# Pseudo periodicity and pseudo almost periodicity in shifts $\delta_\pm$ on time scales

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**Abstract:** By means of the shifts operators $\delta_\pm$, we first define the concepts of pseudo ($v$-pseudo) periodic function in shifts $\delta_\pm$ and pseudo ($v$-pseudo) almost periodic function in shifts $\delta_\pm$ on time scales, then by using the calculus on time scales and the properties of the shift operators $\delta_\pm$, the existence and uniqueness theorems of pseudo ($v$-pseudo) periodic solution in shifts $\delta_\pm$ and pseudo ($v$-pseudo) almost periodic solution in shifts $\delta_\pm$ of a linear dynamic equation are established, respectively. Finally, two delayed dynamic equations are studied on some specific time scales, the corresponding examples are given to illustrate the usefulness of our results.

**Keywords:** pseudo periodicity; pseudo almost periodicity; shift operator; time scale.

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## 1 Introduction

Since the weakly almost periodic function (see Eberlein, 1949) was proposed by Eberlein in 1949, it has been a hot topic in the field of analysis until the 1970s and 1980s. From the weakly almost periodic function on the initial topological group to the weakly almost periodic function on the semi-topological semigroup, a set of abstract analysis theory has been developed. With the developments of the theory, the weakly almost periodic function has exposed some shortcomings in its applications, especially in the applications of differential equations. A basic problem is: under what circumstances is the indefinite integral of a weakly almost periodic function still weakly almost periodic? This problem is difficult to solve by the theory of weakly almost periodic functions. This is because, using the decomposition theorem, $f = f_1 + f_2$, where $f_1$ is an almost periodic function, $f_2$ is a continuous function whose mean is equal to zero at infinity, the difficulty lies in the indefinite integral argument of the second additional term. Based on this, and the consideration of ergodicity, Zhang (1994a, 1994b, 1995, 1997) defined a new function which is called pseudo almost periodic function. Since then, the studies of pseudo almost periodic functions and their applications in differential equations have made a series of important achievements in different fields (see for example, Huang et al., 2021; Xu et al., 2021; Ayachi, 2022; Baroun et al., 2018; Chérif, 2015; Amdouni and Chérif, 2018).

A general time scale is usually not closed under the addition operation, and a large of time scales are bounded or at least one endpoint is bounded, for example, $\widetilde{\mathbb{T}} = \bigcup_{k=0}^{+\infty}[2k, 2k+1]$. Many time scales similar to this type have been applied to the study of dynamic systems of the seventeen-year periodical cicada *magicicada septendecim*, the common mayfly *stenonema canadense* and so on.

However, the existing pseudo almost periodic theory is only applicable to the study of the dynamic equation defined on several special time scales (unbounded and closed under the addition operation). What we are interested in is to construct a theory that can be applied to the study of the pseudo almost periodic of the dynamic equations on more general time scales, especially when the addition operation is not closed or at least one endpoint is bounded.

In recent years, by means of the shift operators $\delta_{\pm}$, the concepts and properties of periodic and almost periodic functions in shifts $\delta_{\pm}$ on time scales have been defined and studied in Adıvar (2013, 2010), Adıvar and Raffoul (2010a, 2010b) and Hu and Wang (2017, 2021, 2022), respectively. The theory of periodic and almost periodic dynamic equations in shifts $\delta_{\pm}$ on time scales have been rapidly developed and applied. On the basis of the above works, to further construct the theory of pseudo periodicity and pseudo almost periodicity on time scales, the main work of this paper is to define the concepts of pseudo ($v$-pseudo) periodic function in shifts $\delta_{\pm}$ and pseudo ($v$-pseudo) almost periodic function in shifts $\delta_{\pm}$ on time scales, and to explore the existence and uniqueness of pseudo ($v$-pseudo) periodic solution in shifts $\delta_{\pm}$ and pseudo ($v$-pseudo) almost periodic solution in shifts $\delta_{\pm}$ of the dynamic equation on time scales as follows:

$$y^{\Delta}(x) = L(x)y(x) + \varphi(x), x \in \mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is a time scale; $L_{n\times n}(x)$ and $\varphi_{n\times 1}(x)$ are rd-continuous functions.

Furthermore, based on the above obtained results, we bring two delayed dynamic equation under investigation on some specific time scales to obtain more general results.

## 2 Preliminaries

The theory of time scales and its applications on dynamic equations (see Bohner and Peterson, 2003).

*Lemma 1 (Bohner and Peterson, 2003):* If $\alpha \in \mathcal{R}$, then

1 $e_0(x,z) \equiv 1, e_\alpha(x,x) \equiv 1$.

2 $e_\alpha(\sigma(x),z) = (1+\mu(x)\alpha(x))e_\alpha(x,z)$.

3 $e_\alpha(x,z) = \frac{1}{e_\alpha(z,x)} = e_{\ominus\alpha}(z,x)$.

4 $e_\alpha(x,z)e_\alpha(z,r) = e_\alpha(x,r)$.

5 $(e_{\ominus\alpha}(x,z))^\Delta = (\ominus\alpha)(x)e_{\ominus\alpha}(x,z)$.

