) e^{(4/(a_1 y + 2))}}{(a_1 y + 2)^2}. \tag{10}$$

- *Subcase 2.6:* If $4a_4 + a_2^2 = 0$ and $a_1 = 0$ is

$$H_2^{(6)}(x, y) = \frac{\left(x^2 + y^2\right) e^{(4/(2 - a_2 x))}}{(2 - a_2 x)^2}. \tag{11}$$

Theorem 3 is proved in Section 2 of Artés et al. (2017).

Our main result is given in the following theorem:

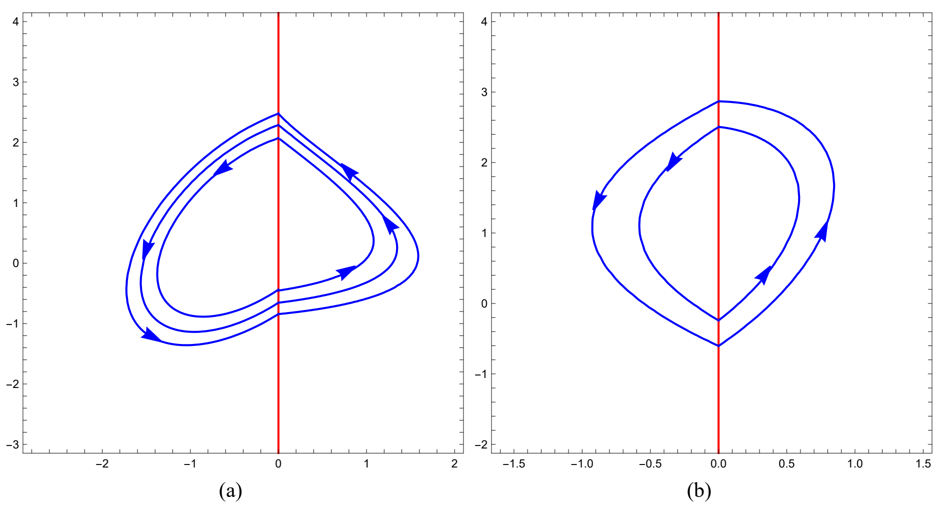
Theorem 4: For a piecewise smooth differential system with two zones, separated by the straight line $x = 0$, and formed by an arbitrary linear centre and an arbitrary cubic uniform isochronous centre the maximum number of limit cycles is at most

1 One if $a_1^2 + a_2^2 = 0$, and there are systems of this type with exactly one limit cycle, see Figure 1(a).

2 Three if $4a_4 - a_1^2 < 0$ and $a_2 = 0$, or $4a_4 + a_2^2 > 0$ and $a_1 = 0$; four if $4a_4 - a_1^2 > 0$ and $a_2 = 0$, or $4a_4 + a_2^2 < 0$ and $a_1 = 0$; two if $4a_4 - a_1^2 = 0$ and $a_2 = 0$, or $4a_4 + a_2^2 = 0$ and $a_1 = 0$. There are systems of these types with exactly three limit cycles shown in Figure 1(b), three limit cycles shown in Figure 2(a), and two limit cycles shown in Figure 2(b),

Where a_1, a_2 and a_4 are the parameters of the cubic differential system when transformed into its normal form (4).

Figure 2 (a) The three limit cycles of the discontinuous piecewise differential system (32)–(33) (b) The two limit cycles of the discontinuous piecewise differential system (34)–(35) (see online version for colours)



Theorem 4 is proved in Section 3.

2 The arbitrary cubic uniform isochronous centres

Now we give the expression of the cubic uniform isochronous centres (4) with its corresponding first integrals after doing an arbitrary affine change of variables

$\{x \to \alpha_1 x + \beta_1 y + \gamma_1, y \to \alpha_2 x + \beta_2 y + \gamma_2\}$, with $\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0$. In this way we obtain the expression of all cubic uniform isochronous centres.

System (4) becomes

$$
\begin{aligned}
\dot{x} = {} & \frac{1}{\alpha_2\beta_1 - \alpha_1\beta_2}(\gamma_2(a_1\beta_1\gamma_1 \\
& - \beta_2\gamma_1(a_2 + a_4\gamma_1) + \beta_2) \\
& + x^2(a_1\alpha_1(\alpha_2\beta_1 - \alpha_1\beta_2) \\
& + a_2\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) \\
& + a_4(\alpha_2\gamma_1(\alpha_2\beta_1 - 2\alpha_1\beta_2) \\
& + \alpha_1\gamma_2(2\alpha_2\beta_1 - \alpha_1\beta_2) \\
& + y(\alpha_1\beta_2 + \alpha_2\beta_1)(\alpha_2\beta_1 - \alpha_1\beta_2))) \\
& + x(\alpha_2(a_1\beta_1(\gamma_1 + \beta_1 y) \\
& + (a_2 + a_4(\gamma_1 + \beta_1 y))(2\beta_1\gamma_2 - \beta_2\gamma_1 \\
& + \beta_1\beta_2 y) + \beta_2) - \alpha_1(\beta_1(\beta_2 y \\
& - \gamma_2)(a_1 + a_4(\gamma_2 + \beta_2 y)) \\
& + 2a_1\beta_2\gamma_1 + \beta_2(a_2 + 2a_4\gamma_1)(\gamma_2 + \beta_2 y) - \beta_1)) \\
& + y(\beta_1\beta_2(a_2\gamma_2 - a_1\gamma_1) \\
& + \beta_1^2\gamma_2(a_1 + a_4\gamma_2) - \beta_2^2\gamma_1(a_2 + a_4\gamma_1) \\
& + \beta_1^2 + \beta_2^2) + \gamma_1(\beta_1 - a_1\beta_2\gamma_1) \\
& + \beta_1\gamma_2^2(a_2 + a_4\gamma_1) + \alpha_1\alpha_2 a_4 x^3(\alpha_2\beta_1 - \alpha_1\beta_2) \\
& + a_4\beta_1\beta_2 y^2(\beta_1\gamma_2 - \beta_2\gamma_1)), \\
\dot{y} = {} & \frac{1}{\alpha_2\beta_1 - \alpha_1\beta_2}(\gamma_1(a_1\alpha_2\gamma_1 - \alpha_1) - a_1\alpha_1\gamma_1\gamma_2 \\
& - x(\alpha_1^2 + \alpha_1\alpha_2(a_2(\gamma_2 + \beta_2 y) - a_1(\gamma_1 \\
& + \beta_1 y)) + \alpha_1^2(\gamma_2 + \beta_2 y)(a_1 + a_4(\gamma_2 + \beta_2 y)) \\
& + \alpha_2^2 - \alpha_2^2(\gamma_1 + \beta_1 y)(a_2 + a_4(\gamma_1 + \beta_1 y))) \\
& + y^2(a_1\beta_1(\alpha_2\beta_1 - \alpha_1\beta_2) + \beta_2(-\alpha_1 a_2\beta_2 \\
& + a_2\alpha_2\beta_1 - \alpha_1 a_4\beta_2\gamma_1 + 2\alpha_2 a_4\beta_1\gamma_1) \\
& + a_4\beta_1\gamma_2(\alpha_2\beta_1 - 2\alpha_1\beta_2)) \\
& - \alpha_1 y(\beta_1\gamma_2(a_1 + a_4\gamma_2) + a_1\beta_2\gamma_1 \\
& + 2\beta_2\gamma_2(a_2 + a_4\gamma_1) + \beta_1) \\
& + \alpha_2 y(2\beta_1\gamma_1(a_1 + a_4\gamma_2) \\
& + \beta_2\gamma_1(a_2 + a_4\gamma_1) + a_2\beta_1\gamma_2 - \beta_2) \\
& - \alpha_1\gamma_2^2(a_2 + a_4\gamma_1) + \alpha_2\gamma_2(\gamma_1(a_2 + a_4\gamma_1) - 1) \\
& + \alpha_1\alpha_2 a_4 x^2(-\alpha_1\gamma_2 + \alpha_2\gamma_1 \\
& - \alpha_1\beta_2 y + \alpha_2\beta_1 y) + a_4\beta_1\beta_2 y^3(\alpha_2\beta_1 - \alpha_1\beta_2)),
\end{aligned}
\tag{12}
$$

its corresponding first integrals are given as follows:

Case 1: $a_1^2 + a_2^2 = 0$, the corresponding first integral (5) becomes

$$H_1(x,y) = \frac{(\gamma_1 + \alpha_1 x + \beta_1 y)^2 + (\gamma_2 + \alpha_2 x + \beta_2 y)^2}{1 - a_4(\gamma_1 + \alpha_1 x + \beta_1 y)^2}. \tag{13}$$

Case 2: $a_1^2 + a_2^2 \neq 0$, the first integral (6) now writes

$$\begin{aligned} H_2^{(1)}(x,y) &= (-a_1 - 2a_4(\gamma_2 + \alpha_2 x + \beta_2 y) + S)^{-\frac{(S+a_1)}{a_1}} \\ &\quad (a_1 + 2a_4(\gamma_2 + \alpha_2 x + \beta_2 y) + S)^{\frac{(a_1 - S)}{a_1}} \\ &\quad \left((\gamma_1 + \alpha_1 x + \beta_1 y)^2 + (\gamma_2 + \alpha_2 x + \beta_2 y)^2\right)^{S/a_1}, \end{aligned} \tag{14}$$

where $S = \sqrt{a_1^2 - 4a_4}$, if $4a_4 - a_1^2 < 0$ and $a_2 = 0$.

The first integral (7) becomes

$$\begin{aligned} H_2^{(2)}(x,y) &= (-a_2 - 2a_4(\gamma_1 + \alpha_1 x + \beta_1 y) + S)^{-\frac{S+a_2}{a_2}} \\ &\quad (a_2 + 2a_4(\gamma_1 + \alpha_1 x + \beta_1 y) + S)^{\frac{a_2 - S}{a_2}} \\ &\quad \left((\gamma_1 + \alpha_1 x + \beta_1 y)^2 + (\gamma_2 + \alpha_2 x + \beta_2 y)^2\right)^{S/a_2}, \end{aligned} \tag{15}$$

where $S = \sqrt{4a_4 + a_1^2}$, if $4a_4 + a_2^2 > 0$ and $a_1 = 0$.

