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Analysis of (MAP, PH)/(PH, PH)/1 retrial queueing model with standby server, collision of orbital customers, breakdowns, two-way communication, phase type repairs, constant retrial rate and impatient behaviour of customers

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Abstract: Our system was modelled using standby server (SS) whenever a main server (MS) is unavailable due to breakdowns and analysed the constant retrial policy as well as collision for the orbital customers. The incoming arrival (IA) of customers follows the Markovian arrival process (MAP). The outgoing arrival (OA) of customers, service for both incoming and outgoing arrival of customers, and repairs are all based on the phase-type (PH) distributions. Using matrix analytic method, we investigate the steady state probability vector of the system. We described the busy period as well as cost analysis of the system and some performance of the system measures. Then, we examine some of the numerical as well as graphical representations for this model.

Keywords: PH distribution; constant retrial rate; Markovian arrival process; MAP; two-way communication; standby server; impatient behaviour.

MSC codes: 60K25, 68M20, 90B22.

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1 Introduction

Classified as repeated attempts queues, they are distinguished by the fact that arriving customers who notice that the server is busy join a virtual group of blocked customers, referred to as orbit, and retry their requests in a random sequence and at random time intervals. The behaviour of those unsuccessful demands whose service cannot begin at the time of their arrival is a distinguishing characteristic of queueing systems with retrials. There are many different real-world circumstances where retrial queues might occur, including our daily activities, telephone switching systems, telecommunications and computer networks, call centres, cellular and local area networks, and so forth. Artalejo (2010) has provided a comprehensive accessible bibliography on the progress made on the retrial queues between the years 2000 and 2009. Neuts (1979) has introduced and investigate the versatile Markovian point process (VMPP), and he also studied a number of other point processes, including Markov-modulated Poisson processes, PH renewal processes, and some semi-Markov point processes. Chakravarthy (2010) provided a detailed description of the MAP, as well as special cases of the Markovian arrival process (MAP) and phase-type (PH) distributions. This depiction of the MAP is a specific case of the batch Markovian arrival process (BMAP). Lucantoni et al. (1990) investigated the concepts MAP and BMAP to express the VMPP using a much more intuitive notation. It has now established itself as a standard notation for describing MAP and BMAP structures. The reason for using the acronym BMAP rather than the initially coined name VMPP is that it was believed at the time that the VMPP is a specific class of BMAP. The point process defined by the MAP is a specific type of semi-Markov processes with a transition probability matrix provided by

$$\int_0^t e^{D_0 x} D_1 dx = [I - e^{D_0 t}](-D_0)^{-1} D_1.$$

Chakravarthy (2009) described a single server queueing model in which arrivals follow a MAP and the system is susceptible to catastrophic failures in which all customers are lost. He also investigated his model's stationary waiting time distribution. Jain and Jain (2010) investigated a single server working vacation queueing system in which the server is subject to interruption owing to a variety of breakdowns and is assigned for repair as soon as the problem is discovered. They also discussed about the matrix-geometric approach to computing the stationary queue length distribution, which they found to be quite significant. Phung-Duc et al. (2009) explored the multi-server retrial queueing system incorporating Bernoulli abandonment. They utilise continued fraction approach to derive the stationary joint distribution of queue length in the system as well as in the queue. Ayyappan and Udayageetha (2018) investigated a retrial queueing model with priority services that is subject to collisions, orbital search, balking, start-up/closedown times, modified Bernoulli vacation with general (arbitrary) vacation periods, and system could become defective at any point of time while it is in operation. Alternatively, when a system is failing, rather than suspending service entirely, service is actually provided at a slower rate to high priority customers. Alem et al. (2019) explored the non-Markovian retrial queueing system with two-way communication, which included both incoming and outgoing calls, and described the n types of outgoing calls. Furthermore, the performance of the considered system can be estimated using the metrics of an $M/M/1$ retrial queueing system with two-way communication. Using a non-Markovian batch arrival queueing system with two types of heterogeneous services, standby server (SS), balking, breakdown and repair, Ayyappan and Karpagam (2016) described a system in which the main server (MS) goes on vacation after closedown completion if there are less than the desired number of customers in the system.

Choudhury and Madan (2007) explored a non-Markovian batch arrival system with a random setup time, Bernoulli schedule vacations, and a restricted admissibility policy for arriving batches. They also investigated the model's expected queue size and busy period. An $M/M/1$ queueing system with setup times and a single working vacation has been studied by Xu et al. (2009). The stationary queue length and waiting time distributions were calculated utilising the quasi-birth-and-death (QBD) process and the matrix-geometric method. Majid et al. (2021) probed a Markovian queueing system with impatient customers and Bernoulli vacation interruption. They also developed a cost analysis for optimising their model. In order to minimise the work load of brokendown machines in the system, Kumar et al. (2019) investigated the machine repairable F-policy, warm standbys, and two unreliable servers in which the server one repairs brokendown machines as early as they got breakdown, but the server two only initiates the repair service when there are more than a predetermined number of brokendown machines available in the system. They also discussed the average cost function for their model. Vadivu and Arumuganathan (2016) described a single server finite queueing system in which arrival follows the MAP and service follows general distribution with two service phases under single and multiple vacation policies. They have obtained the distribution of waiting times for their model. Kumar and Som (2015) examined a single server Markovian queueing system incorporating the notion of reverse reneging and investigated a sensitivity analysis on it. Deepak et al. (2013) incorporated a single server non-Markovian retrial queueing model with two types of customer orbit searches: type I searches for a single customer and type II searches for a batch of customers from the orbit. They also described the cost function for analysing the system's operational quality.

Haridass and Arumuganathan (2012) described the bulk service queueing model, which includes the server's option of admitting reservice, setup time, and multiple vacations. They created a cost model in order to optimise their model. Using a non-Markovian retrial queueing system with two-way communication, Artalejo and Phung-Duc (2013) found that there are two types of arrivals: regular customers making incoming calls, and the server making outgoing calls when the server is idle. Furthermore, they evaluated their model using exponential retrials, utilising the Markovian arrival process for incoming arrivals (IAs) and the PH distribution for outgoing arrivals (OAs) and service times. In the context of orbital search and multiple vacations, Gao and Wang (2020) discussed a single server non-Markovian retrial queueing model with pre-emptive concepts. They also discussed about their model's cost optimisation analysis. Morozov and Phung-Duc (2018) conducted an investigation on a single server retrial queueing system with two-way communication, constant retrial rate, and general service times. Wu and He (2020) explored a double-sided queueing system with marked Markovian arrival processes as well as finite abandonment times. They examined the queueing system using Markov modulated fluid flow processes. They also obtained the queue lengths and waiting times for both sides of the queueing system. Ayyappan et al. (2019) investigated the non-Markovian retrial queueing system with priority services, working breakdown (when a system failures, instead of terminating service completely, the service is extended to the interrupted customer but at a slower rate than before), collision, starting failures, and repairs were all investigated.

Ghosh et al. (2021) explored a single-server batch queueing model with finite buffer, in which arrival follows the BMAP and service follows the Markovian service process. They also discussed regarding their model's queue length distribution. Chakravarthy and Agnihothri (2008) studied a finite capacity queueing model with one MS and a backup server, in which the arrival follows a MAP, the service follows a PH distribution, and the server backup policy is a threshold-type server backup policy with two pre-determined lower and upper thresholds. They explained that when the buffer size reaches the upper threshold, a backup server is generated, and the backup server is dismissed from the system when the buffer size reaches the lower threshold or less at the backup server's service completion. They also discussed a cost model for preferring the optimum threshold values to reduce average total cost per unit time under various system situations. Kuki et al. (2019) investigated a particular two-way communication system that was represented with the help of finite and infinite sources retrial queueing model with searching for customers and a server that was subject to unpredictable breakdowns. Panta et al. (2020) examined the multi-server Markovian queueing system in a fuzzy environment with impatient customer behaviour such as reneging. They also created a cost model for optimising their model. Revathi et al. (2020) explored a single server retrial queue with optional re-service and modified Bernoulli vacation, as well as a server that is prone to breakdowns and repairs. The stationary probability distribution is calculated using the supplementary variable method. A single server bulk queue non-Markovian model with a limited number of vacations has been studied by Jeyakumar and Arumuganathan (2011). They obtained the PGF of queue size and system performance measures using the supplementary variable technique. They also discussed about their model's cost analysis.

Aissani and Artalejo (1998) have investigated the queueing system that incorporates the concepts of active server breakdowns as well as the concepts of independent server breakdowns. They also established the concept of a fundamental server period, as well as an auxiliary queueing system with breakdowns and the ability to exit the system altogether. A single server Markovian retrial queueing system under constant retrial queue with joining strategy and impatient behaviour of retrial customers was described by Gao et al. (2017). They also examined the two different forms of rewards: one is the reward obtained by the customer who leaves the system after completing the service, and the other is the reward acquired by the customer who is forced to leave the system as a result of a retrial failure. Kumar and Sharma (2019) incorporated a multi-server queueing model with the notion of retention of reneging customers. They also derived the system's transient state probabilities by employing the probability generating function technique. In the context of server breakdowns, backup servers, repairs, and vacations, Chakravarthy et al. (2019) examined the queueing model in which arrival follows a versatile point process and service follows PH distributions. They also examined the sojourn time distribution for their model. Arivudainambi et al. (2009) investigated a single server non-preemptive retrial queueing system with Bernoulli vacations. They utilised the supplementary variable technique to determine the probability generating functions of the number of customers present in the priority queue as well as the retrial group. Miyazawa (2004) described the Markov renewal strategy to a single server non-Markovian queueing model with a countably number of background states, which is based on the Markov renewal technique.

Madheswari and Suganthi (2021) explored a single server retrial queueing model with starting failure, optional service, and k Bernoulli vacation types. They utilised the supplementary variable technique to determine the probability generating function of system size. A single server retrial queueing system with arrivals following the MAP and service times following the PH distribution was studied by Artalejo and Chakravarthy (2006), who incorporated the algorithmic analysis of the system in their model. Dragieva and Phung-Duc (2019) incorporated a single server non-Markovian finite-source retrial queueing model with outgoing calls, in which the server not only receives incoming calls but also makes outgoing calls after a period of exponentially distributed idle time. The stationary and transient analysis of a Markovian queueing system with finite capacity subject to disasters and subsequent repair have been described by Vijayashree and Janani (2017). They used techniques such as the Laplace transform and the generating function. Lan and Tang (2020) analysed a single-server discrete time retrial queueing model with probabilistic preemptive priority, repair time replacement, and impatient customer behaviour. Additionally, they explored cost analysis for optimisation. Jailaxmi et al. (2013) explored a non-Markovian retrial queueing system with two-phase service, two types of repair, and vacations under N-policy. Utilising supplementary variable technique, they also acquired the system's probability generating function. With the provision of S dissimilar warm standby units that are subject to failure and repair facility, Jain (2016) described a redundant repairable system with imperfect switching. Wu et al. (2005) described a non-Markovian retrial queueing system with impatient customer behaviour such as balking. They also created the model's busy period as well as waiting time distribution. Barbhuiya and Gupta (2020) investigated a single server non-Markovian bulk arrival queueing system with N threshold policy and renewal input concepts. For their model, they have also done a waiting time analysis.