6 $\left(\frac{1}{e_\alpha(\cdot,z)}\right)^\Delta = -\frac{\alpha(x)}{e_\alpha^\sigma(\cdot,z)}$.

*Remark 1:* By Lemma 1, if $\alpha \in \mathcal{R}^+$, then

$$e_\alpha(x,z) \le \exp\left(\int_z^x \alpha(\zeta)\Delta\zeta\right),$$

for all $x \ge z$.

A comprehensive review on periodicity and almost periodicity in shifts $\delta_{\pm}$ on time scales, see Adıvar (2013), Hu and Wang (2017, 2021, 2022) and Hu (2016).

Consider the corresponding homogeneous equation (1),

$$y^\Delta(x) = L(x)y(x), x \in \mathbb{T}. \tag{2}$$

*Definition 1 (Li and Wang, 2011; Zhang et al., 2010):* Suppose that $e_L(x)$ is the fundamental solution matrix of equation (2), if there exist a projection $P$ and positive constants $k$ and $\alpha$ such that

$$\begin{aligned}
&|e_L(x)Pe_L^{-1}(\sigma(z))| \le ke_{\ominus\alpha}(x,\sigma(z)),\\
&z,x \in \mathbb{T}, x \ge \sigma(z),\\
&|e_L(x)(I-P)e_L^{-1}(\sigma(z))| \le ke_{\ominus\alpha}(\sigma(z),x),\\
&z,x \in \mathbb{T}, x \le \sigma(z),
\end{aligned}$$

then equation (2) satisfies exponential dichotomy on $\mathbb{T}$, $|\cdot|$ is the Euclidean norm.

*Lemma 2 (Bohner and Peterson, 2003):* Suppose that $\psi : \mathbb{T} \to \mathbb{R}$ is strictly increasing, $\tilde{\mathbb{T}} := \psi(\mathbb{T})$ is a time scale. If $g : \tilde{\mathbb{T}} \to \mathbb{R}$, $\psi^\Delta(x)$ and $g^{\tilde{\Delta}}(\psi(x))$ exist for $x \in \mathbb{T}^k$, then

$$(g \circ \psi)^\Delta = (g^{\tilde{\Delta}} \circ \psi)\psi^\Delta.$$

*Lemma 3 (Bohner and Peterson, 2003):* Suppose that $\psi : \mathbb{T} \to \mathbb{R}$ is strictly increasing, $\tilde{\mathbb{T}} := \psi(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, and $\psi^\Delta$ is rd-continuous, then

$$\int_c^d f(z)\psi^\Delta(z)\Delta z = \int_{\psi(c)}^{\psi(d)} f(\psi^{-1}(z))\tilde{\Delta}z.$$

In the following sections, suppose that $\mathbb{Y}$ is a Banach space with the norm $|\cdot|$, $BC(\mathbb{T},\mathbb{Y})$ is a set of all $\mathbb{Y}$-valued bounded continuous functions.

## 3 Pseudo almost periodicity in shifts $\delta_{\pm}$

Define the sets:

$$\begin{aligned}
&APS(\mathbb{T},\mathbb{Y})\\
&=\{\xi:\mathbb{T}\to\mathbb{Y},\xi \text{ is almost periodic in shifts } \delta_{\pm}\};\\
&APS^{\Delta}(\mathbb{T},\mathbb{Y})\\
&=\{\xi:\mathbb{T}\to\mathbb{Y},\xi \text{ is } \Delta\text{−almost periodic in shifts } \delta_{\pm}\};\\
&PAPS_0(\mathbb{T},\mathbb{Y})=\{\xi\in BC(\mathbb{T},\mathbb{Y}):\\
&\lim_{X\to+\infty}\frac{1}{(\delta_{+}^{X}(x_0)-\delta_{-}^{X}(x_0))}\int_{\delta_{-}^{X}(x_0)}^{\delta_{+}^{X}(x_0)}|\xi(x)|\Delta x=0\}.
\end{aligned}$$

*Definition 2:* A function $f:\mathbb{T}\to\mathbb{Y}$ is called pseudo almost periodic in shifts $\delta_{\pm}$, if $f=f_1+f_2$, and $f_1\in APS(\mathbb{T},\mathbb{Y})$, $f_2\in PAPS_0(\mathbb{T},\mathbb{Y})$. Let $PAPS(\mathbb{T},\mathbb{Y})$ is the set of all pseudo almost periodic functions in shifts $\delta_{\pm}$.

Now we study the existence of pseudo almost periodic solution in shifts $\delta_{\pm}$ of equation (1). We first consider the case of $\mathbb{T}$ is unbounded below and above, that is, $x\in(-\infty,+\infty)_{\mathbb{T}}$.