The first integral (8) now writes

$$\begin{aligned} H_2^{(3)}(x,y) &= e^{-2\arctan(R_1(x,y))} \\ &\quad \left(\frac{(\gamma_1 + \alpha_1 x + \beta_1 y)^2 + (\gamma_2 + \alpha_2 x + \beta_2 y)^2}{a_1(\gamma_2 + \alpha_2 x + \beta_2 y) + a_4(\gamma_2 + \alpha_2 x + \beta_2 y)^2 + 1}\right)^{S/a_1}, \end{aligned} \tag{16}$$

with $R_1(x,y) = \dfrac{1}{S}(a_1 + 2a_4(\gamma_2 + \alpha_2 x + \beta_2 y))$ and $S = \sqrt{4a_4 - a_1^2}$, if $4a_4 - a_1^2 > 0$ and $a_2 = 0$.

The first integral (9) becomes

$$\begin{aligned} H_2^{(4)}(x,y) &= e^{-2\arctan(R_2(x,y))} \\ &\quad \left(\frac{(\gamma_1 + \alpha_1 x + \beta_1 y)^2 + (\gamma_2 + \alpha_2 x + \beta_2 y)^2}{a_2(\gamma_1 + \alpha_1 x + \beta_1 y) + a_4(\gamma_1 + \alpha_1 x + \beta_1 y)^2 - 1}\right)^{S/a_2}, \end{aligned} \tag{17}$$

with $R_2(x,y) = \dfrac{1}{S}(a_2 + 2a_4(\gamma_1 + \alpha_1 x + \beta_1 y))$ and $S = \sqrt{-a_2^2 - 4a_4}$, if $4a_4 + a_2^2 < 0$ and $a_1 = 0$.

The first integral (10) becomes

$$H_2^{(5)}(x,y) = \frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{(a_1(\gamma_2+\alpha_2 x+\beta_2 y)+2)^2} e^{R_3(x,y)}, \tag{18}$$

where $R_3(x,y) = \dfrac{4}{a_1(\gamma_2+\alpha_2 x+\beta_2 y)+2}$, if $4a_4 - a_1^2 = 0$ and $a_2 = 0$.

The first integral (11) becomes

$$H_2^{(6)}(x,y) = \frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{(2-a_2(\gamma_1+\alpha_1 x+\beta_1 y))^2} e^{R_4(x,y)}, \tag{19}$$

where $R_4(x,y) = \dfrac{4}{2-a_2(\gamma_1+\alpha_1 x+\beta_1 y)}$, if $4a_4 + a_2^2 = 0$ and $a_1 = 0$.

3 Proof of Theorem 4

Here we are going to show the upper bound number of limit cycles for the discontinuous piecewise differential systems with an arbitrary linear and cubic uniform isochronous centres separated by $x=0$.

In the right half-plane Γ^+ we consider the linear differential centre (2) in the first integral is $H(x,y)$ of the form (3). In the left half-plane Γ^- we consider system (12), with its first integrals $H_j^{(k)}(x,y)$ with $k=1,\ldots,6$ and $j=1,2$, where $H_j^{(k)}(x,y)=H_1(x,y)$ if $j=1$.

The next system of equations must be verified if the discontinuous piecewise differential systems (2)–(12) have a limit cycle that intersects the line $x=0$ in the two points $(0,y)$ and $(0,Y)$, with $y \neq Y$

$$\begin{aligned} E_1 &= H(0,y) - H(0,Y) \\ &= (y-Y)\left(8\alpha d_1 - 4\beta^2 y - y\omega^2 - 4\beta^2 Y - \omega^2 Y\right) \\ &= 0, \\ E_2 &= H_j^{(k)}(0,y) - H_j^{(k)}(0,Y) = h_j^{(k)}(y,Y) = 0. \end{aligned} \tag{20}$$

By solving $E_1=0$, we get $Y = \dfrac{8\alpha d_1}{4\beta^2+\omega^2} - y$ and by replacing it in $E_2=0$ we get an equation $F(y)=0$ with the variable y, that changes depending on the first integrals $H_j^{(k)}(x,y)$ of system (12).

Proof of statement 1 of Theorem 4: We start the proof of this statement for the discontinuous piecewise differential system separated by $x=0$ and formed by the arbitrary linear differential centre (2) and the arbitrary cubic uniform isochronous differential centre (12) satisfying $a_1^2+a_2^2=0$, where $H_1^{(k)}(x,y)=H_1(x,y)$ and

$$\begin{aligned} F(y) = {} & \left(4\beta^2+\omega^2\right)(\beta_2\gamma_2(a_4\gamma_1^2-1) \\ & + a_4\beta_1^2\beta_2\gamma_2 y^2 - \beta_1\gamma_1(a_4\gamma_2^2 + a_4\beta_2^2 y^2 + 1)) \\ & - 4\alpha d_1(\beta_2^2(1-a_4\gamma_1^2) \\ & + \beta_1^2(a_4\gamma_2(\gamma_2+2\beta_2 y)+1) \\ & - 2a_4\beta_1\beta_2^2\gamma_1 y). \end{aligned}$$

The quadratic equation $F(y) = 0$ has at most two real solutions. Consequently, system (20) can have at most two real solutions $(y_1, F(y_1))$ and $(y_2, F(y_2))$. Since $(y_1, F(y_1)) = (F(y_2), y_2)$ these solutions provide the same limit cycle for the discontinuous piecewise differential system (2)–(12). Consequently, the planar discontinuous piecewise differential system (2)–(12) can have at most one limit cycle under the condition $a_1^2 + a_2^2 = 0$.

To complete the proof of this statement we present a discontinuous piecewise differential system that has only one limit cycle and satisfies $a_1^2 + a_2^2 = 0$. We take the linear differential centre in the half-plane Γ^+.

$$\dot{x} = -x + \frac{13y}{8} + \frac{1}{10}, \quad \dot{y} = -2x + y + \frac{3}{10}, \tag{21}$$

with the first integral

$$H(x, y) = 80x^2 - 8x(10y + 3) + y(65y + 8).$$

In the half-plane Γ^- we consider the cubic uniform isochronous centre

$$\begin{aligned} \dot{x} &= \frac{1}{900}(x\left(45x^2 - 84x - 1036\right) + 24(3x + 4)y^2 \\ &\quad - 6x(33x + 56)y + 904y + 208), \\ \dot{y} &= \frac{1}{900}(15x^2(3y + 4) - 2x(3y(33y + 68) + 746) \\ &\quad + 4y(6y(3y + 1) + 115) - 992), \end{aligned} \tag{22}$$

with the first integral

$$H_1(x, y) = \frac{13x^2 + 2x(8 - 7y) + 2y(5y + 4) + 16}{(5x - 2y + 14)(5x - 2(y + 3))}.$$

The unique real solution of system (20) is $(y_1, y_2) = (-1.32287.., 1.1998..)$ which produces the unique limit cycle for the discontinuous piecewise differential system (21)–(22), see Figure 1(a). Then statement 1 holds. □

Proof of statement 2 of Theorem 4: Here, we demonstrate the statement for the discontinuous piecewise differential systems separated by $x = 0$ and formed by the arbitrary linear differential centre (2) and the arbitrary cubic uniform isochronous differential centres (12) satisfying $a_1^2 + a_2^2 \neq 0$, and we distinguish the following subcases:

Subcase 2.1: If $4a_4 - a_1^2 < 0$ and $a_2 = 0$, then $k = 1$ and $j = 2$ in system (20), then $H_2^{(1)}(x, y)$ the first integral of system (12) is given by (14). In this case finding the solution of the equation $F(y) = 0$ is equivalent to solving the $f_1(y) = g_1(y)$ equation such that

$$\begin{aligned} f_1(y) &= \left(\frac{k_0 + k_1\ y + k_2\ y^2}{G_0 + G_1\ y + k_2\ y^2}\right)^r \quad \text{and} \\ g_1(y) &= \left(\frac{m_1 + m_2\ y}{m_3 - m_2\ y}\right)^p \left(\frac{n_1 + m_2\ y}{n_3 - m_2\ y}\right)^q, \end{aligned}$$

where

$$
\begin{aligned}
k_0 &= \gamma_1^2+\gamma_2^2+\frac{64\alpha^2 d_1^2}{(4\beta^2+\omega^2)^2}\left(\beta_1^2+\beta_2^2\right)\\
&\quad+\frac{16\alpha d_1}{4\beta^2+\omega^2}(\beta_1\gamma_1+\beta_2\gamma_2),\\
k_1 &= -2(\beta_1\gamma_1+\beta_2\gamma_2)-\frac{16\alpha d_1}{4\beta^2+\omega^2}\left(\beta_1^2+\beta_2^2\right),\\
k_2 &= \beta_1^2+\beta_2^2,\\
G_0 &= \gamma_1^2+\gamma_2^2,\quad G_1=2(\beta_1\gamma_1+\beta_2\gamma_2),\\
S &= \sqrt{a_1^2-4a_4},\quad r=\frac{S}{a_1}\\
m_1 &= S-a_1-2a_4\gamma_2,\quad m_2=-2a_4\beta_2,\\
m_3 &= S-a_1-2a_4\gamma_2-\frac{16\alpha a_4\beta_2 d_1}{4\beta^2+\omega^2},\quad p=-\frac{S+a_1}{a_1},\\
n_1 &= -S-a_1-2a_4\gamma_2,\\
n_3 &= -S-a_1-2a_4\gamma_2-\frac{16\alpha a_4\beta_2 d_1}{4\beta^2+\omega^2},\\
q &= \frac{a_1-S}{a_1}.
\end{aligned}
$$

The maximum number of the real solutions of system (20) is equivalent to the maximum number of the intersection points of the graphics of the function $f_1(y)$ with the ones of $g_1(y)$.

We denote by $g_1'(y)$ the first derivative of the function $g_1(y)$, given by

$$
g_1'(y)=\frac{(m_1+m_2\ y)^{p-1}(n_1+m_2\ y)^{q-1}}{(m_3-m_2\ y)^{p+1}(n_3-m_2\ y)^{q+1}}\ P_1(y),
$$

where

$$
\begin{aligned}
P_1(y) &= m_2n_1n_3p(m_1+m_3)+m_1m_2m_3q(n_1+n_3)\\
&\quad+(m_2^2(q(m_3-m_1)(n_1+n_3)\\
&\quad-p(m_1+m_3)(n_1-n_3)))y\\
&\quad+(-m_2^3(p(m_1+m_3)+q(n_1+n_3)))\ y^2.
\end{aligned}
$$

In all the graphics of the functions $f_i(y)$ and $g_i(y)$, with $i=1,2,3$, the dashed lines represent the vertical asymptote straight lines, and the horizontal straight line is the y-axis.