Gupta and Kumar (2021) explored a single server Markovian retrial queueing model with working vacation, interruption, and the server undergoing breakdown and repair. They also discussed the cost analysis for model optimisation. Manoharan and Jeeva (2018) examined a Markovian queueing system with impatient customers, long setup times, and a single vacation day. The queue length distribution and sojourn time distribution were obtained utilising the QBD process and the matrix geometric method. Sikdar (2019) examined a $MAP/G/1/N$ queueing model with a finite number of vacation policies, revealing that after a certain number of vacations, the server will remain dormant until the clients arrive. He also discussed about the waiting time analysis. A retrial queueing system with classical retrial policy, general renewal input and general class-dependent service times was investigated by Morozov and Phung-Duc (2017). An investigation into a non-Markovian queueing system with two stages of service process subject to server failures and the extended Bernoulli Vacation was carried out by Ayyappan and Supraja (2017). In addition, they investigated the queue size distribution for their model. A single-server retrial queueing model with two-way communication, in which arrival is governed by a Markov modulated Poisson process, was explored by Nazarov et al. (2019). Wu and He (2020) explored the G system in terms of repair equipment procurement and redundant dependencies. They analysed their model using a matrix-analytical method, and the repair times of broken components follow PH distributions. Chakravarthy (2012) investigated a single server queueing model in which arrivals follow a MAP and the server deteriorates and possibly fails as a result of external shocks. He also investigated an optimisation problem involves different costs. Gao (2021) examined a fault-tolerant retrial system with a second optional repair service and warm standbys. Using the Laplace transform method, he obtained the system's reliability function and mean time to first failure.

Kumar and Jain (2014) investigated a degraded multi-component machine repair system with multiple standbys, start-up, vacation, and two unreliable repairmen operating under a bi-level (N, L) switch over policy in which one of them turns on when N-failed units accumulate in the system and the other repairmen turns on when L-failed units accumulate in the system. Lee et al. (2020) investigated a single server non-Markovian retrial queueing system with a two-way communication concept. They also discussed about their model's waiting time distribution for arbitrary ingoing calls. Paul and Phung-Duc (2018) investigated a single server retrial queueing system with two-way communication, an unreliable server, and afterservice for calls that were interrupted. A machine repair system with mixed standbys, two modes of failure, discouragement, switching failure, balking, and reneging has been described by Jain and Preeti (2014). They also discussed about cost analysis for optimisation. Ammar (2020) probed a single server Markovian queueing system with vacation in a random environment. He utilised the generating function technique to derive the transient probabilities and also described the sojourn times for his model. The repairable systems with imperfect repairs were described by Hajeeh (2015). He also examined the cost analysis for optimisation. Phung-Duc (2014) described a multi-server retrial queueing system with two types of nonpersistent customers, which was inspired by a call centre operating in a consortium of cooperative call centres. Jacob (2020) investigated a retrial queue with customer-induced interruption. He examined the model's behaviour utilising the QBD process and the matrix analytical method. Gao et al. (2020) examined a single server non-Markovian retrial queueing model with two sorts of breakdowns and delayed repairs. For their model, they also investigated an arbitrary customer's sojourn times.

Meena et al. (2019) examined the multi-component machine repair system, which included operating as well as standby support, vacation, and also provided the cost analysis for their model. Sakurai and Phung-Duc (2015) described a single server Markovian retrial queue with various types of outgoing calls, as well as a single server non-Markovian retrial queue with multiple types of outgoing calls whose durations follow distinct arbitrary distributions. With the notions of thresholds model in the context of the classical retrial queue and PH retrial times, Chakravarthy (2020) investigated the multi server retrial queueing model in which an arriving customer, finding all c servers busy, will enter into an infinite capacity retrial buffer. Under the N-policy and modified Bernoulli schedule vacation with the unreliable server, Jain and Upadhyaya (2010) investigated the optimal repairable non-Markovian queueing model with essential as well as multi-optional services. They utilise supplementary variable technique and the generating function to derive the system size distribution. Bouchentouf et al. (2019) examined a single server Markovian queueing system with the concepts of customer impatience, feedback, and vacations. They also described their model's cost model. Using the notion of transmission errors and collisions, Lakaour et al. (2019)

explored a single server Markovian queueing model with the concept of collisions occurring when a primary arriving customer discovers the server is busy. For both single server $M/G/1$ retrial queues and multi-server $M/M/c$ retrial queues, Artalejo and Falin (2002) conducted a comparative analysis of standard queueing systems and retrial queueing systems, as well as differences and similarities between the two types of queueing systems. Khalaf et al. (2012) investigated a non-Markovian queueing model with server vacations, delay times, and random breakdowns under SS circumstances. They have determined the expected queue size and waiting time for their model. On dealing with a controlled retrial queueing model with K heterogeneous servers, Efrosinin and Breuer (2006) investigated the threshold policies for an arrival process that is based on the MAP and a service is based on the PH-distribution. Nazarov et al. (2020) investigated a single server Markovian finite-source retrial queueing system with collision of customers as well as server subject to breakdowns and repairs. They also discussed about the tagged customer's orbital sojourn time for their model.

This manuscript concepts are suitable for call centers and telecommuncations, etc. Though, the motivation for this article comes from online shopping on such web portals like Amazon, Flipkart, Nykaa and so forth. Online shopping is the process of purchasing goods and services from merchants who operate websites that sell their products and services on the internet. Businesses have been attempting to market their products to people who use the internet since the establishment of the World Wide Web. People who prefer to buy online can do so while sitting in front of a computer in the comfort of their home. From online retailers, customers can purchase a wide range of products. Individuals can, in fact, acquire almost anything from companies that sell their items on the internet. Any number of things/products available for purchase through an online store include: books, cosmetics, home appliances, toys, hardware, software, and medical coverage, to name a few. For the sake of convenience and time saving, many people choose to do their shopping online. Online shopping is accomplished by connecting to the internet, visiting a retailer's website and selecting the things on the basis of customer's interest. The items are kept in a virtual shopping cart until the customer is ready to complete the order. Virtually all online stores are open 24 hours a day, seven days a week. The individuals who wish to do shopping on online first they have to setup their accounts with their personal informations like house address, phone number, e-mail ID, etc. on any one of the relevant portals and then they can place their orders through online payment or by cash on delivery, depending on their preference (IAs). Nonetheless, the limited-time offers will be sent by the relevant web portals where the customers have accounts, either through messages or through e-mails (OAs). However, during this time period, when the orders placed by a specific person (incoming or OA service) will receive the date when their ordered products will be delivered, that delivery date may change due to any number of issues (server breakdown) and send the message about the delay to the concerned customer, then that customer will receive service on the changed date with some delay from the fixed date (considered the service offered by SS or collision occurs). Each of the ordering items will go through a number of stages, including order placement, confirmation of order, shipment, out-of-delivery status, and delivery (PH service). Some other people may be interested in purchasing a specific item that is not presently available for purchase in the stock, and they will check to see if the item is available for purchase on that online portal before purchasing the item they desire (retrial of IA of customers). It is possible that some persons who intend to purchase any of the products but are experiencing impatient behaviour circumstances (such as balking) will be unable to complete their purchases and will abandon the website at that time.

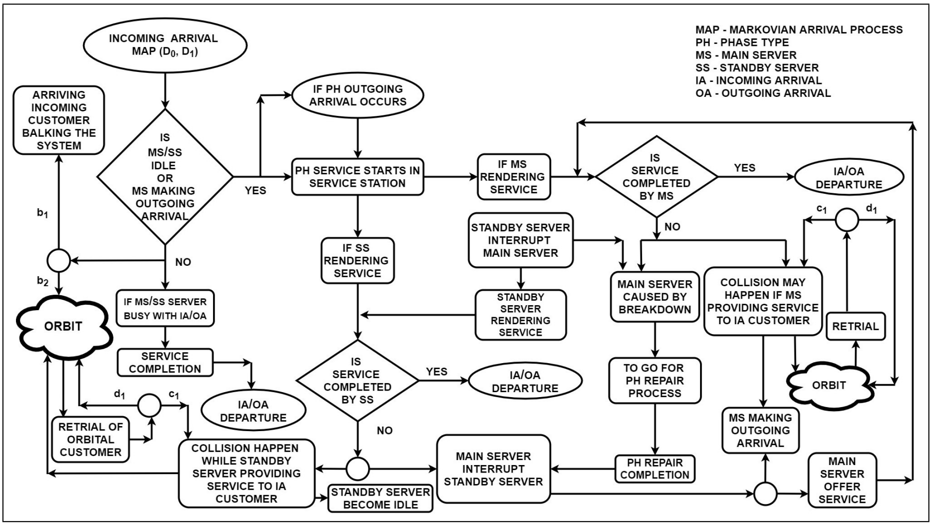
1.1 Structure of the model

The following is the rest of the manuscript. In Section 2, the mathematical model is described in detail. The model's matrix formulation and notations are given in Section 3. The stability condition, stationary probability vector, and the rate matrix are discussed in Section 4. Sections 5 and 6 characterise the system's busy period and the features of the system performance measures, respectively. Section 7 focuses into the cost analysis for this suggested model. Section 8 is devoted to presenting enthralling numerical, graphical, as well as comparative representations of the main and the SS service scenarios. The article's conclusion is presented in Section 9.

2 Description of the model

In this article, we analyse the IA of customers who follow MAP with representation (D_0, D_1) of order m_1, where D_0 indicates that no arrival occurs and D_1 indicates that an arrival occurs to the system. If the incoming arriving customer determines that the server is idle, the service will begin or, if the server is already busy, the arriving incoming customer will join the orbit with infinite retrial capacity. The service times for the IA of customers follows the PH distributions with depiction (α_1, T_1) of order n_1 with $T_1^0 + T_1 e = 0$ such that $T_1^0 = -T_1 e$. When each service completion epoch the MS become idle and making outgoing customers' arrival based on PH-distribution with depiction (β, S) of order m_2 with $S^0 + Se = 0$ such that $S^0 = -Se$. The service for OA also follows PH-distribution with depiction (α_2, T_2) of order n_2 with $T_2^0 + T_2 e = 0$ such that $T_2^0 = -T_2 e$. While the MS rendering service to IA or OA, the MS may caused by breakdown immediately to go for PH repair with depiction (γ, U) of order l with $U^0 + Ue = 0$ such that $U^0 = -Ue$. The MS's breakdowns follow an exponential distribution with parameter τ. Whenever the MS struck with breakdown, the SS will interrupt the MS and carry over the service up to repair completion of the MS. The SS also providing service under the PH-distribution for IA of customers with representation $(\alpha_{11}, \theta_1 T_1)$ of order n_1 where $0 < \theta_1 < 1$ and for OA of customers with representation $(\alpha_{21}, \theta_2 T_2)$ of order n_2 where $0 < \theta_2 < 1$. The MS only makes the OA and in the case of standby sever's service completion epoch the SS will be idle until the MS's repair completion or incoming customer arrives. During the service rendering by main or SS the arriving incoming customer may balk the system with probability b_1 or will join the orbit for retrial with probability b_2 such that $b_1 + b_2 = 1$. When a customer makes retrial from orbit while the server rendering service to IA of customers, that retriving orbital customer may collide with the IA customer then both being shifted to the orbit with probability c_1 otherwise that orbital customer may not collide the IA customer and once again join the orbit for retrial with probability d_1 such that $c_1 + d_1 = 1$. This collision will only happen when the incoming customer receiving service or else it does not make any change while the OA receiving service from the server. Inter-retrial times of orbital customers also follows the exponential distribution with the rate σ.