*Theorem 4:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x)\in APS(\mathbb{T},\mathbb{R}^{n\times n})$, $\varphi(x)=\varphi_1(x)+\varphi_2(x)$, and $\varphi_1(x)\in APS^{\Delta}(\mathbb{T},\mathbb{R}^n)$, $\varphi_2(x)\in PAPS_0(\mathbb{T},\mathbb{R}^n)$, then equation (1) exists exactly one solution $y(x)\in PAPS(\mathbb{T},\mathbb{R}^n)$, and

$$\begin{aligned}
y(x)=&\int_{-\infty}^{x}e_L(x)Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&-\int_{x}^{+\infty}e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z. \qquad (3)
\end{aligned}$$

*Proof:* Firstly, we prove that $y(x)$ is a solution of equation (1). In fact,

$$\begin{aligned}
&y^{\Delta}(x)-L(x)y(x)\\
&=e_L^{\Delta}(x)\int_{-\infty}^{x}Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&+e_L(\sigma(x))Pe_L^{-1}(\sigma(x))\varphi(x)\\
&-e_L^{\Delta}(x)\int_{x}^{+\infty}(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&+e_L(\sigma(x))(I-P)e_L^{-1}(\sigma(x))\varphi(x)\\
&-L(x)e_L(x)\int_{-\infty}^{x}Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&+L(x)e_L(x)\int_{x}^{+\infty}(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&=e_L(\sigma(x))(P+I-P)e_L^{-1}(\sigma(x))\varphi(x)\\
&=\varphi(x).
\end{aligned}$$

Next, we show that $y(x)$ is pseudo almost periodic in shifts $\delta_\pm$.

$$\begin{aligned}
y(x) &= \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi(x)\Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(x)\Delta z \\
&= \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi_1(x)\Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi_1(x)\Delta z \\
&\quad + \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi_2(x)\Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi_2(x)\Delta z.
\end{aligned}$$

Let $y(x) = y_1(x) + y_2(x)$, $y_1(x) = \phi_1(x) + \phi_2(x)$, where

$$\begin{aligned}
y_1(x) &= \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi_1(x)\Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi_1(x)\Delta z, \\
y_2(x) &= \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi_2(x)\Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi_2(x)\Delta z,
\end{aligned}$$

and

$$\begin{aligned}
\phi_1(x) &= \int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi_1(z)\Delta z, \\
\phi_2(x) &= \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi_1(z)\Delta z.
\end{aligned}$$

By Lemma 3,

$$\begin{aligned}
\phi_1(\delta_\pm^q(x)) &= \int_{-\infty}^{\delta_\pm^q(x)} e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(z))\varphi_1(z)\Delta z \\
&= \int_{-\infty}^{x} e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z))) \\
&\quad \varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z)\Delta z; \\
\phi_2(\delta_\pm^q(x)) &= \int_{\delta_\pm^q(x)}^{+\infty} e_L(\delta_\pm^q(x))(I-P)e_L^{-1}(\sigma(z)) \\
&\quad \varphi_1(z)\Delta z
\end{aligned}$$

$$= \int_x^{+\infty} e_L(\delta_\pm^q(x))(I-P)e_L^{-1}(\sigma(\delta_\pm^q(z)))$$
$$\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z)\Delta z.$$

For $q > 0$ and $\varepsilon > 0$,

$$\begin{aligned}
&|\phi_1(\delta_\pm^q(x)) - \phi_1(x)| \\
&= \bigg| \int_{-\infty}^x e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z)))\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z)\Delta z \\
&- \int_{-\infty}^x e_L(x)Pe_L^{-1}(\sigma(z))\varphi_1(z)\Delta z\bigg| \\
&= \bigg| \int_{-\infty}^x e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z)))[\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z) \\
&-\varphi_1(z)]\Delta z + \int_{-\infty}^x [e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z))) \\
&-e_L(x)Pe_L^{-1}(\sigma(z))]\varphi_1(z)\Delta z\bigg| \\
&= \bigg| \int_{-\infty}^x e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z)))[\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z) \\
&-\varphi_1(z)]\Delta z\bigg| + \bigg|\bigg(\int_{-\infty}^{x-\varepsilon} + \int_{x-\varepsilon}^x\bigg)[e_L(\delta_\pm^q(x))Pe_L^{-1} \\
&(\sigma(\delta_\pm^q(z))) - e_L(x)Pe_L^{-1}(\sigma(z))]\varphi_1(z)\Delta z\bigg|,
\end{aligned}$$

and then,

$$\begin{aligned}
&|\phi_1(\delta_\pm^q(x)) - \phi_1(x)|^2 \\
&\le 3\bigg(\int_{-\infty}^x |e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z)))| \\
&|\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z) - \varphi_1(z)|\Delta z\bigg)^2 \\
&+3\bigg(\int_{-\infty}^{x-\varepsilon} |e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z))) \\
&-e_L(x)Pe_L^{-1}(\sigma(z))||\varphi_1(z)|\Delta z\bigg)^2 \\
&+3\bigg(\int_{x-\varepsilon}^x |e_L(\delta_\pm^q(x))Pe_L^{-1}(\sigma(\delta_\pm^q(z))) \\
&-e_L(x)Pe_L^{-1}(\sigma(z))||\varphi_1(z)|\Delta z\bigg)^2 \\
&\le 3k^2\bigg(\int_{-\infty}^x e_{\ominus\alpha}(x,\sigma(z))|\varphi_1(\delta_\pm^q(z))\delta_\pm^{\Delta q}(z) \\
&-\varphi_1(z)|\Delta z\bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&+3\varepsilon^2\bigg(\int_{-\infty}^{x-\varepsilon} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))|\varphi_1(z)|\Delta z\bigg)^2 \\
&+3\varepsilon^2\bigg(\int_{x-\varepsilon}^{x} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))|\varphi_1(z)|\Delta z\bigg)^2.
\end{aligned}
$$