Since $p\neq 0\neq q$, for $p,q>0$, and from the geometric study, the function $g_1(y)$ has two distinct vertical asymptote straight lines $y_1=\dfrac{m_3}{m_2}$ and $y_2=\dfrac{n_3}{m_2}$. Its variation depends on the sign of its first derivative, the nature of the parameters p and q, the roots of the quadratic polynomial $P_1(y)$ with their possible positions concerning y_1 and y_2, and for the two roots $r_1=\dfrac{-m_1}{m_2}$ and $r_2=\dfrac{-n_1}{m_2}$ of $g_1'(y)$.

So if we suppose that $p > q > 0$ and $y_1 < y_2$, the possible positions of the two real roots r_1 and r_2 concerning the vertical asymptote y_1 and y_2 can be as follows.

1 $r_1 < r_2 < y_1 < y_2$ with its symmetric $y_1 < y_2 < r_1 < r_2$

2 $r_1 < y_1 < r_2 < y_2$ with its symmetric $y_1 < r_1 < y_2 < r_2$

3 $r_1 < y_1 < y_2 < r_2$

4 $y_1 < r_1 < r_2 < y_2$.

Similarly we find the same possible positions if $y_2 < y_1$.

Now we will analyse the possible positions of the real roots for the quadratic polynomial $P_1(y)$ with respect to y_1, y_2, r_1 and r_2. We denote by

$$\begin{aligned}\Delta = {} & m_2^4(p(m_1+m_3)(n_1-n_3)\\ & +q(m_1-m_3)(n_1+n_3))^2\\ & +4m_2^2(m_2p(m_1+m_3)\\ & +m_2q(n_1+n_3))(m_2n_1n_3p(m_1+m_3)\\ & +m_1m_2m_3q(n_1+n_3)),\end{aligned}$$

the discriminant of $P_1(y)$, and by using the expressions of y_1, y_2, r_1 and r_2 we can write Δ in the form

$$\begin{aligned}\Delta = {} & m_2^8(r_1-y_1)(r_2-y_2)\big(p^2(r_1-y_1)(r_2-y_2)\\ & +2pq(r_1(r_2-2y_1+y_2)+r_2y_1\\ & +q^2(r_1-y_1)(r_2-y_2)-2r_2y_2+y_1y_2)\big).\end{aligned}$$

If the polynomial $P_1(y)$ has a pair of distinct real roots r_3 and r_4, i.e., $\Delta > 0$, their expressions are given by

$$\begin{aligned} r_3 &= \frac{p(r_1-y_1)(r_2+y_2)+q(r_1+y_1)(r_2-y_2)+\sqrt{\Delta/m_2^8}}{2(p(r_1-y_1)+q(r_2-y_2))},\\ r_4 &= \frac{p(r_1-y_1)(r_2+y_2)+q(r_1+y_1)(r_2-y_2)-\sqrt{\Delta/m_2^8}}{2(p(r_1-y_1)+q(r_2-y_2))}.\end{aligned}$$

If we suppose that $r_1 < r_2 < y_1 < y_2$ it results that $r_3 < r_4$ because $p(r_1-y_1)+q(r_2-y_2) < 0$. By fixing the position of r_3 between r_1 and r_2 with $r_1 < r_3 < r_2$, then we obtain the position of r_4.

The first inequality $r_1 < r_3$ is equivalent to

$$\begin{aligned}&(r_1-y_1)(-y_2(p+q)+2pr_1-pr_2+qr_2)\\ &\quad -\sqrt{\Delta/m_2^8} > 0.\end{aligned} \tag{23}$$

The second inequality $r_3 < r_2$ is equivalent to

$$\begin{aligned}&-(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))\\ &\quad +\sqrt{\Delta/m_2^8} > 0,\end{aligned} \tag{24}$$

by summing the two inequalities (23) and (24) we get $(r_1-r_2)(p(r_1-y_1)+q(r_2-y_2))>0$, which is satisfied for all $r_1<r_2<y_1<y_2$. So to find the position of r_4, we assume that $r_3<r_4<r_2$, it is clear that the first inequality $r_3<r_4$ holds.

For $r_4<r_2$ we get

$$\begin{aligned}&-(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))\\&\quad-\sqrt{\Delta/m_2^8}>0.\end{aligned}\tag{25}$$

A necessary condition so that this last inequality holds is $p(r_1-y_1)-q(r_1-2r_2+y_1)>0$, i.e., $p(r_1-y_1)>q(r_1-2r_2+y_1)=q(r_1-y_1)+2q(y_1-r_2)$. We know that $p\geq q>0$ and $r_1-y_1<0$, then $p(r_1-y_1)\leq q(r_1-y_1)<0$. Since $2q(y_1-r_2)>0$ then $p(r_1-y_1)\leq q(r_1-y_1)<q(r_1-y_1)+2q(y_1-r_2)=q(r_1-2r_2+y_1)$, which is a contradiction. Then the position $r_1<r_3<r_4<r_2<y_1<y_2$ is not possible.

Now we assume that $r_2<r_4<y_1$, the inequality $r_2<r_4$ is equivalent to

$$\begin{aligned}&(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))\\&\quad+\sqrt{\Delta/m_2^8}>0,\end{aligned}\tag{26}$$

and $r_4<y_1$ is equivalent to

$$\begin{aligned}&(r_1-y_1)(p(r_2-2y_1+y_2)+q(r_2-y_2))\\&\quad-\sqrt{\Delta/m_2^8}>0.\end{aligned}\tag{27}$$

So (26)+(27) implies $(y_1-r_2)(p(r_1-y_1)+q(r_2-y_2))>0$ which is a contradiction, because $y_1-r_2>0$, $r_1-y_1<0$ and $r_2-y_2<0$. Then the position $r_1<r_3<r_2<r_4<y_1<y_2$ is not possible.

Now we assume that $y_1<r_4<y_2$, the inequality $y_1<r_4$ is equivalent to

$$\begin{aligned}&(r_1-y_1)(p(r_2-2y_1+y_2)+q(r_2-y_2))\\&\quad-\sqrt{\Delta/m_2^8}<0,\end{aligned}\tag{28}$$

and $r_4<y_2$ equivalent to

$$\begin{aligned}&-(r_2-y_2)(p(r_1-y_1)+q(r_1+y_1-2y_2))\\&\quad+\sqrt{\Delta/m_2^8}<0.\end{aligned}\tag{29}$$

So (28)–(29) gives $(y_1-y_2)(p(y_1-r_1)+q(y_2-r_2))<0$. Since $y_1-y_2<0$, $y_1-r_1>0$ and $y_2-r_2>0$, this inequality holds. Similarly we provide all the positions of the real roots of $P_1(y)$ concerning the vertical asymptote y_1 and y_2 and to r_1 and r_2 that are given in what follows:

1 $r_1<r_3<r_2<y_1<r_4<y_2$ with its symmetric $y_1<r_3<y_2<r_1<r_4<r_2$

2 $y_1<r_3<r_4<r_1<y_2<r_2$ with its symmetric $r_1<y_1<r_2<r_3<r_4<y_2$

3 $r_1<r_3<r_4<y_1<r_2<y_2$ with its symmetric $y_1<r_1<y_2<r_3<r_4<r_2$

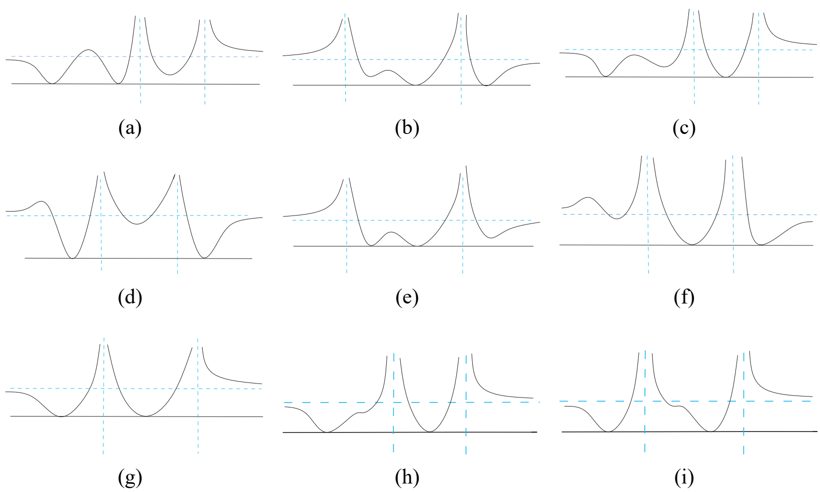
4 $r_3 < r_1 < y_1 < r_4 < y_2 < r_2$ with its symmetric $r_1 < y_1 < r_3 < y_2 < r_2 < r_4$

5 $y_1 < r_1 < r_3 < r_2 < y_2 < r_4$ with its symmetric $r_3 < y_1 < r_1 < r_4 < r_2 < y_2$

6 $r_3 < r_4 < y_1 < r_1 < y_2 < r_2$ with its symmetric $r_1 < y_1 < r_2 < y_2 < r_3 < r_4$.

If $P_1(y)$ has a pair of complex roots, we have only the position $r_1 < y_1 < r_2 < y_2$ together with its symmetric $y_1 < r_1 < y_2 < r_2$.

Figure 3 The graphics of the function $g_1(y)$ if p and q are even, or if p even and $q = \dfrac{k_1}{2k_2+1}$ with $k_1, k_2 \in \mathbb{N}$ (see online version for colours)



If $P_1(y)$ has a double real root r_0, the possible positions of this double root with respect to y_1, y_2, r_1 and r_2 are

1 $r_1 < r_0 < y_1 < r_2 < y_2$ together with its symmetric $y_1 < r_1 < y_2 < r_0 < r_2$

2 $r_1 < y_1 < r_2 < r_0 < y_2$ together with its symmetric $y_1 < r_0 < r_1 < y_2 < r_2$.