Figure 1 Pictorial representation of the proposed model



3 Analysis of the model

In this part, we discuss the notation of our model in order to generate the QBD process.

3.1 Notations for the matrix generation

- $\otimes$: Kronecker multiplication of two matrices.
- $\oplus$: Kronecker addition of two matrices.
- I_m: It denotes an $(m \times m)$-dimensional identity matrix.
- **0**: It denotes zero matrices in the appropriate order.
- e: A column vector of required dimension with each of its entries as 1.
- e_{n_1}: A column vector whose dimension is n_1 with each of its entries as 1.
- $e_1 = e_{m_1m_2+n_1m_1+n_2m_1+lm_1+ln_1m_1+ln_2m_1}$.
- $e_1(1)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with first $\{m_1m_2\}$ entries as 1 and the rest of the entries are zero.
- $e_1(2)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with $\{m_1m_2 + 1\}$ to $\{m_1m_2 + n_1m_1\}$ entries as 1 and the rest of the entries are zero.
- $e_1(3)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with $\{m_1m_2 + n_1m_1 + 1\}$ to $\{m_1m_2 + n_1m_1 + n_2m_1\}$ entries as 1 and the rest of the entries are zero.

- $e_1(4)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with $\{m_1m_2 + n_1m_1 + n_2m_1 + 1\}$ to $\{m_1m_2 + n_1m_1 + n_2m_1 + lm_1\}$ entries as 1 and the rest of the entries are zero.
- $e_1(5)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with $\{m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + 1\}$ to $\{m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + ln_1m_1\}$ entries as 1 and the rest of the entries are zero.
- $e_1(6)$: A column vector of dimension $\{(m_2 + n_1 + n_2 + l + ln_1 + ln_2)m_1 \times 1\}$ with $\{m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + ln_1m_1 + 1\}$ to $\{m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + ln_1m_1 + ln_2m_1\}$ entries as 1 and the rest of the entries are zero.
- Let us denote λ_1 be the fundamental arrival rate and is defined as $\lambda_1 = \pi_1 D_1 e_{m_1}$, where π_1 is the generator matrix $D = D_0 + D_1$, governs transitions of the MAP. To easily find the π_1 such that $\pi_1 D = 0$, $\pi_1 e_{m_1} = 1$.
- The normal service rate for both types of arriving customers the service offered by the MS and the SS are indicated by δ_1, δ_2 and $\theta_1\delta_1$, $\theta_2\delta_2$ where $\delta_1 = [\alpha_1(-T_1)^{-1}e_{n_1}]^{-1}$, $\delta_2 = [\alpha_2(-T_2)^{-1}e_{n_2}]^{-1}$.
- The arrival rate of the OA is indicated as λ_2, where $\lambda_2 = [\beta_1(-S)^{-1}e_{m_2}]^{-1}$.
- The repair rate of the MS is indicated as Ψ, where $\Psi = [\gamma(-U)^{-1}e_l]^{-1}$.
- $N(t)$ denotes the number of customers in the orbit,
- $V(t)$ denotes the server status at time t, where

$$V(t) = \begin{cases} 0, & \text{if the main server making outgoing arrival,} \\ 1, & \text{if the main server providing service to incoming arrival of customers,} \\ 2, & \text{if the main server providing service to outgoing arrival of customers,} \\ 3, & \text{if the standby server being idle while the main server under the PH repair,} \\ 4, & \text{if the standby server providing service to incoming arrival of customers while the main server under the PH repair,} \\ 5, & \text{if the standby server providing service to outgoing arrival of customers while the main server under the PH repair.} \end{cases}$$

- $I(t)$ denotes the phase for the repair process.
- $J_1(t)$ denotes the service phase for the IA of customers.
- $J_2(t)$ denotes the service phase for the OA of customers.
- $M_1(t)$ denotes the phase for IA of customers under MAP.
- $M_2(t)$ denotes the phase for OA of customers under PH.

- Let $\{(N(t),\, V(t),\, I(t),\, J_1(t),\, J_2(t),\, M_1(t),\, M_2(t))\colon t \geq 0\}$ is the continuous time Markov chain (CTMC) with state level independent QBD process such that state space is as follows:

  $$\mathbf{\Omega} = \bigcup_{p=0}^{\infty} l(p)$$

  where

  $$\begin{aligned} l(p) = {} & \{(p,0,s_1,s_2) : 1 \leq s_1 \leq m_1; 1 \leq s_2 \leq m_2\} \\ & \cup \{(p,1,r_1,s_1) : 1 \leq r_1 \leq n_1; 1 \leq s_1 \leq m_1\} \\ & \cup \{(p,2,r_2,s_1) : 1 \leq r_2 \leq n_2; 1 \leq s_1 \leq m_1\} \\ & \cup \{(p,3,q_2,s_1) : 1 \leq q_2 \leq l; 1 \leq s_1 \leq m_1\} \\ & \cup \{(p,4,q_2,r_1,s_1) : 1 \leq q_2 \leq l; 1 \leq r_1 \leq n_1; 1 \leq s_1 \leq m_1\} \\ & \cup \{(p,5,q_2,r_2,s_1) : 1 \leq q_2 \leq l; 1 \leq r_2 \leq n_2; 1 \leq s_1 \leq m_1\} \end{aligned}$$

The infinitesimal matrix formation of the QBD process can be expressed as follows:

$$Q = \begin{bmatrix} B_0 & B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\cdots \\ A_2 & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\cdots \\ \mathbf{0} & A_2 & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \cdots\cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 & A_1 & A_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots\ \ddots \end{bmatrix}$$

The following are the entries of the Q block matrices:

$$B_0 = \begin{bmatrix} D_0 \oplus S & \alpha_1 \otimes D_1 \otimes e_{m_2} & \alpha_2 \otimes I_{m_1} \otimes S^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ T_1^0 \otimes I_{m_1} \otimes \beta & [T_1 \oplus (D_0 + b_1 D_1) - \tau I_{n_1 m_1}] & \mathbf{0} & \mathbf{0} & \gamma \otimes \tau I_{n_1} \otimes I_{m_1} & \mathbf{0} \\ T_2^0 \otimes I_{m_1} \otimes \beta & \mathbf{0} & [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] & \mathbf{0} & \mathbf{0} & \gamma \otimes \tau I_{n_2} \otimes I_{m_1} \\ U^0 \otimes I_{m_1} \otimes \beta & \mathbf{0} & \mathbf{0} & U \oplus D_0 & I_l \otimes \alpha_{11} \otimes D_1 & \mathbf{0} \\ \mathbf{0} & U^0 \otimes I_{n_1} \otimes I_{m_1} & \mathbf{0} & I_l \otimes \theta_1 T_1^0 \otimes I_{m_1} & [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1)] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & U^0 \otimes I_{n_2} \otimes I_{m_1} & I_l \otimes \theta_2 T_2^0 \otimes I_{m_1} & \mathbf{0} & [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n_1} \otimes b_2 D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_2} \otimes b_2 D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_1} \otimes b_2 D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_2} \otimes b_2 D_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} & A_1^{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_1^{21} & A_1^{22} & \mathbf{0} & \mathbf{0} & A_1^{25} & \mathbf{0} \\ A_1^{31} & \mathbf{0} & A_1^{33} & \mathbf{0} & \mathbf{0} & A_1^{36} \\ A_1^{41} & \mathbf{0} & \mathbf{0} & A_1^{44} & A_1^{45} & \mathbf{0} \\ \mathbf{0} & A_1^{52} & \mathbf{0} & A_1^{54} & A_1^{55} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1^{63} & A_1^{64} & \mathbf{0} & A_{66} \end{bmatrix},$$

where

$$
\begin{aligned}
&A_1^{11} = D_0 \oplus S - \sigma I_{m_1 m_2};\; A_1^{12} = \alpha_1 \otimes D_1 \otimes e_{m_2};\; A_1^{13} = \alpha_2 \otimes I_{m_1} \otimes S^0;\\
&A_1^{21} = T_1^0 \otimes I_{m_1} \otimes \beta;\; A_1^{22} = T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1};\\
&A_1^{25} = \gamma \otimes \tau I_{n_1} \otimes I_{m_1};\; A_1^{31} = T_2^0 \otimes I_{m_1} \otimes \beta;\; A_1^{33} = T_2 \oplus (D_0 + b_1 D_1)\\
&-\tau I_{n_2 m_1};\; A_1^{36} = \gamma \otimes \tau I_{n_2} \otimes I_{m_1};\; A_1^{41} = U^0 \otimes I_{m_1} \otimes \beta;\\
&A_1^{44} = U \oplus D_0 - \sigma I_{l m_1};\; A_1^{45} = I_l \otimes \alpha_{11} \otimes D_1;\; A_1^{52} = U^0 \otimes I_{n_1} \otimes I_{m_1};\\
&A_1^{54} = I_l \otimes \theta_1 T_1^0 \otimes I_{m_1};\; A_1^{55} = U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1};\\
&A_1^{63} = U^0 \otimes I_{n_2} \otimes I_{m_1};\; A_1^{64} = I_l \otimes \theta_2 T_2^0 \otimes I_{m_1};\\
&A_1^{66} = U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1).
\end{aligned}
$$

$$
A_2 = \begin{bmatrix}
\mathbf{0} & \sigma(\alpha_1 \otimes I_{m_1} \otimes e_{m_2}) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma(I_l \otimes \alpha_{11} \otimes I_{m_1}) & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix},
$$

$$
A_0 = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
c_1\sigma e_{n_1} \otimes I_{m_1} \otimes \beta & I_{n_1} \otimes b_2 D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & I_{n_2} \otimes b_2 D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes c_1\sigma e_{n_1} \otimes I_{m_1} & I_l \otimes I_{n_1} \otimes b_2 D_1 & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \otimes I_{n_2} \otimes b_2 D_1
\end{bmatrix},
$$

4 Stability condition

We examine our model under certain conditions to ensure whether the system is stable.

4.1 Stability analysis

Let us describe the matrix A as $A = A_0 + A_1 + A_2$. It reveals that the order of the square matrix A is $(m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + ln_1m_1 + ln_2m_1)$, and that this matrix is an irreducible infinitesimal generator matrix.