Using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&|\phi_1(\delta_{\pm}^q(x))-\phi_1(x)|^2 \\
&\leq 3k^2\bigg(\int_{-\infty}^{x} e_{\ominus\alpha}(x,\sigma(z))\Delta z\bigg) \\
&\times\bigg(\int_{-\infty}^{x} e_{\ominus\alpha}(x,\sigma(z))|\varphi_1(\delta_{\pm}^q(z))\delta_{\pm}^{\Delta q}(z)-\varphi_1(z)|^2\Delta z\bigg) \\
&+3\varepsilon^2\bigg(\int_{-\infty}^{x-\varepsilon} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))\Delta z\bigg) \\
&\bigg(\int_{-\infty}^{x-\varepsilon} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))|\varphi_1(z)|^2\Delta z\bigg) \\
&+3\varepsilon^2\bigg(\int_{x-\varepsilon}^{x} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))\Delta z\bigg) \\
&\bigg(\int_{x-\varepsilon}^{x} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))|\varphi_1(z)|^2\Delta z\bigg) \\
&\leq 3k^2\varepsilon^2\bigg(\int_{-\infty}^{x} e_{\ominus\alpha}(x,\sigma(z))\Delta z\bigg)^2 \\
&+3\varepsilon^2\bigg(\int_{-\infty}^{x-\varepsilon} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))\Delta z\bigg)^2\hat{\varphi}_1 \\
&+3\varepsilon^2\bigg(\int_{x-\varepsilon}^{x} e_{\ominus\frac{\alpha}{2}}(x,\sigma(z))\Delta z\bigg)^2\hat{\varphi}_1,
\end{aligned}
$$

then, by Lemma 1 and Remark 1,

$$
\begin{aligned}
&|\phi_1(\delta_{\pm}^q(x))-\phi_1(x)|^2 \\
&\leq \frac{3k^2\varepsilon^2}{\inf(\ominus\alpha)^2}+\frac{12\hat{\varphi}_1\varepsilon^2}{\inf(\ominus\alpha)^2}+3\hat{\varphi}_1\varepsilon^4. \qquad (4)
\end{aligned}
$$

Similarly, we can obtain

$$
|\phi_2(\delta_{\pm}^q(x))-\phi_2(x)|^2 \leq \frac{3k^2\varepsilon^2}{\alpha^2}+\frac{12\hat{\varphi}_1\varepsilon^2}{\alpha^2}+3\hat{\varphi}_1\varepsilon^4, \qquad (5)
$$

where $\hat{\varphi}_1=\sup_{x\in\mathbb{T}}|\varphi_1(x)|$.

From equations (4) and (5), $y_1(x)$ is almost periodic in shifts $\delta_{\pm}$.

On the other hand, equation (2) satisfies exponential dichotomy,

$$
\begin{aligned}
|y_2(x)| &= \Big| \int_{-\infty}^{x} e_L(x) P e_L^{-1}(\sigma(z)) \varphi_2(z) \Delta z \\
&\quad - \int_{x}^{+\infty} e_L(x)(I-P) e_L^{-1}(\sigma(z)) \varphi_2(z) \Delta z \Big| \\
&\le k \Big( \int_{-\infty}^{x} e_{\ominus\alpha}(x, \sigma(z)) |\varphi_2(x)| \Delta z \\
&\quad + \int_{x}^{+\infty} e_{\ominus\alpha}(\sigma(z), x) |\varphi_2(x)| \Delta z \Big),
\end{aligned}
$$

then, by Remark 1,

$$
\begin{aligned}
&\frac{1}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} |y_2(x)| \Delta x \\
&\le \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{-\infty}^{x} e_{\ominus\alpha}(x, \sigma(z)) |\varphi_2(x)| \Delta z \Delta x \\
&+ \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{x}^{+\infty} e_{\ominus\alpha}(\sigma(z), x) |\varphi_2(x)| \Delta z \Delta x \\
&\le \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{-\infty}^{x} (1 + \mu(z)\alpha) \exp\{-\alpha(x - z)\} \\
&|\varphi_2(x)| \Delta z \Delta x \\
&+ \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{x}^{+\infty} \exp\left\{ \frac{-\alpha}{1 + \mu(z)\alpha} (\sigma(z) - x) \right\} \\
&|\varphi_2(x)| \Delta z \Delta x \\
&\le \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{-\infty}^{x} (1 + \hat{\mu}\alpha) \exp\{-\alpha(x - z)\} |\varphi_2(x)| \Delta z \Delta x \\
&+ \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))}
\end{aligned}
$$

$$\int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_x^{+\infty} \exp\left\{\frac{-\alpha}{1+\hat{\mu}\alpha}(z-x)\right\}|\varphi_2(x)|\Delta z\Delta x$$
$$\triangleq y_2^{(1)}(X) + y_2^{(2)}(X),$$

where $\hat{\mu} = \sup_{z\in\mathbb{T}} \mu(z)$.