Remark 5: In these proofs we only give the graphics of the functions, when the first derivative's sign of all the functions started when $y \to -\infty$ with a positive sign and also with a negative sign when $y \to -\infty$. The cases that we omit to consider explicitly will be called the symmetric cases of the ones that we considered.
Figures 3, 4 and 5 the possible graphics for the function $g_1(y)$. Indeed, if p and q are even integers or if p is an even integer and $q = \dfrac{2l_1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$, or if $p = \dfrac{2l_1}{2l_2+1}$ and $q = \dfrac{2l'_1}{2l'_2+1}$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 3. If $P_1(y)$ has a pair of distinct real roots r_3 and r_4 which can take the position (1) where the graphic of $g_1(y)$ is given in Figure 3(a), or either the position (2), or (3), or (4), or (5) or (6), where graphics of $g_1(y)$ are given either in Figure 3(b), or Figure 3(c), or Figure 3(d), or Figure 3(e) or Figure 3(f), respectively. If $P_1(y)$ has a

pair of complex roots, the graphic of $g_1(y)$ is illustrated in Figure 3(g). If $P_1(y)$ has a double real root taking either the position (1) or (2), then the graphics of $g_1(y)$ are given by Figure 3(h) or Figure 3(i).

If p and q are odd integers, or if p is an odd integer and $q = \dfrac{2l_1+1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$, or if $p = \dfrac{2l_1+1}{2l_2+1}$ and $q = \dfrac{2l_1'+1}{2l_2'+1}$ with $l_1, l_2, l_1', l_2' \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 4. If $P_1(y)$ has a pair of distinct real roots r_3 and r_4 either in the position (1), or (2), or (3), or (4), or (5) or (6), then the graphics of $g_1(y)$ are given either in Figures 4(a) and 4(b), or Figures 4(c) and 4(d), or Figures 4(e) and 4(f), or Figures 4(g) and 4(h), or Figures 4(i) and 4(j), or Figures 4(k) and 4(l), respectively. If $P_1(y)$ has a pair of complex roots, then the graphics of $g_1(y)$ are shown in Figures 4(m) and 4(n). If $P_1(y)$ has a double real root r_0 either in the position (1) or (2), then the graphics of $g_1(y)$ are given either in Figures 4(o) and 4(p), or Figures 4(q) and 4(r), respectively.

In a similar way we obtain that if p is an odd integer and q is an even integer, or if $p = \dfrac{2l_1+1}{2l_2+1}$ and $q = \dfrac{2l_1'}{2l_2'+1}$ with $l_1, l_2, l_1', l_2' \in \mathbb{N}$, or if p is an even integer and $q = \dfrac{2l_1+1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$, or if p is an odd integer and $q = \dfrac{2l_1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 5.

If p is an odd integer and q is either irrational or $q = \dfrac{l_1}{2l_2}$ with $l_1, l_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_2, y_2)$ and the sign of $g_1'(y)$ depends on the sign of the polynomial $P_1(y)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} when both p and q are odd integers.

If p is an even integer and q is either irrational or $q = \dfrac{l_1}{2l_2}$ with $l_1, l_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_2, y_2)$ and the sign of $g_1'(y)$ is determined by the sign of $P_1(y)$ and on the sign of the product $(n_1 + n_2 y_2)(n_1 + n_3 y_2)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} in which one of the integers p and q is odd and the other is even.

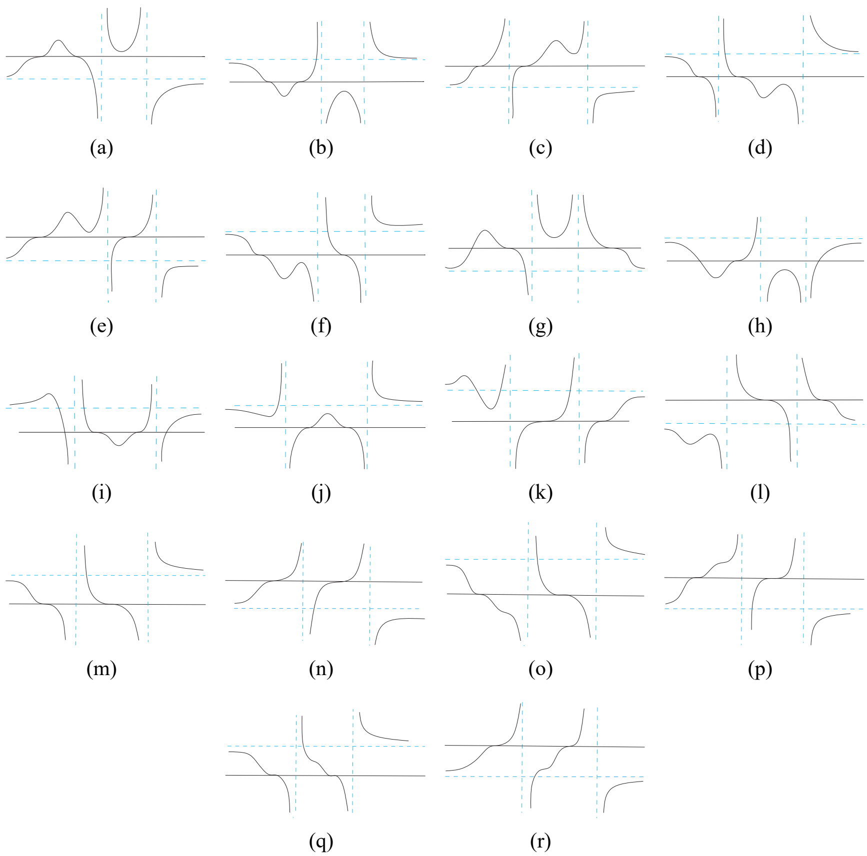
If p is either irrational or $p = \dfrac{l_1}{2l_2}$ and q is either irrational or $q = \dfrac{l_1'}{2l_2'}$ with $l_1, l_2, l_1', l_2' \in \mathbb{N}$ and $l_2 \neq 0 \neq l_2'$, or if p is either irrational or $p = \dfrac{l_1}{2l_2}$ and $q = \dfrac{2l_1'+1}{2l_2'+1}$ with $l_1, l_2, l_1', l_2' \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_1, y_1) \cap [r_2, y_2)$ and the sign of $g_1'(y)$ is determined on sign of $P_1(y)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} when both p and q are odd integers.

If p is either irrational or $p = \dfrac{l_1}{2l_2}$ and $q = \dfrac{2l_1'}{2l_2'+1}$ with $l_1, l_2, l_1', l_2' \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_1, y_1) \cap [r_2, y_2)$ and the sign of $g_1'(y)$ is determined by the sign of $P_1(y)$ and $(n_1 + m_2\, y)(n_3 - m_2\, y)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} in which one of the integers p and q is odd and the other is even.

For the case $p, q < 0$ or $pq < 0$, we found the same graphics by the same way.

Now for the function $f_1(y)$ we denote by Δ_1 and Δ_2 the discriminant of the quadratic polynomials $G_0 + G_1\, y + k_2\, y^2$ and $k_0 + k_1\, y + k_2\, y^2$, respectively. We have $\Delta = \Delta_1 = \Delta_2 = -4(\beta_2\gamma_1 - \beta_1\gamma_2)^2 \leq 0$.

Figure 4 The graphics of the function $g_1(y)$ if p and q are odd (see online version for colours)



If $\Delta = 0$, the function $f_1(y)$ becomes

$$f_1(y) = \left(\frac{G_1 + 2k_2\ y}{k_1 + 2k_2\ y} \right)^{2r},$$

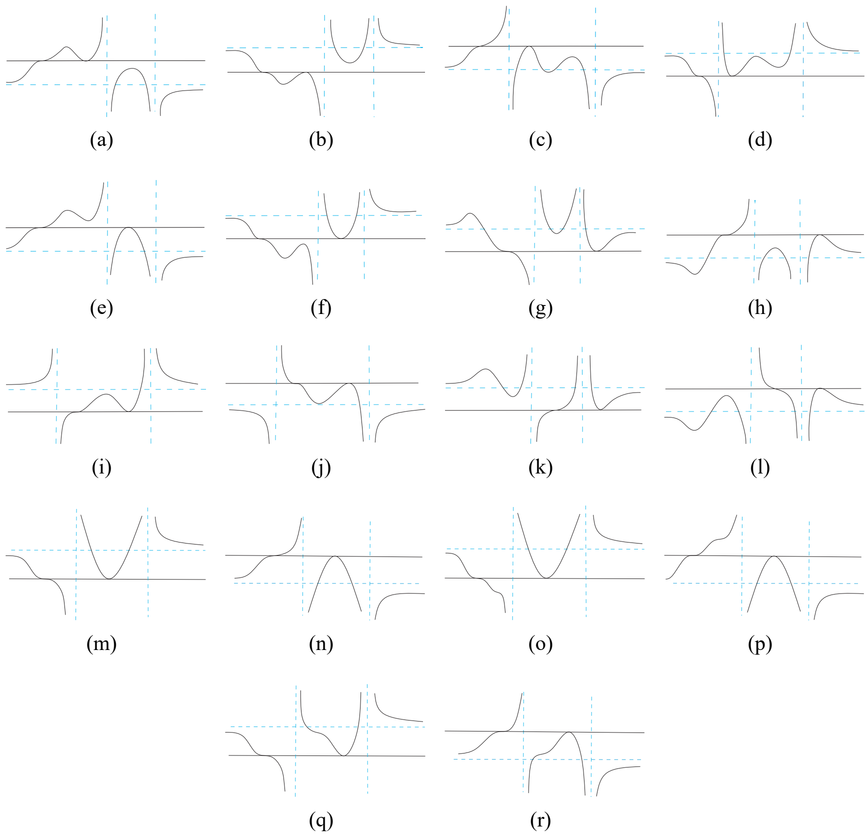
and its first derivative has the form

$$f_1'(y) = \frac{\eta(G_1 + 2k_2\ y)^{2r-1}}{(k_1 + 2k_2\ y)^{2r+1}},$$

with $\eta = 4rk_2(k_1 - G_1)$. Then to draw all the graphics of the function $f_1(y)$ with $\Delta = 0$ we have to study the sign of its derivative which depends on η and the nature of the parameter $2r$.

For $r > 0$ it is clear that $f_1'(y)$ vanish at $z_1 = -\dfrac{G_1}{2k_2}$ that can have only one possible position with respect to the vertical asymptote straight line $z_2 = -\dfrac{k_1}{2k_2}$. Thus in this case we only draw the graphics of the function $f_1(y)$ when $z_1 < z_2$, and we omit the case when $z_2 < z_1$ as in the graphics of $g_1(y)$.

Figure 5 The graphics of the function $g_1(y)$ if p odd and q even, or p odd and $q = \dfrac{l_1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$ (see online version for colours)



If r is a natural number or $r = \dfrac{l_1}{2l_2+1}$ with $l_1, l_2 \in \mathbb{N}$, the sign of $f_1'(y)$ depends on the sign of the product $\eta(G_1 + 2k_2\, y)(k_1 + 2k_2\, y)$. So the only possible graphic of this function is shown in Figure 6(a).