The vector $\boldsymbol{\xi}$ is described by $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$. Let the vector $\boldsymbol{\xi}$ be the steady-state probability vector of the matrix A satisfying $\boldsymbol{\xi}\ A = 0$ and $\boldsymbol{\xi}\ e = 1$, where ξ_0 is of dimension m_1m_2, ξ_1 is of dimension n_1m_1, ξ_2 is of dimension n_2m_1, ξ_3 is of dimension lm_1, ξ_4 is of dimension ln_1m_1 and ξ_5 is of dimension ln_2m_1. The Markov process has the QBD structure, there exits stability of the model must satisfy the condition $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$, which is the necessary and the sufficient condition of a quasi-birth-and-death process. The vector $\boldsymbol{\xi}$ is acquired by simplifying the following expressions:

$$\xi_0[D_0 \oplus S - \sigma I_{m_1 m_2}] + \xi_1[(T_1^0 \otimes I_{m_1} \otimes \beta) + (c_1 \sigma e_{n_1} \otimes I_{m_1} \otimes \beta)]$$
$$+\xi_2[T_2^0 \otimes I_{m_1} \otimes \beta] + \xi_3[U^0 \otimes I_{m_1} \otimes \beta] = 0,$$

$$\xi_0[(\alpha_1 \otimes D_1 \otimes e_{m_2}) + (\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2})]$$
$$+\xi_1[T_1 \oplus D + (d_1 \sigma - (\sigma + \tau)) I_{n_1 m_1}] + \xi_4[U^0 \otimes I_{n_1} \otimes I_{m_1}] = 0,$$

$$\xi_0[\alpha_2 \otimes I_{m_1} \otimes S^0] + \xi_2[T_2 \oplus D - \tau I_{n_2 m_1}] + \xi_5[U^0 \otimes I_{n_2} \otimes I_{m_1}] = 0,$$

$$\xi_3[U \oplus D_0 - \sigma I_{l m_1}] + \xi_4[(I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}) + (I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1})]$$
$$+\xi_5[I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}] = 0,$$

$$\xi_1[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] + \xi_3[(I_l \otimes \alpha_{11} \otimes D_1) + (I_l \otimes \sigma \alpha_{11} \otimes I_{m_1})]$$
$$+\xi_4[U \oplus \theta_1 T_1 \oplus D + (d_1 \sigma - \sigma) I_{l n_1 m_1}] = 0,$$

$$\xi_2[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] + \xi_5[U \oplus \theta_2 T_2 \oplus D] = 0.$$

subject to the normalising condition

$$\xi_0 e_{m_1 m_2} + \xi_1 e_{n_1 m_1} + \xi_2 e_{n_2 m_1} + \xi_3 e_{l m_1} + \xi_4 e_{l n_1 m} + \xi_5 e_{l n_2 m} = 1.$$

After some algebraic manipulation, then the stability condition $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$ which is turns to be

$$\{\xi_1[(c_1 \sigma e_{n_1} \otimes e_{m_1} \otimes \beta e_{m_2}) + (e_{n_1} \otimes b_2 D_1 e_{m_1})] + \xi_2[e_{n_2} \otimes b_2 D_1 e_{m_1}]$$
$$+\xi_4[(e_l \otimes c_1 \sigma e_{n_1} \otimes e_{m_1}) + (e_l \otimes e_{n_1} \otimes b_2 D_1 e_{m_1})] + \xi_5[e_l \otimes e_{n_2} \otimes b_2 D_1 e_{m_1}]\}$$
$$< \{\xi_0[\sigma \alpha_{11} \otimes e_{m_1} \otimes e_{m_2}] + \xi_3[e_l \otimes \sigma \alpha_{11} \otimes e_{m_1}]\}$$

4.2 Analysis of stationary probability vector

Let us consider the variable $\mathbf{x}$ be the steady-state probability vector of Q and is subdivided as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$. Mention that $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, ... are of dimension $(m_1 m_2 + n_1 m_1 + n_2 m_1 + l m_1 + l_1 n_1 m_1 + l n_1 m + l n_2 m_1)$. Then, the vector $\mathbf{x}$ satisfies the condition $\mathbf{x}Q = 0$ and $\mathbf{x}e = 1$.

However, once the stability condition is satisfied and the sub-vectors of $\mathbf{x}$ except for $\mathbf{x}_0$ and $\mathbf{x}_1$, corresponding to the distinct level states are indicated by the expression

$$\mathbf{x}_j = \mathbf{x}_1 R^{j-1}, \qquad j \geqslant 2$$

where the rate matrix R indicates the minimum non-negative solution of the matrix quadratic equation as $R^2 A_2 + R A_1 + A_0 = 0$ (see Neuts, 1979).

Hence, the rate matrix R is a square matrix of order is $(m_1 m_2 + n_1 m_1 + n_2 m_1 + l m_1 + l_1 n_1 m_1 + l n_1 m + l n_2 m_1)$ and it satisfies the expression $R A_2 e = A_0 e$.

The sub vectors of $\mathbf{x}_0$ and $\mathbf{x}_1$ are acquired by solving the following expressions:

$$\mathbf{x}_0 B_0 + \mathbf{x}_1 A_2 = 0.$$
$$\mathbf{x}_0 B_1 + \mathbf{x}_1 (A_1 + R A_2) = 0.$$

The normalising condition is subjected to

$$\mathbf{x}_0 e + \mathbf{x}_1 (I - R)^{-1} e = 1.$$

4.3 Calculation of R matrix

There are various algorithms available for determining the rate matrix R. However, we have provided two algorithms in this section.

4.3.1 Iterative algorithm

The rate matrix can be easily evaluated using the recursive procedure described below:

Step 0:
$R(0) = 0$.
Step 1:
$R(n+1) = A_0(A_1)^{-1} + R^2(n)A_2(-A_1)^{-1}, n = 0, 1, 2, 3, ...$
Continue Step 1 Until $||R(n+1) - R(n)||_\infty < \epsilon$

4.3.2 Logarithmic reduction algorithm

The logarithmic reduction algorithm, created by Latouche and Ramaswami (1993), is known for its fast convergence. The following are the essential steps involved in the logarithmic reduction algorithm, as described by Latouche and Ramaswami (1993):

(i) Step 0:
$H \leftarrow (-A_1)^{-1}A_0,\ L \leftarrow (-A_1)^{-1}A_2,\ G = L$, and $T = H$.
(ii) Step 1:
$U = HL + LH$
$M = H^2$
$H \leftarrow (I - U)^{-1}M$
$M \leftarrow L^2$
$L \leftarrow (I - U)^{-1}M$
$G \leftarrow G + TL$
$T \leftarrow TH$
Continue Step 1 Until $||e - Ge||_\infty < \epsilon$
(iii) Step 2:
$R = -A_0(A_1 + A_0G)^{-1}$.

Theorem: The structure of the rate matrix R is

$$R = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \quad (1)$$

Proof: The calculation of the R matrix demonstrates definitively that the R matrix should have the structure specified in for our model (1), in which the MS may experience a breakdown while providing service, requiring immediate repair. In this situation, the SS will interrupt service and provide service at a lower rate than the MS.

When the server has recovered from the repair, the MS is restored back into service. The zeros in the first and fourth rows are noticeable in this set of results. In addition, we will provide R's construction in this section. The matrix quadratic expression $R^2A_2 + RA_1 + A_0 = 0$ can be rewritten as

$$R = (A_0 + R^2A_2)(-A_1)^{-1}$$

It can easily verify that the structure of the matrix $(-A_1)^{-1}$ as follows:

$$(-A_1)^{-1} = \frac{1}{V}\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \tag{2}$$

where the elements of $(-A_1)^{-1}$ as follows:

$$\begin{aligned} V = &\big[[T_2 \oplus (D_0 + b_1D_1) - \tau I_{n_2m_1}][U \oplus \theta_2T_2 \oplus (D_0 + b_1D_1)] \\ &- [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big] \times \Big[\big[[U \oplus D_0 - \sigma I_{lm_1}][U \\ &\oplus \theta_1T_1 \oplus (D_0 + b_1D_1) + [d_1\sigma - \sigma]I_{ln_1m_1}] - [I_l \otimes \alpha_{11} \otimes D_1] \\ &\times [I_l \otimes \theta_1T_1^0 \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1m_2}][T_1 \oplus (D_0 + b_1D_1) \\ &+ [d_1\sigma - (\sigma + \tau)]I_{n_1m_1}] - [\alpha_1 \otimes D_1 \otimes e_{m_2}][T_1^0 \otimes I_{m_1} \otimes \beta]\big] \\ &- [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big[[D_0 \oplus S - \sigma I_{m_1m_2}][U \oplus D_0 - \sigma I_{lm_1}] \times [U^0 \otimes I_{n_1} \\ &\otimes I_{m_1}] + [\alpha_1 \otimes D_1 \otimes e_{m_2}][U^0 \otimes I_{m_1} \otimes \beta][I_l \otimes \theta_1T_1^0 \otimes I_{m_1}]\big]\Big] \\ &+ [\alpha_2 \otimes I_{m_1} \otimes S^0] \times \Big[\big[[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2T_2^0 \otimes I_{m_1}]\big]\big[[T_1^0 \otimes I_{m_1} \\ &\otimes \beta][I_l \otimes \alpha_{11} \otimes D_1][U^0 \otimes I_{n_1} \otimes I_{m_1}] - [T_1 \oplus (D_0 + b_1D_1) \\ &+ [d_1\sigma - (\sigma + \tau)]I_{n_1m_1}][U^0 \otimes I_{m_1} \otimes \beta][U \oplus \theta_1T_1 \oplus (D_0 + b_1D_1) \\ &+ [d_1\sigma - \sigma]I_{ln_1m_1}]\big] + \big[[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}][U^0 \otimes I_{n_1} \otimes I_{m_1}]\big]\big[[T_2^0 \otimes I_{m_1} \\ &\otimes \beta][U \oplus D_0 - \sigma I_{lm_1}] \times [U \oplus \theta_2T_2 \oplus (D_0 + b_1D_1)] \\ &- [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2T_2^0 \otimes I_{m_1}][U^0 \otimes I_{m_1} \otimes \beta]\big] \\ &- [T_1 \oplus (D_0 + b_1D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1m_1}]\big[[T_2^0 \otimes I_{m_1} \\ &\otimes \beta][U \oplus \theta_2T_2 \oplus (D_0 + b_1D_1)] \times \big\{[U \oplus D_0 - \sigma I_{lm_1}][U \\ &\oplus \theta_1T_1 \oplus (D_0 + b_1D_1) + [d_1\sigma - \sigma]I_{ln_1m_1}] - [I_l \otimes \alpha_{11} \otimes D_1] \\ &\times [I_l \otimes \theta_1T_1^0 \otimes I_{m_1}]\big\}\big]\Big], \end{aligned}$$

$$\begin{aligned} f_{11} = &\Big[[T_1 \oplus (D_0 + b_1D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1m_1}]\big[[U \oplus D_0 \\ &- \sigma I_{lm_1}][U \oplus \theta_1T_1 \oplus (D_0 + b_1D_1) + [d_1\sigma - \sigma]I_{ln_1m_1}] \\ &- [I_l \otimes \theta_1T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 \\ &- \sigma I_{lm_1}] \times [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\Big] \times \big[[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] \\ &- [T_2 \oplus (D_0 + b_1D_1) - \tau I_{n_2m_1}] \times [U \oplus \theta_2T_2 \oplus (D_0 + b_1D_1)]\big], \end{aligned}$$

$$\begin{aligned}f_{12} &= [\alpha_1 \otimes D_1 \otimes e_{m_2}]\Big[[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \\ &\quad \oplus (D_0 + b_1 D_1)]] \times \big\{[U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) \\ &\quad + [d_1 \sigma - \sigma] I_{l n_1 m_1}] - [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1]\big\}\Big] \\ &\quad + [U^0 \otimes I_{n_1} \otimes I_{m_1}]\Big[[[\alpha_2 \otimes I_{m_1} \otimes S^0][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]] \\ &\quad \times \big\{[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big\}\Big],\end{aligned}$$