Now, we prove that $\lim_{X\to+\infty} y_2^{(1)}(X) = \lim_{X\to+\infty} y_2^{(2)}(X) = 0$. In fact,

$$\begin{aligned}
0 &\le \lim_{X\to+\infty} y_2^{(1)}(X) \\
&= \lim_{X\to+\infty} \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\quad \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_{-\infty}^{x} (1+\hat{\mu}\alpha)\exp\{-\alpha(x-z)\} \\
&\quad |\varphi_2(x)|\Delta z\Delta x \\
&= \lim_{X\to+\infty} \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\quad \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_0^{+\infty} (1+\hat{\mu}\alpha)\exp\{-\alpha\zeta\}|\varphi_2(x)|\Delta\zeta\Delta x \\
&\le (1+\hat{\mu}\alpha)\lim_{X\to+\infty} \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\quad \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} \int_0^{+\infty} |\varphi_2(x)|\Delta\zeta\Delta x \\
&= (1+\hat{\mu}\alpha)\int_0^{+\infty} \lim_{X\to+\infty} \frac{k}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \\
&\quad \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} |\varphi_2(x)|\Delta x\Delta\zeta = 0,
\end{aligned}$$

that is $\lim_{X\to+\infty} y_2^{(1)}(X) = 0$.

Similarly, we can obtain $\lim_{X\to+\infty} y_2^{(2)}(X) = 0$. Hence, $y_2(x) \in PAPS_0(\mathbb{T}, \mathbb{R}^n)$. This completes the proof. □

Next we study the existence of $v$-pseudo almost periodic solution in shifts $\delta_{\pm}$ of equation (1).

Let $\mathbb{U}$ is a set of functions (weightz) $v : \mathbb{T} \to (0, +\infty)$, and

$$u(X, v) = \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} v(x)\Delta x,$$
$$\mathbb{U}_\infty = \{v \in \mathbb{U} : \lim_{X\to+\infty} u(X, v) = +\infty\},$$
$$\mathbb{U}_B = \left\{v \in \mathbb{U}_\infty : v \text{ is bounded with } \inf_{x\in\mathbb{T}} v(x) > 0\right\},$$

then $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$. For $v \in \mathbb{U}_\infty$, set

$$PAPS_0(\mathbb{T},\mathbb{Y},v) = \{\xi \in BC(\mathbb{T},\mathbb{Y}) : \lim_{X\to+\infty} \frac{1}{u(X,v)} \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} |\xi(x)|v(x)\Delta x = 0\}.$$

*Definition 3:* Let $v \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{T},\mathbb{Y})$ is called $v$-pseudo almost periodic in shifts $\delta_\pm$ or weighted pseudo almost periodic in shifts $\delta_\pm$, if $f = f_1 + f_2$, and $f_1 \in APS(\mathbb{T},\mathbb{Y})$, $f_2 \in PAPS_0(\mathbb{T},\mathbb{Y},v)$. Let $PAPS(\mathbb{T},\mathbb{Y},v)$ is the set of all $v$-pseudo almost periodic functions in shifts $\delta_\pm$.

*Theorem 5:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in APS(\mathbb{T},\mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in APS^\Delta(\mathbb{T},\mathbb{R}^n)$, $\varphi_2(x) \in PAPS_0(\mathbb{T},\mathbb{R}^n,v)$, then equation (1) exists exactly one solution $y(x) \in PAPS(\mathbb{T}, \mathbb{R}^n, v)$, and

$$\begin{aligned} y(x) = &\int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z. \end{aligned} \tag{6}$$

*Remark 2:* Theorem 5 can be proved in a similar way as the proof of Theorem 4.

If the time scale $\mathbb{T}$ is bounded below, that is, $x \in [x_0,+\infty)_\mathbb{T}$, there are the following results.

*Theorem 6:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in APS(\mathbb{T},\mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in APS^\Delta(\mathbb{T},\mathbb{R}^n)$, $\varphi_2(x) \in PAPS_0(\mathbb{T},\mathbb{R}^n)$, then equation (1) exists exactly one solution $y(x) \in PAPS(\mathbb{T},\mathbb{R}^n)$, and

$$\begin{aligned} y(x) = &\int_{x_0}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z. \end{aligned} \tag{7}$$

*Theorem 7:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in APS(\mathbb{T},\mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in APS^\Delta(\mathbb{T},\mathbb{R}^n)$, $\varphi_2(x) \in PAPS_0(\mathbb{T},\mathbb{R}^n,v)$, then equation (1) exists exactly one solution $y(x) \in PAPS(\mathbb{T}, \mathbb{R}^n, v)$, and

$$\begin{aligned} y(x) = &\int_{x_0}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z. \end{aligned} \tag{8}$$