If $r = \dfrac{2l_1+1}{4l_2+2}$ with $l_1, l_2 \in \mathbb{N}$, the sign of $f_1'(y)$ is relates only on the parameter η. Here the graphics of this function are shown in Figure 6(b) if $\eta < 0$ and in Figure 6(c) if $\eta > 0$.

If either r is irrational or $r = \dfrac{2l_1+1}{4l_2}$ with $l_1, l_2 \in \mathbb{N}$ and $k_2 \neq 0$, the function $f_1(y)$ is well defined on $D_{f_1} = [z_1, z_2)$, and the sign of $f_1'(y)$ is related only on the nature of η. Thus the graphics of $f_1(y)$ are the parts drawn on D_{f_1} in Figure 6(b) if $\eta < 0$ and in Figure 6(c) if $\eta > 0$.

Similarly when $r < 0$, we obtain the identical graphics as $r > 0$.

If $\Delta < 0$ the function $f_1(y)$ has two extremums. Thus Figures 6(d) and 6(e) are the only possible graphics for the function $f_1(y)$.

Figure 6 The graphics of the function $f_1(y)$ (see online version for colours)

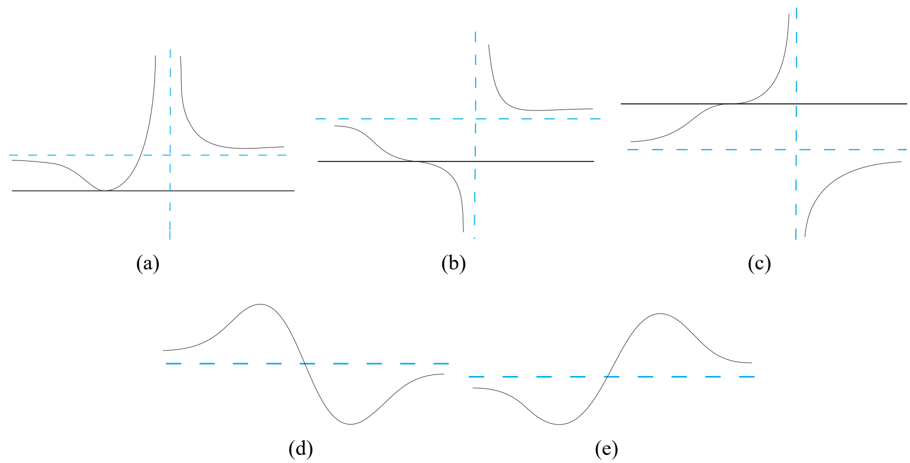
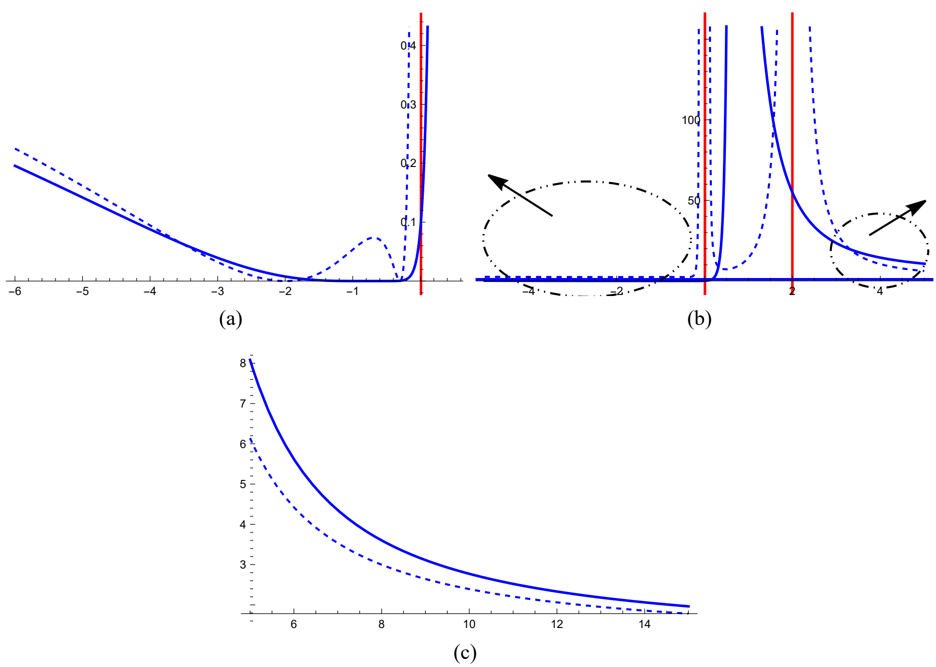


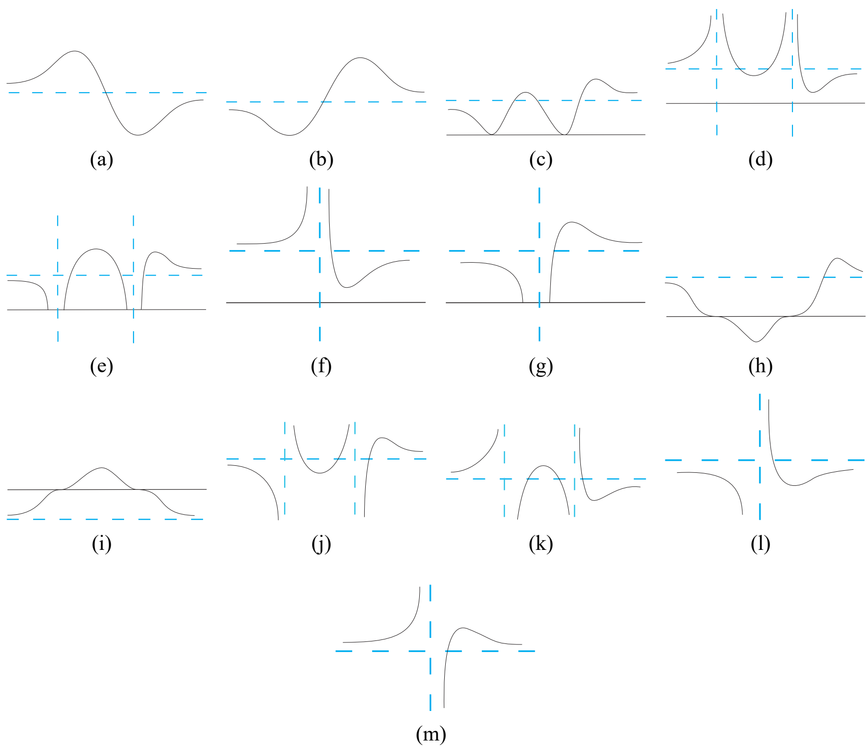
Figure 7 The seven intersection points between the graphics of $f_1(y)$ presented in a continuous line and $g_1(y)$ presented in a dashed line (see online version for colours)



Note: The vertical lines represent the asymptote's straight lines.

For the function $g_1(y)$ when both integers p and q are even, we notice that the derivative's sign only changes at most seven times, but in the other cases it changes at most five times, which guarantees that the even case is the one that gives the maximum number of the intersection points of the function $f_1(y)$ with $g_1(y)$, so to provide this maximum it is sufficient to obtain the upper bound number of the intersection points of $f_1(y)$ with $g_1(y)$ when p and q are even integers. Then we are only interested in the graphics of the function $g_1(y)$ drawn in Figure 3. Since $y = 1$ is the common horizontal asymptote straight line for these two functions, it ensures that no intersection points exist between these graphics at infinity. Then the graphics of $f_1(y)$ and $g_1(y)$ can intersect in at most seven points. Consequently system (20) can have at most seven real solutions. It is simple to demonstrate that if (y, Y) is a solution of system (20), then the symmetry (Y, y) is also a solution of that system. Therefore the maximum number of limit cycles of the discontinuous piecewise differential system (2)–(12) for $4a_4 - a_1^2 < 0$ and $a_2 = 0$ is at most three.

Figure 8 The graphics of the function $f_2(y)$ (see online version for colours)



By considering $\{m_1,\ m_2,\ m_3,\ n_1,\ n_3,\ p,\ q\ ,k_0,\ k_1,\ k_2,\ G_0,\ G_1,\ r\} \longrightarrow \{0.3,\ 1,$ $0.01,\ 2,\ 2,\ 2,\ 2,\ 0.333333..,\ 1,\ 0.75,\ 0.75,\ -1.5, 3\}$ we build an example in which the graphics of $f_1(y)$ and $g_1(y)$ interesect in seven points, see Figure 7.

Subcase 2.2: If $4a_4 + a_2^2 > 0$ and $a_1 = 0$, then $k = 2$ and $j = 2$ in system (20), and $H_2^{(2)}(x, y)$ given by (15) represent the first integral of system (12). Thus the solutions of $F(y) = 0$ are identical to the solutions of $f_1(y) = g_1(y)$ given in subcase 2.1, with

$$m_1 = S - a_2 - 2a_4\gamma_1, \quad m_2 = -2a_4\beta_1,$$

$$m_3 = S - a_2 - 2a_4\gamma_1 - \frac{16\alpha a_4\beta_1 d_1}{4\beta^2+\omega^2},$$
$$n_1 = -S - a_2 + 2a_4\gamma_1,$$
$$n_3 = -S - a_2 - 2a_4\gamma_1 - \frac{16\alpha a_4\beta_1 d_1}{4\beta^2+\omega^2},$$
$$p = -\frac{S+a_2}{a_2}, \quad q = \frac{a_2 - S}{a_2},$$
$$r = \frac{S}{a_2}, \quad S = \sqrt{a_2^2 + 4a_4},$$

Therefore the maximum number of the intersection points between the graphics of $f_1(y)$ and $g_1(y)$ is at most five. Then as in the previous subcase the maximum number of limit cycles of the discontinuous piecewise differential system (2)–(12) with $4a_4 + a_2^2 > 0$ and $a_1 = 0$ is at most three.

To prove that our results are reached we will give an example of three limit cycles for the class formed by a linear centre and a cubic uniform isochronous centre with $4a_4 + a_2^2 > 0$ and $a_1 = 0$.