$$\begin{aligned}f_{13} &= \big[[\alpha_2 \otimes I_{m_1} \otimes S^0][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[T_1 \oplus (D_0 + b_1 D_1) \\ &\quad + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}] \times \big\{[U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) \\ &\quad + [d_1 \sigma - \sigma] I_{l n_1 m_1}] - [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1]\big\} \\ &\quad - [U \oplus D_0 - \sigma I_{l m_1}][U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big],\end{aligned}$$

$$\begin{aligned}f_{14} &= [\alpha_1 \otimes D_1 \otimes e_{m_2}]\big[[I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\{[T_2 \oplus (D_0 \\ &\quad + b_1 D_1) - \tau I_{n_2 m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] - [U^0 \otimes I_{n_2} \\ &\quad \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big\}\big] + \big[[\alpha_2 \otimes I_{m_1} \otimes S^0] \times [I_l \otimes \theta_2 T_2^0 \\ &\quad \otimes I_{m_1}]\big]\big[[T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] \\ &\quad \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - \sigma] I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \\ &\quad \otimes \tau I_{n_1} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big],\end{aligned}$$

$$\begin{aligned}f_{15} &= \big[[\alpha_1 \otimes D_1 \otimes e_{m_2}][U \oplus D_0 - \sigma I_{l m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big]\big[[U^0 \otimes I_{n_2} \\ &\quad \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] - [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \\ &\quad \oplus (D_0 + b_1 D_1)]\big] - [T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}]\big[[\alpha_2 \\ &\quad \otimes I_{m_1} \otimes S^0][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\{[I_l \otimes \alpha_{11} \otimes D_1] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big\}\big],\end{aligned}$$

$$\begin{aligned}f_{16} &= \big[[\alpha_2 \otimes I_{m_1} \otimes S^0][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[T_1 \oplus (D_0 + b_1 D_1) \\ &\quad + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}] \times \big\{[I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1] \\ &\quad - [U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - \sigma] I_{l n_1 m_1}]\big\} \\ &\quad + [U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[U \oplus D_0 - \sigma I_{l m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big],\end{aligned}$$

$$\begin{aligned}f_{21} &= \big[[T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[U \oplus D_0 - \sigma I_{l m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big\} \\ &\quad + [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\{[U^0 \otimes I_{m_1} \otimes \beta] \times [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] \\ &\quad - [T_1^0 \otimes I_{m_1} \otimes \beta][I_l \otimes \alpha_{11} \otimes D_1]\big\}\big]\big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] \\ &\quad \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] - [U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big],\end{aligned}$$

$$
\begin{aligned}
f_{22} = \big[&[U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{ln_1m_1}] \\
&- [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1]\big]\big[[U \oplus \theta_2 T_2 \oplus (D_0 \\
&+ b_1 D_1)]\big\{[T_2^0 \otimes I_{m_1} \otimes \beta][\alpha_2 \otimes I_{m_1} \otimes S^0] - [D_0 \oplus S - \sigma I_{m_1m_2}][T_2 \\
&\oplus (D_0 + b_1 D_1) - \tau I_{n_2m_1}]\big\} + [D_0 \oplus S - \sigma I_{m_1m_2}] \times \big\{[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \\
&\otimes \tau I_{n_2} \otimes I_{m_1}]\big\}\big] + [U^0 \otimes I_{m_1} \otimes \beta]\big[[\alpha_2 \otimes I_{m_1} \otimes S^0][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}] \\
&\times \big\{[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{ln_1m_1}]\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{23} = \big[&[\alpha_2 \otimes I_{m_1} \otimes S^0][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[I_l \\
&\otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1] - [U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \\
&\oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{ln_1m_1}]\big\} - [U^0 \otimes I_{m_1} \otimes \beta]\big\{[I_l \otimes \theta_1 T_1^0 \\
&\otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{24} = \big[&[I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1m_2}]\big\{[U^0 \\
&\otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] - [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2m_1}][U \\
&\oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\} + [T_2^0 \otimes I_{m_1} \otimes \beta] \times [\alpha_2 \otimes I_{m_1} \otimes S^0][U \\
&\oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big] - [T_1^0 \otimes I_{m_1} \otimes \beta]\big[[\alpha_2 \otimes I_{m_1} \otimes S^0] \\
&\times [I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) \\
&+ [d_1\sigma - \sigma] I_{ln_1m_1}]\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{25} = \big[&[U \oplus D_0 - \sigma I_{lm_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1m_2}]\big\{[T_2 \\
&\oplus (D_0 + b_1 D_1) - \tau I_{n_2m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] - [U^0 \\
&\otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big\} - [T_2^0 \otimes I_{m_1} \otimes \beta] \times [\alpha_2 \otimes I_{m_1} \\
&\otimes S^0][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big] + \big[[\alpha_2 \otimes I_{m_1} \otimes S^0][I_l \otimes \theta_2 T_2^0 \\
&\otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta][I_l \otimes \alpha_{11} \otimes D_1] \\
&- [U^0 \otimes I_{m_1} \otimes \beta][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{26} = \big[&[\alpha_2 \otimes I_{m_1} \otimes S^0][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[U \oplus D_0 \\
&- \sigma I_{lm_1}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{ln_1m_1}] \\
&- [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big\} + [U^0 \otimes I_{m_1} \\
&\otimes \beta][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{31} = \big[&[I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[T_1^0 \\
&\otimes I_{m_1} \otimes \beta][I_l \otimes \alpha_{11} \otimes D_1] - [U^0 \otimes I_{m_1} \otimes \beta][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\} \\
&+ [U^0 \otimes I_{m_1} \otimes \beta][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)] I_{n_1m_1}][U \\
&\oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{ln_1m_1}]\big] + \big[[T_2^0 \otimes I_{m_1} \otimes \beta]
\end{aligned}
$$

$$\times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}] \times \big\{[U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}] - [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1]\big\} - [U^0 \otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{lm_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big],$$

$$f_{32} = \big[[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big]\big[[U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}] - [D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1]\big] + [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[\alpha_1 \otimes D_1 \otimes e_{m_2}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \times \big\{[I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1] - [U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}]\big\}\big],$$

$$f_{33} = [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big[\big\{[T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}] - [D_0 \oplus S - \sigma I_{m_1 m_2}] \times [T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}]\big\}\big\{[U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}] - [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big\} + [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] \times \big\{[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{lm_1}] + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big],$$

$$f_{34} = \big[[I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}]\big\{[U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] - [T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}]\big\} + [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}]\big] - [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[\alpha_1 \otimes D_1 \otimes e_{m_2}][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\}\big],$$

$$f_{35} = \big[[I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}] - [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}]\big] + \big[[\alpha_1 \otimes D_1 \otimes e_{m_2}] \times [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[\alpha_2 \otimes I_{m_1} \otimes S^0][U \oplus D_0 - \sigma I_{lm_1}] + [U^0 \otimes I_{m_1} \otimes \beta][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big],$$

$$f_{36} = [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big[\big\{[U \oplus D_0 - \sigma I_{lm_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{ln_1 m_1}] - [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1]\big\}\big\{[D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}] - [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}]\big\} - \big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] \times [D_0 \oplus S - \sigma I_{m_1 m_2}]\big\}\big\{[U^0 \otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{lm_1}] + [\alpha_1 \otimes D_1 \otimes e_{m_2}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big],$$

$$f_{41} = \big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] - [U^0 \otimes I_{n_2} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[T_1^0 \otimes I_{m_1} \otimes \beta][I_l \otimes \alpha_{11} \otimes D_1] - [U^0 \otimes I_{m_1} \otimes \beta] \times [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\} + [U^0 \otimes I_{m_1} \otimes \beta][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}]\big],$$

$$f_{42} = \big[[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] - [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}][I_l \otimes \alpha_{11} \otimes D_1] + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}]\big] + [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[U^0 \otimes I_{n_1} \otimes I_{m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big\{[I_l \otimes \alpha_{11} \otimes D_1] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\}\big],$$

$$f_{43} = \big[[\alpha_2 \otimes I_{m_1} \otimes S^0][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] - [T_1^0 \otimes I_{m_1} \otimes \beta][I_l \otimes \alpha_{11} \otimes D_1]\big\} - [T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}]\big],$$

$$f_{44} = \big[[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] - [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big] \times \big[[D_0 \oplus S - \sigma I_{m_1 m_2}]\big\{[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\} - [T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[\alpha_1 \otimes D_1 \otimes e_{m_2}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}]\big\}\big] + [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[\alpha_2 \otimes I_{m_1} \otimes S^0] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\{[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big],$$

$$f_{45} = \big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] - [U^0 \otimes I_{n_2} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}] \times [I_l \otimes \alpha_{11} \otimes D_1] - [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][I_l \otimes \alpha_{11} \otimes D_1] + [U^0 \otimes I_{m_1} \otimes \beta] \times [\alpha_1 \otimes D_1 \otimes e_{m_2}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big] - [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big\{[I_l \otimes \alpha_{11} \otimes D_1][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\}\big],$$

$$f_{46} = \big[[\alpha_2 \otimes I_{m_1} \otimes S^0][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\} + [U^0 \otimes I_{m_1} \otimes \beta]\big\{[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big],$$

$$
\begin{aligned}
f_{51} = \big[&[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}] - [T_2 \oplus (D_0 + b_1 D_1) \\
&- \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta][U^0 \otimes I_{n_1} \\
&\otimes I_{m_1}][U \oplus D_0 - \sigma I_{l m_1}] + [U^0 \otimes I_{m_1} \otimes \beta] \times [T_1 \oplus (D_0 + b_1 D_1) \\
&+ [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{52} = \big[&[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \\
&- [U^0 \otimes I_{n_2} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \\
&\otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{l m_1}] + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][I_l \\
&\otimes \theta_1 T_1^0 \otimes I_{m_1}]\big] - \big[[U^0 \otimes I_{n_1} \otimes I_{m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big] \\
&\times \big[[T_2^0 \otimes I_{m_1} \otimes \beta][U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \\
&+ [U^0 \otimes I_{m_1} \otimes \beta] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{53} = \big[&[U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)][\alpha_2 \otimes I_{m_1} \otimes S^0]\big]\big[[T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[U^0 \\
&\otimes I_{n_1} \otimes I_{m_1}] \times [U \oplus D_0 - \sigma I_{l m_1}]\big\} + [U^0 \otimes I_{m_1} \otimes \beta]\big\{[T_1 \oplus (D_0 \\
&+ b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{54} = \big[&[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \\
&- [U^0 \otimes I_{n_2} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}]\big\{[T_1 \\
&\oplus (D_0 + b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\} \\
&- [T_1^0 \otimes I_{m_1} \otimes \beta]\big\{[\alpha_1 \otimes D_1 \otimes e_{m_2}][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big] + [\alpha_2 \otimes I_{m_1} \\
&\otimes S^0]\big[[T_1^0 \otimes I_{m_1} \otimes \beta][U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2 T_2^0 \\
&\otimes I_{m_1}]\big\} - [T_2^0 \otimes I_{m_1} \otimes \beta][T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - (\sigma \\
&+ \tau)] I_{n_1 m_1}]\big\{[I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)]\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{55} = \big[&[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] \\
&- [U^0 \otimes I_{n_2} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_2} \otimes I_{m_1}]\big]\big[[U \oplus D_0 - \sigma I_{l m_1}]\big\{[T_1^0 \\
&\otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}] - [D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 \\
&+ b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}]\big\}\big] + \big[[T_1 \oplus (D_0 + b_1 D_1) \\
&+ [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big]\big[[T_2^0 \otimes I_{m_1} \otimes \beta][U \\
&\oplus D_0 - \sigma I_{l m_1}] \times [U \oplus \theta_2 T_2 \oplus (D_0 + b_1 D_1)] + [U^0 \otimes I_{m_1} \\
&\otimes \beta][\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big],
\end{aligned}
$$