## 4 Pseudo periodicity in shifts $\delta_\pm$

Define the sets:

$$\begin{aligned}
&PS(\mathbb{T},\mathbb{Y}) = \{\xi : \mathbb{T} \to \mathbb{Y}, \xi \text{ is periodic in shifts } \delta_\pm\};\\
&PS^\Delta(\mathbb{T},\mathbb{Y})\\
&= \{\xi : \mathbb{T} \to \mathbb{Y}, \xi \text{ is } \Delta\text{−periodic in shifts } \delta_\pm\};\\
&PPS_0(\mathbb{T},\mathbb{Y}) = \{\xi \in BC(\mathbb{T},\mathbb{Y}) :\\
&\lim_{X\to+\infty} \frac{1}{(\delta_+^X(x_0) - \delta_-^X(x_0))} \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} |\xi(x)|\Delta x = 0\}.
\end{aligned}$$

*Definition 4:* A function $f : \mathbb{T} \to \mathbb{Y}$ is called pseudo periodic in shifts $\delta_\pm$, if $f = f_1 + f_2$, and $f_1 \in PS(\mathbb{T},\mathbb{Y})$, $f_2 \in PPS_0(\mathbb{T},\mathbb{Y})$. Let $PPS(\mathbb{T},\mathbb{Y})$ is the set of all pseudo periodic functions in shifts $\delta_\pm$.

Now we study the existence of pseudo periodic solution in shifts $\delta_\pm$ of equation (1). We first consider the case of $\mathbb{T}$ is unbounded below and above, that is, $x \in (-\infty,+\infty)_\mathbb{T}$.

*Theorem 8:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in PS^\Delta(\mathbb{T},\mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in PS^\Delta(\mathbb{T},\mathbb{R}^n)$, $\varphi_2(x) \in PPS_0(\mathbb{T},\mathbb{R}^n)$, then equation (1) exists exactly one solution $y(x) \in PPS(\mathbb{T},\mathbb{R}^n)$, and

$$\begin{aligned}
y(x) = &\int_{-\infty}^{x} e_L(x)Pe_L^{-1}(\sigma(z))\varphi(z)\Delta z\\
&- \int_{x}^{+\infty} e_L(x)(I-P)e_L^{-1}(\sigma(z))\varphi(z)\Delta z. \qquad (9)
\end{aligned}$$

*Proof:* From the proof of Theorem 4, $y(x)$ is a solution of equation (1).

Let $y(x) = y_1(x) + y_2(x)$, where $y_1(x), y_2(x)$ have been defined in Section 3.
Since $y_1(x)$ is a solution of

$$y^\Delta(x) = L(x)y(x) + \varphi_1(x), \qquad (10)$$

then $y_1(\delta_\pm^\beta(x))$ is also a solution of equation (10). In fact, $\delta_\pm^\beta(x)$ is strictly increasing, let $\gamma = \delta_\pm^\beta(x)$, by Lemma 2,

$$\begin{aligned}
y_1^\Delta(\delta_\pm^\beta(x)) &= y_1^\Delta(\gamma)\gamma^\Delta\\
&= L(\delta_\pm^\beta(x))y_1(\delta_\pm^\beta(x))\delta_\pm^{\Delta\beta}(x)\\
&+ \varphi_1(\delta_\pm^\beta(x))\delta_\pm^{\Delta\beta}(x)\\
&= L(x)y_1(\delta_\pm^\beta(x)) + \varphi_1(x).
\end{aligned}$$

Besides, equation (2) satisfies exponential dichotomy, then $y_1(x) = y_1(\delta_\pm^\beta(x))$, that is, $y_1(x)$ is periodic in shifts $\delta_\pm$.

Similar to the proof of Theorem 4, we can get $y_2(x) \in PPS_0(\mathbb{T}, \mathbb{R}^n)$. This completes the proof. $\square$

Next we study the existence of $v$-pseudo periodic solution in shifts $\delta_\pm$ of equation (1).

For $v \in \mathbb{U}_\infty$, set

$$PPS_0(\mathbb{T}, \mathbb{Y}, v) = \{\xi \in BC(\mathbb{T}, \mathbb{Y}) : \\ \lim_{X \to +\infty} \frac{1}{u(X, v)} \int_{\delta_-^X(x_0)}^{\delta_+^X(x_0)} |\xi(x)| v(x) \Delta x = 0\}.$$

*Definition 5:* Let $v \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{T}, \mathbb{Y})$ is called $v$-pseudo periodic in shifts $\delta_\pm$ or weighted pseudo periodic in shifts $\delta_\pm$, if $f = f_1 + f_2$, and $f_1 \in PS(\mathbb{T}, \mathbb{Y})$, $f_2 \in PPS_0(\mathbb{T}, \mathbb{Y}, v)$. Let $PPS(\mathbb{T}, \mathbb{Y}, v)$ is the set of all $v$-pseudo periodic functions in shifts $\delta_\pm$.