In the region Γ^+ we consider the linear differential centre

$$\dot{x} = x - \frac{41y}{32} + \frac{9}{10}, \quad \dot{y} = 2x - y + 1, \tag{30}$$

with the first integral

$$H(x,y) = -\frac{8}{5}(10x+9)y + 16x(x+1) + \frac{41y^2}{4}.$$

In the region Γ^- we consider the cubic isochronous centre

$$\begin{aligned}\dot{x} &= x(y(0.0880169.. - 0.0000688837..y)\\ &\quad - 1.99445..) + 0.0000375..x^3\\ &\quad + x^2(0.000336918..y + 0.00960476..)\\ &\quad + (-0.0179787..y - 17.9585..)y\\ &\quad + 12.6236..,\\ \dot{y} &= y(y(0.00207603.. - 0.0000688837..y)\\ &\quad + 1.94435..) + x^2(0.0000375..y - 0.00108607..)\\ &\quad + x((0.000336918..y - 0.00994057..)y\\ &\quad + 0.27169..) + 13.1497..,\end{aligned} \tag{31}$$

with the first integral

$$H_2^{(2)}(x,y) = \frac{0.00015625..\begin{pmatrix} x^2 + x(14.6152..y + 98.7172..) \\ +y(67.4919..y - 94.8194..) + 14255.7.. \end{pmatrix}^{2.}}{\begin{bmatrix} (0.00075..x - 0.00015..y + 0.300095..)^3 \\ (-0.00075..x + 0.00015..y + 0.0999052..)^{1.}, \end{bmatrix}}$$

Figure 9 The two possible graphics of the function $g_2(y)$ (see online version for colours)

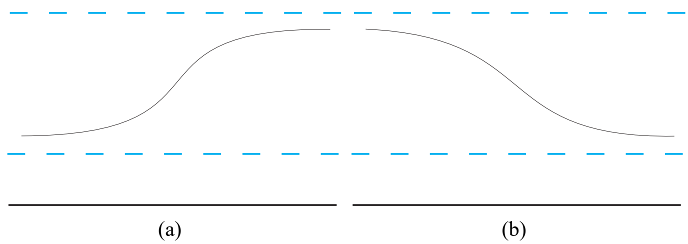
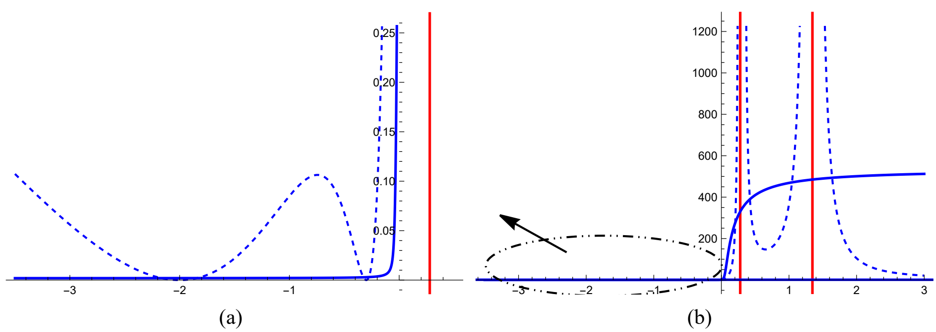
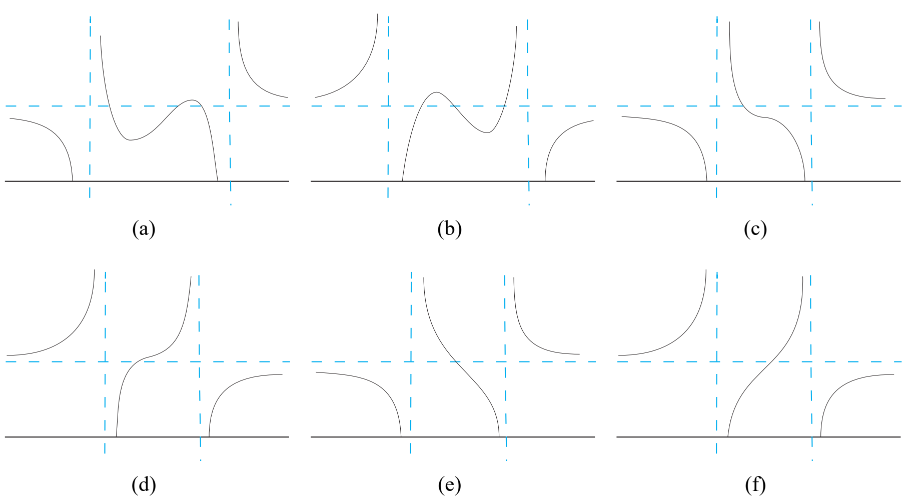


Figure 10 The nine points intersection between the graphics of $f_2(y)$ presented in a continuous line and $g_2(y)$ presented in a dashed line (see online version for colours)



Note: The vertical lines represent the asymptote's straight lines.

Figure 11 The graphics of the function $g_3(y)$ (see online version for colours)



For the discontinuous piecewise differential system (30)–(31), system (20) has the three solutions $(y_1, y_2) = (-0.426236.., 1.83111..)$, $(y_3, y_4) = (-0.173113.., 1.57799..)$ and

$(y_5, y_6) = (0.193249.., 1.21163..)$ which provide the three limit cycles intersecting the separation straight line $x = 0$ in the six points $(0, y_j)$ with $j = 1, \ldots, 6$, see Figure 1(b).

Subcase 2.3: If $4a_4 - a_1^2 > 0$ and $a_2 = 0$, then $k = 3$ and $j = 2$ in system (20), and the first integral $H_2^{(3)}(x, y)$ of (12) is given by (16). The solutions of the equation $F(y) = 0$ are identically equivalent to the ones of the equation $f_2(y) = g_2(y)$ such that

$$\begin{aligned} f_2(y) &= \left(\frac{k_0 + k_1\ y + k_2\ y^2}{G_0 + G_1\ y + G_2\ y^2}\right)^r \quad \text{and} \\ g_2(y) &= e^{2(\arctan(m_1 + m_2\ y) - \arctan(m_3 - m_2\ y))}, \end{aligned}$$

where

$$\begin{aligned} k_0 &= \frac{(\gamma_1^2 + \gamma_2^2)}{(4\beta^2 + \omega^2)^2}(16\beta^4(\gamma_2(a_1 + a_4\gamma_2) + 1) \\ &\quad + 8\beta^2(\omega^2(\gamma_2(a_1 + a_4\gamma_2) + 1) \\ &\quad + 4\alpha\beta_2 d_1(a_1 + 2a_4\gamma_2)) + a_1\gamma_2\omega^4 + 8\alpha a_1\beta_2 d_1\omega^2 \\ &\quad + a_4\left(\gamma_2\omega^2 + 8\alpha\beta_2 d_1\right)^2 + \omega^4), \\ k_1 &= \frac{16\alpha\beta_2 d_1}{4\beta^2 + \omega^2}(a_1\beta_1\gamma_1 + a_1\beta_2\gamma_2 + 2a_4\beta_1\gamma_1\gamma_2 \\ &\quad - a_4\beta_2\gamma_1^2 + a_4\beta_2\gamma_2^2) + 2a_1\beta_1\gamma_1\gamma_2 - a_1\beta_2\gamma_1^2 \\ &\quad + a_1\beta_2\gamma_2^2 + 2a_4\beta_1\gamma_1\gamma_2^2 - 2a_4\beta_2\gamma_1^2\gamma_2 \\ &\quad + \frac{128\alpha^2 a_4\beta_2^2 d_1^2}{(4\beta^2 + \omega^2)^2}(\beta_1\gamma_1 + \beta_2\gamma_2) + 2\beta_1\gamma_1 + 2\beta_2\gamma_2, \\ k_2 &= \frac{8\alpha\beta_2 d_1}{4\beta^2 + \omega^2}(a_1\beta_1^2 + a_1\beta_2^2 - 4a_4\beta_1\beta_2\gamma_1 \\ &\quad + 2a_4\gamma_2(\beta_1 - \beta_2)(\beta_1 + \beta_2)) + a_1\beta_1^2\gamma_2 \\ &\quad - 2a_1\beta_1\beta_2\gamma_1 - a_1\beta_2^2\gamma_2 + a_4\beta_1^2\gamma_2^2 - 4a_4\beta_1\beta_2\gamma_1\gamma_2 \\ &\quad + a_4\beta_2^2\gamma_1^2 - 2a_4\beta_2^2\gamma_2^2 + \frac{64\alpha^2 a_4\beta_2^2 d_1^2}{(4\beta^2 + \omega^2)^2}\left(\beta_1^2 + \beta_2^2\right) \\ &\quad + \beta_1^2 + \beta_2^2, \\ G_0 &= \frac{\gamma_2(a_1 + a_4\gamma_2) + 1}{(4\beta^2 + \omega^2)^2}((4\beta^2 + \omega^2)^2(\gamma_1^2 + \gamma_2^2) \\ &\quad + 64\alpha^2 d_1^2(\beta_1^2 + \beta_2^2) \\ &\quad + 16\alpha d_1(4\beta^2 + \omega^2)(\beta_1\gamma_1 + \beta_2\gamma_2)), \\ G_1 &= -2\beta_1\gamma_1(\gamma_2(a_1 + a_4\gamma_2) + 1) \\ &\quad + \beta_2(a_1(\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) + 2\gamma_2(a_4\gamma_1^2 - 1)) \\ &\quad + \frac{64\alpha^2\beta_2 d_1^2}{(4\beta^2 + \omega^2)^2}(\beta_1^2 + \beta_2^2)(a_1 + 2a_4\gamma_2) \\ &\quad - \frac{16\alpha d_1}{4\beta^2 + \omega^2}(\beta_1^2(\gamma_2(a_1 + a_4\gamma_2) + 1) \\ &\quad - \beta_1\beta_2\gamma_1(a_1 + 2a_4\gamma_2) + \beta_2^2(1 - a_4\gamma_2^2)), \end{aligned}$$

$$
\begin{aligned}
G_2 = &-\frac{16\alpha\beta_2 d_1}{4\beta^2+\omega^2}(a_1(\beta_1^2+\beta_2^2)\\
&+a_4(2\beta_1^2\gamma_2-\beta_1\beta_2\gamma_1+\beta_2^2\gamma_2))\\
&+a_1\beta_1^2\gamma_2-2a_1\beta_1\beta_2\gamma_1\\
&-a_1\beta_2^2\gamma_2+a_4\beta_1^2\gamma_2^2-4a_4\beta_1\beta_2\gamma_1\gamma_2\\
&+a_4\beta_2^2\gamma_1^2-2a_4\beta_2^2\gamma_2^2+\frac{64\alpha^2 a_4\beta_2^2 d_1^2}{(4\beta^2+\omega^2)^2}\left(\beta_1^2+\beta_2^2\right)\\
&+\beta_1^2+\beta_2^2, \quad r=\frac{S}{a_1}, \quad S=\sqrt{4a_4-a_1^2},\\
m_1 = &\frac{1}{S}\left(a_1+2a_4\gamma_2+\frac{16\alpha a_4\beta_2 d_1}{4\beta^2+\omega^2}\right),\\
m_2 = &\frac{2a_4\beta_2}{S}, \quad m_3=\frac{a_1+2a_4\gamma_2}{S}.
\end{aligned}
$$

The first derivative of the function $f_2(y)$ is

$$
f_2'(y)=\frac{\left(k_0+k_1\ y+k_2\ y^2\right)^{r-1}}{(G_0+G_1\ y+G_2\ y^2)^{r+1}}\ P_2(y),
$$

where

$$
\begin{aligned}
P_2(y) = &\ r(G_0k_1-G_1k_0)+r(2G_0k_2-2G_2k_0)y\\
&+r(G_1k_2-G_2k_1)y^2.
\end{aligned}
$$

Now to draw all possible graphics of the function $f_2(y)$, we denote by Δ, Δ_1 and Δ_2 the discriminants of the quadratic polynomials $G_0+G_1\ y+G_2\ y^2$, $k_0+k_1\ y+k_2\ y^2$ and $P_2(y)$, respectively.