$$
\begin{aligned}
f_{56} = -&[\alpha_2 \otimes I_{m_1} \otimes S^0]\big[[\alpha_1 \otimes D_1 \otimes e_{m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[T_2 \\
&\oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] \times [U \oplus D_0 - \sigma I_{l m_1}]\big\} + [U^0 \\
&\otimes I_{m_1} \otimes \beta][T_1 \oplus (D_0 + b_1 D_1) + [d_1 \sigma - (\sigma + \tau)] I_{n_1 m_1}] \\
&\times \big\{[\gamma \otimes \tau I_{n_2} \otimes I_{m_1}][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big],
\end{aligned}
$$

$$f_{61} = [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}][U^0 \otimes I_{n_2} \otimes I_{m_1}] \times \big\{[I_l \otimes \alpha_{11} \otimes D_1][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] - [U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}]\big\}\big] + [U^0 \otimes I_{m_1} \otimes \beta]\big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}] \times \big\{[U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] - [T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}] \times [U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}]\big\}\big] - [U^0 \otimes I_{n_1} \otimes I_{m_1}]\big[[T_2^0 \otimes I_{m_1} \otimes \beta] \times [U^0 \otimes I_{n_2} \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{l m_1}]\big\} + [T_1^0 \otimes I_{m_1} \otimes \beta] \times [T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}]\big\{[I_l \otimes \alpha_{11} \otimes D_1][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\}\big],$$

$$f_{62} = [I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}] \times \big\{[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}] + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}]\big\} - [T_2^0 \otimes I_{m_1} \otimes \beta][U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[\alpha_2 \otimes I_{m_1} \otimes S^0][I_l \otimes \alpha_{11} \otimes D_1]\big\}\big] + [T_2^0 \otimes I_{m_1} \otimes \beta] \times \big[[\alpha_1 \otimes D_1 \otimes e_{m_2}][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big\{[U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}] - [I_l \otimes \alpha_{11} \otimes D_1][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big],$$

$$f_{63} = \big[[U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}] - [I_l \otimes \alpha_{11} \otimes D_1] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_2} \otimes I_{m_1}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}] - [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big] - \big[[U^0 \otimes I_{n_2} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big]\big[[D_0 \oplus S - \sigma I_{m_1 m_2}][U^0 \otimes I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{l m_1}] + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big] + [\alpha_2 \otimes I_{m_1} \otimes S^0] \times \big[[T_1^0 \otimes I_{m_1} \otimes \beta][U^0 \otimes I_{n_1} \otimes I_{m_1}]\big\{[I_l \otimes \alpha_{11} \otimes D_1][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\} + [U^0 \otimes I_{m_1} \otimes \beta] \times [I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\{[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big],$$

$$f_{64} = [D_0 \oplus S - \sigma I_{m_1 m_2}]\big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\{[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)]I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma]I_{l n_1 m_1}] - [U^0 \otimes I_{n_1} \otimes I_{m_1}] \times [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big] + [\alpha_1 \otimes D_1 \otimes e_{m_2}]\big[[T_2^0 \otimes I_{m_1} \otimes \beta][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\} - [T_1^0 \otimes I_{m_1}$$

$$\otimes \beta][T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}]\big\{[U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big\}\big] + \big[[T_2^0 \otimes I_{m_1} \otimes \beta][\alpha_2 \otimes I_{m_1} \otimes S^0]\big] \times \big[[U^0 \otimes I_{n_1} \otimes I_{m_1}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}] - [T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big],$$

$$f_{65} = \big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}][I_l \otimes \theta_2 T_2^0 \otimes I_{m_1}]\big]\big[[U \oplus D_0 - \sigma I_{l m_1}]\big\{[T_1^0 \otimes I_{m_1} \otimes \beta] \times [\alpha_1 \otimes D_1 \otimes e_{m_2}] - [D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}]\big\} - [U^0 \otimes I_{m_1} \otimes \beta]\big\{[\alpha_1 \otimes D_1 \otimes e_{m_2}][\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big\}\big] + [T_2^0 \otimes I_{m_1} \otimes \beta]\big[[T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)] I_{n_1 m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big\{[I_l \otimes \alpha_{11} \otimes D_1][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big\} - [\alpha_1 \otimes D_1 \otimes e_{m_2}][U^0 \otimes I_{n_2} \otimes I_{m_1}]\big\{[\gamma \otimes \tau I_{n_1} \otimes I_{m_1}][U \oplus D_0 - \sigma I_{l m_1}]\big\}\big],$$

$$f_{66} = \big[[I_l \otimes \alpha_{11} \otimes D_1][I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}] - [U \oplus D_0 - \sigma I_{l m_1}][U \oplus \theta_1 T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - \sigma] I_{l n_1 m_1}]\big]\big[[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}]\big\{[D_0 \oplus S - \sigma I_{m_1 m_2}][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma + \tau)] I_{n_1 m_1}] - [T_1^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}]\big\} - [T_2^0 \otimes I_{m_1} \otimes \beta][T_1 \oplus (D_0 + b_1 D_1) + [d_1\sigma - (\sigma+\tau)] I_{n_1 m_1}][\alpha_2 \otimes I_{m_1} \otimes S^0]\big] + [\gamma \otimes \tau I_{n_1} \otimes I_{m_1}]\big[[U^0 \otimes I_{n_1} \otimes I_{m_1}] \times [U \oplus D_0 - \sigma I_{l m_1}]\big\{[D_0 \oplus S - \sigma I_{m_1 m_2}][T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] - [T_2^0 \otimes I_{m_1} \otimes \beta] \times [\alpha_2 \otimes I_{m_1} \otimes S^0]\big\} + [U^0 \otimes I_{m_1} \otimes \beta][\alpha_1 \otimes D_1 \otimes e_{m_2}]\big\{[T_2 \oplus (D_0 + b_1 D_1) - \tau I_{n_2 m_1}] \times [I_l \otimes \theta_1 T_1^0 \otimes I_{m_1}]\big\}\big].$$

In the similar way, pre-multiplying a diagonal block matrix with $(-A_1)^{-1}$ matrix. However, the structure of matrix $A_0(-A_1)^{-1}$ is provided by,

$$A_0(-A_1)^{-1} = \frac{1}{V}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} \end{bmatrix} \tag{3}$$

where the elements of $A_0(-A_1)^{-1}$ is as follows:

$$g_{21} = [c_1\sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{11} + [I_{n_1} \otimes b_2 D_1] f_{21},$$
$$g_{22} = [c_1\sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{12} + [I_{n_1} \otimes b_2 D_1] f_{22},$$

$$g_{23} = [c_1 \sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{13} + [I_{n_1} \otimes b_2 D_1] f_{23},$$
$$g_{24} = [c_1 \sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{14} + [I_{n_1} \otimes b_2 D_1] f_{24},$$
$$g_{25} = [c_1 \sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{15} + [I_{n_1} \otimes b_2 D_1] f_{25},$$
$$g_{26} = [c_1 \sigma e_{n_1} \otimes I_{m_1} \otimes \beta] f_{16} + [I_{n_1} \otimes b_2 D_1] f_{26},$$
$$g_{31} = [I_{n_2} \otimes b_2 D_1] f_{31},$$
$$g_{32} = [I_{n_2} \otimes b_2 D_1] f_{32},$$
$$g_{33} = [I_{n_2} \otimes b_2 D_1] f_{33},$$
$$g_{34} = [I_{n_2} \otimes b_2 D_1] f_{34},$$
$$g_{35} = [I_{n_2} \otimes b_2 D_1] f_{35},$$
$$g_{36} = [I_{n_2} \otimes b_2 D_1] f_{36},$$
$$g_{51} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{41} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{51},$$
$$g_{52} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{42} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{52},$$
$$g_{53} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{43} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{53},$$
$$g_{54} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{44} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{54},$$
$$g_{55} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{45} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{55},$$
$$g_{56} = [I_l \otimes c_1 \sigma e_{n_1} \otimes I_{m_1}] f_{46} + [I_l \otimes I_{n_1} \otimes b_2 D_1] f_{56},$$
$$g_{61} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{61},$$
$$g_{62} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{62},$$
$$g_{63} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{63},$$
$$g_{64} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{64},$$
$$g_{65} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{65},$$
$$g_{66} = [I_l \otimes I_{n_2} \otimes b_2 D_1] f_{66}.$$

Here, we pre-multiplying a block matrix A_2 with $(-A_1)^{-1}$ matrix. Thus, the structure of matrix $A_2(-A_1)^{-1}$ is provided by,

$$A_2(-A_1)^{-1} = \frac{1}{V} \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (4)$$

where the elements of $A_2(-A_1)^{-1}$ is as follows:

$$h_{11} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{21},$$
$$h_{12} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{22},$$
$$h_{13} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{23},$$
$$h_{14} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{24},$$
$$h_{15} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{25},$$
$$h_{16} = [\sigma \alpha_1 \otimes I_{m_1} \otimes e_{m_2}] f_{26},$$
$$h_{41} = [I_l \otimes \sigma \alpha_{11} \otimes I_{m_1}] f_{41},$$

$$h_{42} = [I_l \otimes \sigma\alpha_{11} \otimes I_{m_1}]f_{42},$$
$$h_{43} = [I_l \otimes \sigma\alpha_{11} \otimes I_{m_1}]f_{43},$$
$$h_{44} = [I_l \otimes \sigma\alpha_{11} \otimes I_{m_1}]f_{44},$$
$$h_{45} = [I_l \otimes \sigma\alpha_{11} \otimes I_{m_1}]f_{45},$$
$$h_{46} = [I_l \otimes \sigma\alpha_{11} \otimes I_{m_1}]f_{46}.$$

The sequence of $\{R^{(n)}, n = 0, 1, 2, 3, ...\}$ is described by,

$$R^{(n+1)} = \left[(R^{(n)})^2 A_2 + A_0\right](-A_1)^{-1}, \quad n = 0, 1, 2, 3, ...$$

The minimum non-negative solution of the matrix quadratic expression $R^2A_2 + RA_1 + A_0 = 0$ converges monotonically with $R^{(0)} = 0$. The structure of the first iterative value of the R matrix, i.e., $R^{(1)}$ will be the same as that of $\{A_0(-A_1)^{-1}\}$. Then we can clearly observe that $(R^{(1)})^2A_2(-A_1)^{-1}$ will have the same structure as $R^{(1)}$, implying the generalisation of the fact that $R^{(n)}$ will have the same structure as $\{A_0(-A_1)^{-1}\}$. As a result, the preceding findings are verified.