*Theorem 9:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^n)$, $\varphi_2(x) \in PPS_0(\mathbb{T}, \mathbb{R}^n, v)$, then equation (1) exists exactly one solution $y(x) \in PPS(\mathbb{T}, \mathbb{R}^n, v)$, and

$$\begin{aligned} y(x) = &\int_{-\infty}^{x} e_L(x) P e_L^{-1}(\sigma(z)) \varphi(z) \Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I - P) e_L^{-1}(\sigma(z)) \varphi(z) \Delta z. \end{aligned} \tag{11}$$

*Remark 3:* Theorem 9 can be proved in a similar way as the proof of Theorem 8.

If the time scale $\mathbb{T}$ is bounded below, that is, $x \in [x_0, +\infty)_\mathbb{T}$, there are the following results.

*Theorem 10:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^n)$, $\varphi_2(x) \in PPS_0(\mathbb{T}, \mathbb{R}^n)$, then equation (1) exists exactly one solution $y(x) \in PPS(\mathbb{T}, \mathbb{R}^n)$, and

$$\begin{aligned} y(x) = &\int_{x_0}^{x} e_L(x) P e_L^{-1}(\sigma(z)) \varphi(z) \Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I - P) e_L^{-1}(\sigma(z)) \varphi(z) \Delta z. \end{aligned} \tag{12}$$

*Theorem 11:* Suppose that $\mu(x)$ is bounded on $\mathbb{T}$, equation (2) satisfies exponential dichotomy, $L(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^{n\times n})$, $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, and $\varphi_1(x) \in PS^\Delta(\mathbb{T}, \mathbb{R}^n)$, $\varphi_2(x) \in PPS_0(\mathbb{T}, \mathbb{R}^n, v)$, then equation (1) exists exactly one solution $y(x) \in PPS(\mathbb{T}, \mathbb{R}^n, v)$, and

$$\begin{aligned} y(x) = &\int_{x_0}^{x} e_L(x) P e_L^{-1}(\sigma(z)) \varphi(z) \Delta z \\ &- \int_{x}^{+\infty} e_L(x)(I - P) e_L^{-1}(\sigma(z)) \varphi(z) \Delta z. \end{aligned} \tag{13}$$

## 5 Applications

Consider the following delayed dynamic equation

$$\begin{aligned} y^{\Delta}(x) &= -l(x)y(x) \\ &\quad + h(x)\int_{x_0}^{+\infty} k(z)d(y(\delta_{-}^{z}(x)))\Delta z \\ &\quad + \eta(x), x_0, x \in \mathbb{T}. \end{aligned} \tag{14}$$

Suppose that $l(x) \in APS(\mathbb{T},\mathbb{R})$, $h(x) \in APS^{\Delta}(\mathbb{T},\mathbb{R})$, and $\eta(x) = \eta_1(x) + \eta_2(x)$, where $\eta_1(x) \in APS^{\Delta}(\mathbb{T},\mathbb{R})$ and $\eta_2(x) \in PAPS_0(\mathbb{T},\mathbb{R})$.

Let $\hat{\theta} = \sup\limits_{x\in[x_0,+\infty)_{\mathbb{T}}} |\theta(x)|, \tilde{\theta} = \inf\limits_{x\in[x_0,+\infty)_{\mathbb{T}}} |\theta(x)|$.

We first give some assumptions:

$H_1$ $\tilde{l} > 0$.

$H_2$ $d \in C(\mathbb{R},\mathbb{R})$, $d(0) = 0$ and $|d(y_1) - d(y_2)| \le L_d|y_1 - y_2|$, where $L_d > 0$ is the Lipschitz constant.

$H_3$ $\delta_{+}^{\Delta_{\varsigma}}(\cdot,\varsigma)$ is bounded, and $0 < \delta_{+}^{\Delta_{\varsigma}}(\cdot,\varsigma) \le \varrho$, where $\varrho > 0$ is a constant.

If $H_1$–$H_3$ hold, and $\xi \in PAPS(\mathbb{T},\mathbb{R})$, then $\int_{x_0}^{+\infty} k(z)d(\xi(\delta_{-}^{z}(x)))\Delta z$ is pseudo almost periodic in shifts $\delta_{\pm}$.

Using the Banach fixed point theorem, we can obtain the following theorem.

*Theorem 12:* Suppose that $H_1$–$H_3$ hold, and $\lambda = \frac{\hat{h}L_d}{\tilde{l}} < 1$, then equation (14) exists exactly one solution $y(x) \in PAPS(\mathbb{T},\mathbb{R})$.

*Example 1:* Let $\mathbb{T} = \mathbb{R}$, then $\mu(x) = 0$.

Choose $x_0 = 0$, $y(\delta_{-}^{z}(x)) = y(x - z)$, and

$$\begin{aligned} l(x) &= \frac{1}{2} - \frac{1}{4}\cos(x), h(x) = \sin(\sqrt{2}x), \\ \eta(x) &= \cos(\sqrt{3}x) - \frac{1}{1+x^2}, \\ k(z) &= e^{-2z}, d(y) = \frac{1}{11}(|y+1| - |y-1|). \end{aligned}$$

According to Theorem 12, equation (14) exists exactly one solution $y(x) \in PAPS(\mathbb{T},\mathbb{R})$.