If $\Delta, \Delta_1 > 0$ the function $f_2(y)$ becomes a particular case of $g_1(y)$, i.e., $p=q=r$. Then the graphics of $f_2(y)$ are equivalently the same graphics as the ones of $g_1(y)$ when both p, q are odd or even which provide the graphics shown in Figures 3 and 4.

If $\Delta, \Delta_1 \leq 0$ the function $f_2(y)$ have the same graphics as the function $f_1(y)$. Then the graphics of $f_2(y)$ are illustrated in Figure 6.

For $r > 0$ and according to the sign of the derivative $f_2'(y)$ which is related to the nature of the parameter r and on the sign of discriminates Δ, Δ_1 and Δ_2, we study all the possible graphics of the function $f_2(y)$ in what follows.

If either r is an even integer or $r=\dfrac{k_1}{2k_2+1}$ with k_1 and k_2 in $\mathbb{N}$, the possible graphics of the function $f_2(y)$ are shown in Figures 8(a) and 8(b) if $\Delta<0$, $\Delta_1=0$ and $\Delta_2>0$; or in Figure 8(c) if $\Delta<0$ and $\Delta_1, \Delta_2>0$; or in Figures 8(d) and 8(e) if Δ, $\Delta_2>0$ and $\Delta_1<0$; or in Figures 8(f) and 8(g) if $\Delta=0$, $\Delta_1<0$ and $\Delta_2>0$.

If the integer r is odd, the possible graphics of the function $f_2(y)$ are illustrated in Figures 8(a) and 8(b) if $\Delta < 0$, $\Delta_1 = 0$ and $\Delta_2 > 0$; or in Figures 8(h) and 8(i) if $\Delta < 0$ and $\Delta_1, \Delta_2 > 0$; or in Figures 8(j) and 8(k) if $\Delta, \Delta_2 > 0$ and $\Delta_1 < 0$; or in Figures 8(l) and 8(m) if $\Delta = 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$.

If $r = \dfrac{k_1}{2k_2}$ with k_1, k_2 in $\mathbb{N}$ and $k_1 \neq 0 \neq k_2$, the function $f_2(y)$ is well defined on D_{f_2} when $(k_0 + k_1\ y + k_2\ y^2)(G_0 + G_1\ y + G_2\ y^2) \geq 0$ and $G_0 + G_1\ y + G_2\ y^2 \neq 0$, then the graphics of $f_2(y)$ are the parts drawn on D_{f_2} when r is an odd integer.

Similarly if $r < 0$ we obtain the identical graphics as $r > 0$.

According to the sign of the derivative $g_2'(y)$ given by

$$
\begin{aligned}
g_2'(y) = 2m_2 & \left(\frac{(m_1 + m_2\ y)^2 + (m_3 - m_2\ y)^2 + 2}{((m_1 + m_2\ y)^2 + 1)\,((m_3 - m_2\ y)^2 + 1)} \right) \\
& e^{2\left(\text{arctan}(m_1 + m_2\ y) - \text{arctan}(m_3 - m_2\ y)\right)},
\end{aligned}
$$

which depends only on the parameter m_2, we get the two different possible graphics: Figure 9(a) of if $m_2 > 0$, and Figure 9(b) if $m_2 < 0$.

In this case the upper bound number of the intersection points between $f_2(y)$ and $g_2(y)$ is reached when r is an even integer, $\Delta > 0$ and $\Delta_1 > 0$, and the graphics of $f_2(y)$ are the ones drawn in Figure 3 because in this case the function $f_2(y)$ has the form of the function $g_1(y)$ and we proved in the previous subcases the reason in order that r must be an even integer. According to the graphics of the function $g_2(y)$ shown in Figure 9 and due to the fact there are no intersecting points at infinity, it results that these graphics can intersect at most in nine points. Consequently the maximum number of limit cycles of the discontinuous piecewise differential system (2)–(12) for $4a_4 - a_1^2 > 0$ and $a_2 = 0$ is at most four.

By taking $\{k_0, k_1, k_2, G_0, G_1, G_2, r, s_1, s_2, s_3\} \longrightarrow \{0.3, 1.15, 0.5, 0.15, –0.65, 0.4, 2, 1, 1, –30\}$ we build an example in which the graphics of the two functions $f_2(y)$ and $g_2(y)$ intersect in nine points. These points are shown in Figure 12.

Subcase 2.4: If $4a_4 + a_2^2 < 0$ and $a_1 = 0$, so $k = 4$ and $j = 2$ in system (20), and the first integral of system (12) is $H_2^{(4)}(x, y)$ given by (17). Then to solve $F(y) = 0$ it is equivalent to solve $f_2(y) = g_2(y)$ given in subcase 2.3, with

$$
\begin{aligned}
k_0 = {} & \frac{(\gamma_1^2 + \gamma_2^2)}{(4\beta^2 + \omega^2)^2}(16\beta^4(\gamma_1(a_2 + a_4\gamma_1) - 1) \\
& + 8\beta^2(\omega^2(\gamma_1(a_2 + a_4\gamma_1) - 1) \\
& + 4\alpha\beta_1 d_1(a_2 + 2a_4\gamma_1)) + \omega^4(a_2\gamma_1 - 1) \\
& + 8\alpha a_2\beta_1 d_1\omega^2 + a_4(\gamma_1\omega^2 + 8\alpha\beta_1 d_1)^2), \\
k_1 = {} & \frac{16\alpha\beta_1 d_1}{4\beta^2 + \omega^2}(\beta_1\gamma_1(a_2 + a_4\gamma_1) + \beta_2\gamma_2(a_2 + 2a_4\gamma_1) \\
& - a_4\beta_1\gamma_2^2) + a_2\beta_1\gamma_1^2 - a_2\beta_1\gamma_2^2 + 2a_2\beta_2\gamma_1\gamma_2 \\
& - 2a_4\beta_1\gamma_1\gamma_2^2 + 2a_4\beta_2\gamma_1^2\gamma_2 \\
& + \frac{128\alpha^2 a_4\beta_1^2 d_1^2}{(4\beta^2 + \omega^2)^2}(\beta_1\gamma_1 + \beta_2\gamma_2) - 2\beta_1\gamma_1 - 2\beta_2\gamma_2,
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= \frac{8\alpha\beta_1 d_1}{4\beta^2+\omega^2}(a_2(\beta_1^2+\beta_2^2)\\
&\quad - 2a_4(\beta_1^2\gamma_1 + 2\beta_1\beta_2\gamma_2 - \beta_2^2\gamma_1)) - a_2\beta_1^2\gamma_1\\
&\quad - 2a_2\beta_1\beta_2\gamma_2 + a_2\beta_2^2\gamma_1\\
&\quad - 2a_4\beta_1^2\gamma_1^2 + a_4\beta_1^2\gamma_2^2 - 4a_4\beta_1\beta_2\gamma_1\gamma_2\\
&\quad + a_4\beta_2^2\gamma_1^2 + \frac{64\alpha^2 a_4\beta_1^2 d_1^2}{(4\beta^2+\omega^2)^2}(\beta_1^2+\beta_2^2) - \beta_1^2 - \beta_2^2,\\
G_1 &= \beta_1\gamma_2^2(a_2+2a_4\gamma_1) - 2\beta_2\gamma_2(\gamma_1(a_2+a_4\gamma_1)-1)\\
&\quad + \frac{64\alpha^2\beta_1 d_1^2(a_2+2a_4\gamma_1)}{(4\beta^2+\omega^2)^2}(\beta_1^2+\beta_2^2)\\
&\quad + \frac{16\alpha d_1}{4\beta^2+\omega^2}(\beta_1\beta_2\gamma_2(a_2+2a_4\gamma_1)\\
&\quad - \beta_2^2\gamma_1(a_2+a_4\gamma_1) + \beta_1^2(a_4\gamma_1^2+1) + \beta_2^2)\\
&\quad + \beta_1\gamma_1(2-a_2\gamma_1),\\
G_2 &= -\frac{16\alpha\beta_1 d_1}{4\beta^2+\omega^2}(a_2(\beta_1^2+\beta_2^2)\\
&\quad + a_4(\beta_1^2\gamma_1 - \beta_1\beta_2\gamma_2 + 2\beta_2^2\gamma_1))\\
&\quad - a_2\beta_1^2\gamma_1 - 2a_2\beta_1\beta_2\gamma_2 + a_2\beta_2^2\gamma_1\\
&\quad - 2a_4\beta_1^2\gamma_1^2 + a_4\beta_1^2\gamma_2^2 - 4a_4\beta_1\beta_2\gamma_1\gamma_2 + a_4\beta_2^2\gamma_1^2\\
&\quad + \frac{64\alpha^2 a_4\beta_1^2 d_1^2}{(4\beta^2+\omega^2)^2}(\beta_1^2+\beta_2^2) - \beta_1^2 - \beta_2^2,\\
m_1 &= \frac{a_2+2a_4\gamma_1}{S}, \quad m_2 = \frac{2a_4\beta_1}{S},\\
m_3 &= \frac{1}{S}(a_2 + 2a_4\gamma_1 + \frac{16\alpha a_4\beta_1 d_1}{4\beta^2+\omega^2}),\\
r &= \frac{S}{a_2}, \quad S = \sqrt{-a_2^2 - 4a_4}.
\end{aligned}
$$

Consequently the maximum number of limit cycles of the discontinuous piecewise differential system (2)–(12) for $4a_4 - a_2^2 < 0$ and $a_1 = 0$ is at most four.