5 Busy period analysis

- A busy period can be measured as the time between customers entering an empty system and the system being empty again after the first interval. As a consequence, it is the first passage time between levels 1 and 0. The busy cycle is defined as the first return time to level 0 with at least one visit to any other level.
- We begin with a brief overview of the fundamental period before moving on to the busy period. It is the first passage time from level j to level $j-1, j \geqslant 2$ when the QBD process is taken into account.
- Individual cases j = 0, 1 corresponding to boundary states must be discussed. Mention that $(m_1m_2 + n_1m_1 + n_2m_1 + lm_1 + ln_1m_1 + ln_2m_1)$ states correspond to each and every level j, $j \geqslant 1$. When the states are organised in alphabetical order, the state (j, k) of level j denotes the k^{th} state of level j.
- Let $G_{kk'}(u, \mathrm{x})$ be the conditional probability that the quasi-birth-and-death process visits the level $j-1$ to make changes to u transitions to the left and also enter the state (j, k') with the constraint that it start-up in the state (j, k) at time $t = 0$.

  Let us begin with the joint transform concept.

  $$\widetilde{G}_{kk'}(z, s) = \sum_{u=1}^{\infty} z^u \int_0^{\infty} e^{-s\mathrm{x}} dG_{kk'}(u, \mathrm{x}); \quad | z |\leq 1, Re(s) \geq 0$$

  and the matrix is described as $\widetilde{G}(z, s) = \widetilde{G}_{kk'}(z, s)$ then the above-mentioned matrix $\widetilde{G}(z, s)$ satisfies the expression

  $$\widetilde{G}(z, s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\widetilde{G}^2(z, s)$$

- The matrix of $G = G_{kk'} = \widetilde{G}(1, 0)$ would be taken for the first passage times, exclude for the boundary states. If we have already known the matrix R then we could find the matrix G using the result

  $$G = -(A_1 + RA_2)^{-1}A_2$$

Otherwise, one can utilise the concept of a logarithmic reduction algorithm to obtain the values of the G matrix.

5.1 Notations of boundary level states

- $G^{(1,0)}_{kk'}(u, \mathrm{x})$ denotes the conditional probability that the quasi-birth-and-death process is discussed for the first passage times from level one to level zero at time $t = 0$.
- $G^{(0,0)}_{kk'}(u, \mathrm{x})$ denotes the conditional probability that the quasi-birth-and-death process is discussed for the return time to level zero.
- $\mathbb{F}_{1j}$ indicates the average first passage time from the level j to level $j-1$, given that the process is in the state (j, k) at time $t = 0$.
- $\overrightarrow{\mathbb{F}_1}$ indicates the column vector with entries $\mathbb{F}_{1j}$.
- $\mathbb{F}_{2j}$ indicates the average number of customers to be served during the first passage time from level j to level $j-1$, given that the first passage time begins in the state (j, k).
- $\overrightarrow{\mathbb{F}_2}$ indicates the column vector with entries $\mathbb{F}_{2j}$.
- $\overrightarrow{\mathbb{F}_1}^{(1,0)}$ indicates the average first passage time from level one to level zero.
- $\overrightarrow{\mathbb{F}_2}^{(1,0)}$ denotes the mean number of service completed during the first passage time from level one to level zero.
- $\overrightarrow{\mathbb{F}_1}^{(0,0)}$ denotes the first return time to level zero.
- $\overrightarrow{\mathbb{F}_2}^{(0,0)}$ denotes the mean number of service completion in between first return time to level zero.

The following equations which are given $\widetilde{G}^{(1,0)}(z, s)$ and $\widetilde{G}^{(0,0)}(z, s)$ are for the boundary levels one and zero respectively.

$$\widetilde{G}^{(1,0)}(z, s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\widetilde{G}(z, s)\widetilde{G}^{(1,0)}(z, s)$$
$$\widetilde{G}^{(0,0)}(z, s) = (sI - B_0)^{-1}B_1\widetilde{G}^{(1,0)}(z, s)$$

Thus, the following instances are evaluated utlising the matrices as G, $\widetilde{G}^{(0,0)}(1, 0)$ and $\widetilde{G}^{(1,0)}(1, 0)$ are stochastic.

$$\overrightarrow{\mathbb{F}_1} = -\left.\frac{\partial}{\partial s}\widetilde{G}(z, s)\right|_{z=1,s=0} e = -[A_1 + A_0(I + G)]^{-1}e \tag{5}$$

$$\overrightarrow{\mathbb{F}_2} = \left.\frac{\partial}{\partial z}\widetilde{G}(z, s)\right|_{z=1,s=0} e = -[A_1 + A_0(I + G)]^{-1}A_2 e \tag{6}$$

$$\overrightarrow{\mathbb{F}_1}^{(1,0)} = -\left.\frac{\partial}{\partial s}\widetilde{G}^{(1,0)}(z, s)\right|_{z=1,s=0} e = -[A_1 + A_0G]^{-1}(A_0\overrightarrow{\mathbb{F}_1} + e) \tag{7}$$

$$\overrightarrow{\mathbb{F}_2}^{(1,0)} = \frac{\partial}{\partial z}\widetilde{G}^{(1,0)}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\overrightarrow{\mathbb{F}_2} + A_2 e) \tag{8}$$

$$\overrightarrow{\mathbb{F}_1}^{(0,0)} = -\frac{\partial}{\partial s}\widetilde{G}^{0,0}(z,s)\bigg|_{z=1,s=0} e = -B_0^{-1}[B_1\overrightarrow{\mathbb{F}_1}^{(1,0)} + e] \tag{9}$$

$$\overrightarrow{\mathbb{F}_2}^{(0,0)} = \frac{\partial}{\partial z}\widetilde{G}^{(0,0)}(z,s)\bigg|_{z=1,s=0} e = -B_0^{-1}[B_1\overrightarrow{\mathbb{F}_2}^{(1,0)}]. \tag{10}$$

6 Performance measures

We probe the qualitative behaviour of our model under a steady state. In this part, we included a few performances of system measures, together with their computation expressions, as follows:

- Probability that the MS making OA:

$$P_{MMO} = \sum_{p=0}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} \mathbf{x}_{p0s_1s_2} = \mathbf{x}_0 e_1(1) + \mathbf{x}_1(I-R)^{-1}e_1(1)$$

- Probability that the MS providing service to IA of customers:

$$P_{MBI} = \sum_{p=0}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1} \mathbf{x}_{p1r_1s_1} = \mathbf{x}_0 e_1(2) + \mathbf{x}_1(I-R)^{-1}e_1(2)$$

- Probability that the MS providing service to OA of customers:

$$P_{MBO} = \sum_{p=0}^{\infty}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1} \mathbf{x}_{p2r_2s_1} = \mathbf{x}_0 e_1(3) + \mathbf{x}_1(I-R)^{-1}e_1(3)$$

- Probability that the SS being idle:

$$P_{SI} = \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1} \mathbf{x}_{p3q_2s_1} = \mathbf{x}_0 e_1(4) + \mathbf{x}_1(I-R)^{-1}e_1(4)$$

- Probability that the SS providing service to incoming customers:

$$P_{SBI} = \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1} \mathbf{x}_{p4q_2r_1s_1} = \mathbf{x}_0 e_1(5) + \mathbf{x}_1(I-R)^{-1}e_1(5)$$

- Probability that the SS providing service to outgoing customers:

$$P_{SBO} = \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1} \mathbf{x}_{p5q_2r_2s_1} = \mathbf{x}_0 e_1(6) + \mathbf{x}_1(I-R)^{-1}e_1(6)$$

- Expected orbit size:

$$E_{orbit} = \sum_{p=0}^{\infty} p\, \mathbf{x}_p\, e_1 = \mathbf{x}_1 (I-R)^{-2} e_1$$

- The average system size:

$$\begin{aligned} E_{system} =& \sum_{p=0}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} p\mathbf{x}_{p0s_1s_2} + \sum_{p=0}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1} (p+1)\mathbf{x}_{p1r_1s_1} \\ &+ \sum_{p=0}^{\infty}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1} (p+1)\mathbf{x}_{p2r_2s_1} + \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1} p\mathbf{x}_{p3q_2s_1} \\ &+ \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1} (p+1)\mathbf{x}_{p4q_2r_1s_1} + \sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1} (p+1)\mathbf{x}_{p5q_2r_2s_1} \end{aligned}$$

- The successful rate of retrials:

$$\begin{aligned} \aleph =& \sigma\left[\sum_{p=1}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} \mathbf{x}_{p0s_1s_2} + \sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1} \mathbf{x}_{p3q_2s_1}\right] \\ =& \sigma\Big[\mathbf{x}_1(I-R)^{-1}e_1(1) + \mathbf{x}_1(I-R)^{-1}e_1(4)\Big] \end{aligned}$$

- Probability of a successful retrial:

$$\begin{aligned} P_{SRT} =& \left\{\frac{\sigma}{\sigma+\lambda_1+\lambda_2}\right\}\left[\sum_{p=1}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} \mathbf{x}_{p0s_1s_2} + \sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1} \mathbf{x}_{p3q_2s_1}\right] \\ =& \left\{\frac{\sigma}{\sigma+\lambda_1+\lambda_2}\right\}\Big[\mathbf{x}_1(I-R)^{-1}e_1(1) + \mathbf{x}_1(I-R)^{-1}e_1(4)\Big] \end{aligned}$$

- Average number of a successful retrial:

$$\begin{aligned} E_{SRT} =& \left\{\frac{\sigma}{\sigma+\lambda_1+\lambda_2}\right\}\left[\sum_{p=1}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} p\mathbf{x}_{p0s_1s_2} + \sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1} p\mathbf{x}_{p3q_2s_1}\right] \\ =& \left\{\frac{\sigma}{\sigma+\lambda_1+\lambda_2}\right\}\Big[\mathbf{x}_1(I-R)^{-2}e_1(1) + \mathbf{x}_1(I-R)^{-2}e_1(4)\Big] \end{aligned}$$

- Expected number of customers in the orbit when the MS making OA:

$$E_{MMO} = \sum_{p=0}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2} p\mathbf{x}_{p0s_1s_2} = \mathbf{x}_1(I-R)^{-2}e_1(1)$$

- Average orbit size when the MS providing service to IA of customers:

$$E_{MBI} = \sum_{p=0}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1} p\mathbf{x}_{p1r_1s_1} = \mathbf{x}_1(I-R)^{-2}e_1(2)$$

- Average orbit size when the MS providing service to OA of customers:

$$E_{MBO}=\sum_{p=0}^{\infty}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}p\mathbf{x}_{p2r_2s_1}=\mathbf{x}_1(I-R)^{-2}e_1(3)$$

- Average orbit size when the SS being idle:

$$E_{SI}=\sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1}p\mathbf{x}_{p3q_2s_1}=\mathbf{x}_1(I-R)^{-2}e_1(4)$$

- Average orbit size when the SS providing service to incoming customers:

$$E_{SBI}=\sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1}p\mathbf{x}_{p4q_2r_1s_1}=\mathbf{x}_1(I-R)^{-2}e_1(5)$$

- Average orbit size when the SS providing service to outgoing customers:

$$E_{SBO}=\sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}p\mathbf{x}_{p5q_2r_2s_1}=\mathbf{x}_1(I-R)^{-2}e_1(6)$$