Next we consider the following delayed dynamic equation

$$\begin{aligned} y^{\Delta}(x) &= -l(x)y(x) + h(x)d(y(\delta_{-}^{z}(x))) \\ &\quad + \eta(x), x_0, x \in \mathbb{T}. \end{aligned} \tag{15}$$

Suppose that $l(x) \in PS^{\Delta}(\mathbb{T},\mathbb{R})$, $h(x) = b_1(x) + b_2(x)$, and $\eta(x) = \eta_1(x) + \eta_2(x)$, where $b_1(x), \eta_1(x) \in PS^{\Delta}(\mathbb{T},\mathbb{R})$ and $b_2(x), \eta_2(x) \in PPS_0(\mathbb{T},\mathbb{R})$.

**Figure 1** Numerical solutions of equation (14) (Example 1) with the initial values $y(0) = \{0.5, 1, 1.5\}$ (see online version for colours)

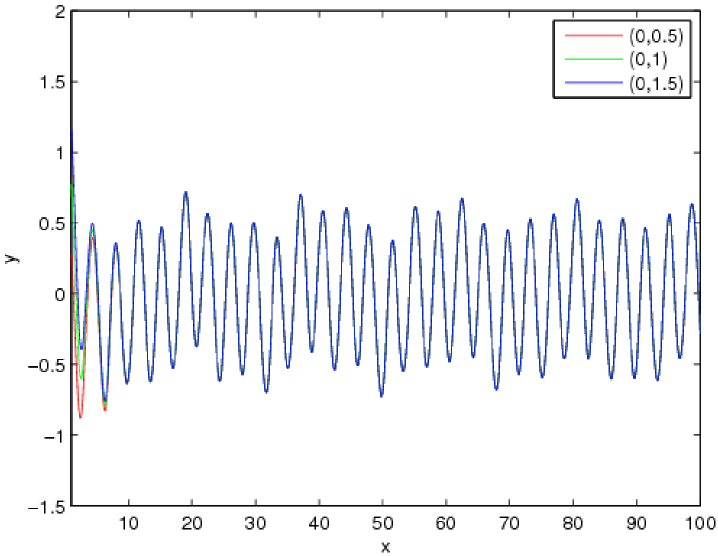
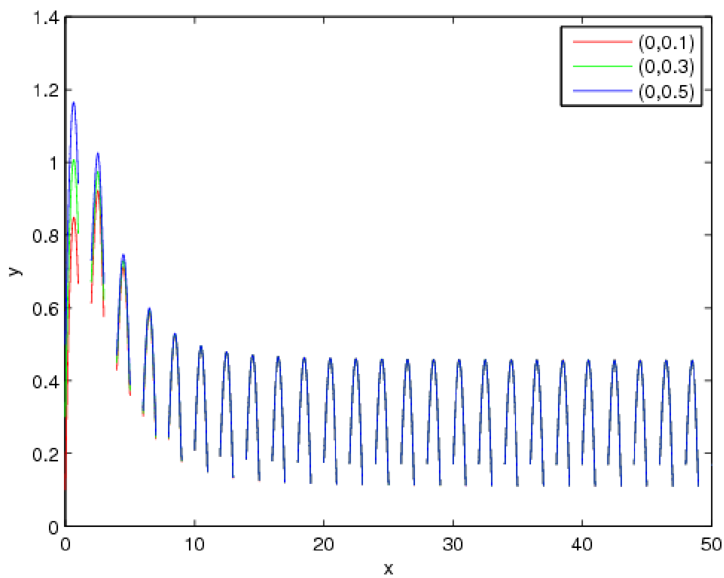



**Figure 2** Numerical solutions of equation (15) (Example 2) with the initial values $y(0) = \{0.1, 0.3, 0.5\}$ (see online version for colours)




Similarly to the proof of Theorem 12, we can obtain the following result.

*Theorem 13:* Suppose that $H_1$–$H_3$ hold, $l(x) \in PS^{\Delta}(\mathbb{T}, \mathbb{R})$, and $\lambda = \frac{\hat{h}L_d}{\tilde{l}} < 1$, then equation (15) exists exactly one solution $y(x) \in PPS(\mathbb{T}, \mathbb{R})$.

*Example 2:* Let $\mathbb{T} = \overline{\bigcup_{k\in\mathbb{Z}}[2k, 2k+1]}$, then $\mu(x) = 0$ if $x \in \bigcup_{k\in\mathbb{Z}}[2k, 2k+1)$, and $\mu(x) = 1$ if $x \in \bigcup_{k\in\mathbb{Z}}\{2k+1\}$.

Choose $x_0 = 0$, $\delta_{-}^{z}(x) = x - z$, $z = 2$, and

$$
\begin{aligned}
&l(x) = \frac{1}{2} - \frac{1}{4}\sin \pi x, h(x) = \sin \pi x - \frac{1}{1+x^2}, \\
&\eta(x) = \cos \pi x + \frac{1}{1+x^2}, \\
&d(y) = \frac{1}{11}(|y+1| - |y-1|).
\end{aligned}
$$

According to Theorem 13, equation (15) exists exactly one solution $y(x) \in PPS(\mathbb{T}, \mathbb{R})$.

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