The maximum number of limit cycles in subcases 2.3 and 2.4 is at most four, but we can only build an example having three limit cycles for the class formed by a linear centre and a cubic uniform isochronous centre with $4a_4 - a_1^2 > 0$ and $a_2 = 0$.

In the region Γ^+ we consider the linear differential centre

$$
\dot{x} = \frac{4x}{5} - \frac{881y}{800} + \frac{9}{10}, \quad \dot{y} = 2x - \frac{4y}{5} + 1, \tag{32}
$$

with the first integral

$$
H(x,y) = 4\left(2x - \frac{4y}{5}\right)^2 + 16\left(x - \frac{9y}{10}\right) + \frac{25y^2}{4}.
$$

In the region Γ^- we consider the cubic isochronous centre

$$\begin{aligned}
\dot{x} &= x^2(1.33839.. - 0.126705..y) \\
&\quad + x(y(5.14213.. - 0.155116..y) - 8.99951..) \\
&\quad - 0.025..x^3 + y(4.56894..y - 16.6475..) \\
&\quad + 11.9886.., \\
\dot{y} &= x^2(-0.025..y - 0.270459..) \\
&\quad + x((-0.126705..y - 0.76873..)y + 0.93212..) \\
&\quad + y((-0.155116..y - 0.268073..)y + 0.502979..) \\
&\quad + 4.79534..,
\end{aligned} \tag{33}$$

with the first integral

$$H_2^{(3)}(x,y) = \frac{\begin{pmatrix} 52x^2 + x(218.821..y - 696.045..) \\ +y(231.874..y - 1428.38..) + 2524.64.. \end{pmatrix}}{x^2 + x(6.y - 14.) + y(9y - 42) + 149} e^{2\arctan(0.1x+0.3y-0.7)},$$

For the discontinuous piecewise differential system (32)–(33), system (20) has the three solutions $(y_1, y_2) = (-0.842999.., 2.47751..)$, $(y_3, y_4) = (-0.654549.., 2.28905..)$ and $(y_5, y_6) = (-0.43812.., 2.07263..)$ which provide the three limit cycles intersecting the separation straight line $x = 0$ in the six points $(0, y_j)$ with $j = 1, ..., 6$, see Figure 2(a).

Subcase 2.5: If $4a_4 - a_1^2 = 0$ and $a_2 = 0$, then $k = 5$ and $j = 2$ in system (20), and $H_2^{(5)}(x,y)$ given by (18) is the first integral of the cubic uniform isochronous centre (12). Now to solve $F(y) = 0$ it is sufficient to solve $f_3(y) = g_3(y)$ with

$$f_3(y) = \frac{k_0 + k_1\ y + k_2\ y^2}{G_0 + G_1\ y + k_2\ y^2} \quad \text{and}$$

$$g_3(y) = \left(\frac{m_1 + m_2\ y}{m_3 - m_2\ y}\right)^2 e^{\frac{1}{(m_1 + m_2\ y)} - \frac{1}{(m_3 - m_2\ y)}},$$

and

$$m_1 = -\frac{1}{4}(a_1\gamma_2 + 2), \quad m_2 = -\frac{1}{4}(a_1\beta_2),$$
$$m_3 = -\frac{1}{4}\left(a_1\gamma_2 + \frac{8\alpha a_1\beta_2 d_1}{4\beta^2 + \omega^2} + 2\right),$$

and k_0, k_1, k_2, G_0, G_1 are the same with subcase 2.1.

It is clear that the function $f_3(y)$ is a particular case of $f_1(y)$ where $r = 1$ and $\Delta = -4(\beta_2\gamma_1 - \beta_1\gamma_2)^2$, then the corresponding graphics of $f_3(y)$ are Figure 6(a) if $\Delta = 0$, and Figures 6(d) and 6(e) if $\Delta < 0$.

For the function $g_3(y)$, the first derivative of this function is

$$g_3'(y) = \frac{P_3(y)}{(m_3 - m_2\ y)^4} e^{\frac{1}{(m_1 + m_2\ y)} - \frac{1}{(m_3 - m_2\ y)}},$$

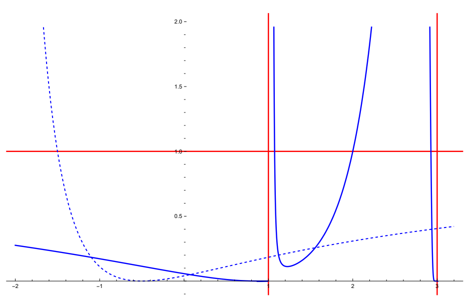
where

$$P_3(y) = m_2\left(2m_1^2m_3 - m_1^2 + 2m_1m_3^2 - m_3^2\right) \\ - 2m_2^2(m_1 - m_3)(m_1 + m_3 + 1)\ y \\ - 2m_2^3(m_1 + m_3 + 1)\ y^2.$$

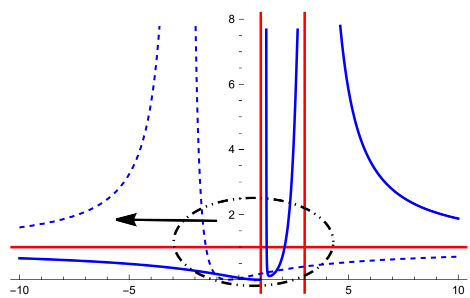
Since $m_2 \neq 0$, the variation of $g_3(y)$ depends on the ones of the quadratic polynomial $P_3(y)$. In Figure 11 we show all the possible graphics of $g_3(y)$, where Figures 11(a) and 11(b) correspond to the case in which the polynomial $P_3(y)$ has two distinct real roots, Figures 11(c) and 11(d) when $P_3(y)$ has one double real root for $P_3(y)$ and (e) and (f) if $P_3(y)$ has two complex roots for $P_3(y)$.

From the graphics of the function $g_3(y)$ shown in Figure 11, and due to the variation of this function it is obvious that we get the maximum number of the intersection points by intersecting Figures 11(a) and 11(b) with the graphics of the function $f_3(y)$. We guarantee that at infinity there are no intersection points, because the two functions $f_3(y)$ and $g_3(y)$ share the same horizontal asymptote straight line $y = 1$. Then we remark that these graphics can intersect at most in five points. Consequently the upper bound of the number of limit cycles in this case for the discontinuous piecewise differential system (2)–(12) for $4a_4 - a_1^2 = 0$ and $a_2 = 0$ is at most two.

Figure 12 The five intersection points between the graphics of $f_3(y)$ presented in a continuous line and $g_3(y)$ presented in a dashed line (see online version for colours)



(a)



(b)

Note: The straight lines represent the asymptotes straight lines.

By taking $\{m_1, m_2, m_3, k_0, k_1, k_2, G_0, G_1\} \longrightarrow \{-2, 2, 6, 1/4, 1, 1, 25/4, 5\}$ we build an example in which the functions $f_3(y)$ and $g_3(y)$ intersects in five points shown in Figure 12.

Subcase 2.6: If $4a_4 + a_2^2 = 0$ and $a_1 = 0$, then $k = 6$ and $j = 2$ in system (20), and $H_2^{(6)}(x, y)$ given by (19) is the first integral of system (12). To solve $F(y) = 0$ it is sufficient to solve the equation $f_3(y) = g_3(y)$ mentioned in subcase 2.5, with

$$m_1 = \frac{1}{4}(a_2\gamma_1 - 2), \quad m_2 = \frac{a_2\beta_1}{4}, \\ m_3 = \frac{1}{4}\left(a_2\gamma_1 + \frac{8\alpha a_2\beta_1 d_1}{4\beta^2 + \omega^2} - 2\right),$$

and the expressions of k_0, k_1, k_2, G_0, G_1 are the same as the ones given in subcase 2.1.

Working in a similar way to the previous subcase the maximum number of limit cycles of the discontinuous piecewise differential system (2)–(12) under the present conditions is at most two.

Finally we construct an example with two limit cycles of the discontinuous piecewise differential system (2)–(12) satisfying $4a_4 + a_2^2 = 0$ and $a_1 = 0$ to reach the result of statement 2. In the right half-plane Γ^+ we consider the linear differential centre

$$\dot{x} = \frac{x}{2} - \frac{461y}{580} + \frac{9}{10}, \quad \dot{y} = \frac{1}{20}(29x - 10y + 30), \tag{34}$$

with the first integral

$$H(x,y) = 4\left(\frac{29x}{20} - \frac{y}{2}\right)^2 + \frac{58}{5}\left(\frac{3x}{2} - \frac{9y}{10}\right) + \frac{361y^2}{100}.$$

In the left half-plane Γ^- we consider the cubic isochronous centre

$$\begin{aligned}
\dot{x} &= 59.0642.. - 2.510^{-9}x^3 + x^2(0.0000162997.. \\
&\quad + 1.25625..10^{-6}y) + x((0.000601877..y \\
&\quad - 0.089255..)y + 2.33048..) \\
&\quad + y(4.41817..y - 57.1156..), \\
\dot{y} &= -81.4419.. + x^2(5.70054..10^{-8} - 2.510^{-9}y) \\
&\quad + x(-0.00260418.. + (6.00594.. \\
&\quad + 1.25626..y)10^{-6}y) \\
&\quad + y((0.000601877..y - 0.112201..)y + 5.23317..),
\end{aligned} \tag{35}$$

with the first integral

$$H_2^{(6)}(x,y) = \frac{\begin{pmatrix} x(26{,}139.5 - 502.503y) + x^2 \\ +y(367{,}006.y - 2.04258 \times 10^7) + 3.28816 \times 10^8 \end{pmatrix}}{(x + 300.y - 19500.)^2} e^{-R(x,y)},$$

with $R(x,y) = \dfrac{40{,}000}{x + 300(y - 65)}$.

In this case system (20) has the two solutions $(y_1, y_2) = (-0.604777.., 2.86942..)$ and $(y_3, y_4) = (-0.242279.., 2.50692..)$ which produce the two limit cycles for the discontinuous piecewise differential system (34)–(35), see Figure 2(b). Then statement 2 is held. $\square$

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