- The overall rate of retrials at which the orbital customers request for service:

$$\begin{aligned}\sigma^*=\sigma\Bigg\{&\sum_{p=1}^{\infty}\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2}\mathbf{x}_{p0s_1s_2}+\sum_{p=1}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1}\mathbf{x}_{p1r_1s_1}+\sum_{p=1}^{\infty}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}\mathbf{x}_{p2r_2s_1}\\&+\sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{s_1=1}^{m_1}\mathbf{x}_{p3q_2s_1}+\sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1}\mathbf{x}_{p4q_2r_1s_1}\\&+\sum_{p=1}^{\infty}\sum_{q_2=1}^{l}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}\mathbf{x}_{p5q_2r_2s_1}\Bigg\}=\sigma\big[\mathbf{x}_1(I-R)^{-1}e_1(1)\\&+\mathbf{x}_1(I-R)^{-1}e_1(2)+\mathbf{x}_1(I-R)^{-1}e_1(3)+\mathbf{x}_1(I-R)^{-1}e_1(4)\\&+\mathbf{x}_1(I-R)^{-1}e_1(5)+\mathbf{x}_1(I-R)^{-1}e_1(6)\big]\end{aligned}$$

- The rate at which the customers balking from the system:

$$\begin{aligned}\Upsilon=b_1\lambda_1\Bigg[&\sum_{p=0}^{\infty}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1}\mathbf{x}_{p1r_1s_1}+\sum_{p=0}^{\infty}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}\mathbf{x}_{p2r_2s_1}\\&+\sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_1=1}^{n_1}\sum_{s_1=1}^{m_1}\mathbf{x}_{p4q_2r_1s_1}+\sum_{p=0}^{\infty}\sum_{q_2=1}^{l}\sum_{r_2=1}^{n_2}\sum_{s_1=1}^{m_1}\mathbf{x}_{p5q_2r_2s_1}\Bigg]\end{aligned}$$

7 Cost analysis

In this section, we assign a cost to some of the system performance measures under consideration. We discuss a cost function TC, which is described as follows:

$$\begin{aligned}TC=&C_HE_{orbit}+C_{MMO}P_{MMO}+C_{MBI}P_{MBI}+C_{MBO}P_{MBO}\\&+C_{SI}P_{SI}+C_{SBI}P_{SBI}+C_{SBO}P_{SBO}+\tau C_1+\delta_1C_2+\delta_2C_3\\&+(\theta_1\delta_1)C_4+(\theta_2\delta_2)C_5+\Psi C_6+\Upsilon C_7\end{aligned}$$

where

- TC – Total cost of the system.
- C_H – Holding cost of each and every customer in the orbit.
- C_{MMO} – Cost per unit time when the MS is making OA of customers.
- C_{MBI} – Cost per unit time while the MS rendering service to IA of customers.
- C_{MBO} – Cost per unit time while the MS rendering service to OA of customers.
- C_{SI} – Cost per unit time during the SS being idle.
- C_{SBI} – Cost per unit time while the SS rendering service to IA of customers.
- C_{SBO} – Cost per unit time while the SS rendering service to OA of customers.
- C_1 – Cost per unit time when the MS caused by breakdowns.
- C_2 – Cost per unit time for the incoming service is rendered by the MS.
- C_3 – Cost per unit time for the outgoing service is rendered by the MS.
- C_4 – Cost per unit time for the incoming service is rendered by the SS.
- C_5 – Cost per unit time for the outgoing service is rendered by the SS.
- C_6 – Cost per unit time for the MS rejuvenated from repair.
- C_7 – Cost per unit time due to the impatient behaviour of IA of customers.

8 Numerical analysis

In this section, we analyse the model's behaviour utilising numerical and graphical representations. The variance and correlation structures of the five different MAP representations are different. The first three arrival processes, ERLA, EXPA, and HYP-EXPA, all correspond to renewal processes, so their correlation is zero. The arrival processes of MAP-NC and MAP-PC are correlated arrivals, with –0.4804 and 0.4804 as the correlation between two successive inter-arrival times. The inter-arrival times of these five arrival processes have coefficients of variation of 0.2500, 1, 14.6365, 1.0408, and 1.0408.

- *Erlang of order 4 (ERLA):*

$$D_0 = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 4&0&0&0 \end{bmatrix}$$

- *Exponential (EXPA):*

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

- *Hyper-exponential (HYP-EXPA):*

$$D_0 = \begin{bmatrix} -8.2 & 0 & 0 \\ 0 & -0.82 & 0 \\ 0 & 0 & -0.082 \end{bmatrix}, \ D_1 = \begin{bmatrix} 5.74 & 2.05 & 0.41 \\ 0.574 & 0.205 & 0.041 \\ 0.0574 & 0.0205 & 0.0041 \end{bmatrix}$$

- *MAP-negative correlation (MAP-NC):*

$$D_0 = \begin{bmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0175 & 0 & 0 & 1.7325 \\ 3.465 & 0 & 0 & 0.035 \end{bmatrix}$$

- *MAP-positive correlation (MAP-PC):*

$$D_0 = \begin{bmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.7325 & 0 & 0 & 0.0175 \\ 0.035 & 0 & 0 & 3.465 \end{bmatrix}$$

Consider the following three PH distributions: IA, service process, and repair process. The arrival times, service times, and repair times with depictions λ_2, δ_1, δ_2 and Ψ are obtained by normalising these three representations. We will use the notations EXPX, ERLX, and HYP-EXPX for exponential, Erlang, and hyper-exponential cases dealing with X-type distributions, where $X = A$, S, or R depending on whether we are concerning about services or repairs. As a result, ERLS stands for Erlang-based services, while EXPR stands for exponential repairs and HYP-EXPA stands for hyper-exponential arrivals.

- *Erlang of order 3 (ERLX):*

$$\beta = \alpha_1 = \alpha_2 = \gamma = \big(1, 0, 0\big), \ S = T_1 = T_2 = U = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

- *Exponential (EXPX):*

$$\beta = \alpha_1 = \alpha_2 = \gamma = \big(1\big), \ S = T_1 = T_2 = U = \big[-1\big]$$

- *Hyper-exponential (HYP-EXPX):*

$$\beta = \alpha_1 = \alpha_2 = \gamma = \big(0.8, 0.2\big), \ S = T_1 = T_2 = U = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$$

8.1 Example illustrative 1

To probe the impact of the retrial rate of incoming arriving customers (σ) on the expected orbit size (E_{orbit}). We fix λ_1 = 3; λ_2 = 2; δ_1 = 11; δ_2 = 10; θ_1 = 0.9; θ_2 = 0.8; Ψ = 7; τ = 2; b_1 = 0.5; b_2 = 0.5; c_1 = 0.4; d_1 = 0.6.

Table 1 Retrial rate of incoming arriving customers versus E_{orbit} – Erlang service

σ	ERLS				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2.0	0.2991	0.8892	11.5019	1.0234	17.5591
2.5	0.2289	0.6526	6.3759	0.7139	12.7905
3.0	0.1861	0.5232	4.4994	0.5555	10.2648
3.5	0.1573	0.4419	3.5324	0.4597	8.6976
4.0	0.1367	0.3862	2.9467	0.3958	7.6293
4.5	0.1212	0.3457	2.5563	0.3502	6.8546
5.0	0.1093	0.3151	2.2794	0.3163	6.2678
5.5	0.0998	0.2912	2.0739	0.2901	5.8088
6.0	0.0920	0.2720	1.9164	0.2693	5.4411
6.5	0.0856	0.2564	1.7925	0.2525	5.1410
7.0	0.0803	0.2435	1.6932	0.2386	4.8926
7.5	0.0757	0.2326	1.6123	0.2271	4.6847
8.0	0.0718	0.2234	1.5455	0.2173	4.5091
8.5	0.0685	0.2155	1.4899	0.2090	4.3598
9.0	0.0655	0.2087	1.4432	0.2019	4.2324
9.5	0.0629	0.2028	1.4037	0.1957	4.1233
10.0	0.0607	0.1976	1.3702	0.1903	4.0297
10.5	0.0587	0.1931	1.3417	0.1856	3.9495

Table 2 Retrial rate of incoming arriving customers versus E_{orbit} – exponential service

σ	EXPS				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2.0	0.6351	1.3386	15.1626	1.3921	23.7953
2.5	0.4719	0.9438	7.7865	0.9447	16.3057
3.0	0.3787	0.7415	5.3388	0.7269	12.6092
3.5	0.3185	0.6186	4.1189	0.5983	10.3914
4.0	0.2765	0.5359	3.3897	0.5134	8.9038
4.5	0.2456	0.4766	2.9055	0.4534	7.8313
5.0	0.2218	0.4319	2.5613	0.4087	7.0178
5.5	0.2031	0.3970	2.3044	0.3741	6.3772
6.0	0.1879	0.3690	2.1057	0.3466	5.8580
6.5	0.1754	0.3461	1.9475	0.3242	5.4276
7.0	0.1649	0.3269	1.8187	0.3057	5.0642
7.5	0.1559	0.3107	1.7120	0.2900	4.7527
8.0	0.1482	0.2968	1.6221	0.2767	4.4824
8.5	0.1416	0.2847	1.5454	0.2651	4.2452
9.0	0.1357	0.2741	1.4792	0.2551	4.0353
9.5	0.1306	0.2648	1.4216	0.2462	3.8480
10.0	0.1260	0.2565	1.3709	0.2384	3.6798
10.5	0.1218	0.2491	1.3261	0.2314	3.5277

Table 3 Retrial rate of incoming arriving customers versus E_{orbit} – hyper-exponential service

σ	HYP-EXPS				
	ERLA	*EXPA*	*HYP-EXPA*	*MAP-NC*	*MAP-PC*
2.0	1.9331	2.8954	25.1219	2.8736	37.4707
2.5	1.2952	1.8119	10.2828	1.7481	21.4097
3.0	0.9920	1.3464	6.5083	1.2837	14.9823
3.5	0.8145	1.0866	4.7813	1.0296	11.4876
4.0	0.6978	0.9204	3.7908	0.8690	9.2738
4.5	0.6150	0.8046	3.1488	0.7581	7.7359
5.0	0.5532	0.7192	2.6992	0.6768	6.5997
5.5	0.5052	0.6536	2.3670	0.6146	5.7228
6.0	0.4668	0.6014	2.1117	0.5654	5.0236
6.5	0.4354	0.5589	1.9097	0.5255	4.4521
7.0	0.4093	0.5236	1.7458	0.4924	3.9760
7.5	0.3871	0.4938	1.6104	0.4646	3.5732
8.0	0.3681	0.4683	1.4966	0.4408	3.2283
8.5	0.3516	0.4462	1.3997	0.4202	2.9300
9.0	0.3372	0.4268	1.3162	0.4023	2.6700
9.5	0.3244	0.4097	1.2436	0.3864	2.4421
10.0	0.3130	0.3945	1.1799	0.3724	2.2411
10.5	0.3028	0.3809	1.1235	0.3598	2.0632

A quick observation from Tables 1, 2 and 3 as follows:

- While enhancing the retrial rate of IA of customers from the orbit reveals the E_{orbit} decreases correspondingly with the distinct groupings of arrival and service times.
- From the viewpoint of service times, the E_{orbit} decreases in the case of HYP-EXPS decreases highly as well as fastly and the ERLS decreases minimally as well as slowly. Now, consider the arrival times, the E_{orbit} decreases in the pairs of arrival such as (ERLA, ERLA), (ERLA, EXPA), (ERLA, HYP-EXPA) are slowly as well as minimally decreases and the pairs of arrival such as (MAP-PC, ERLA), (MAP-PC, EXPA), (MAP-PC, HYP-EXPA) are fastly as well as highly decreases.
- Whenever the server is idle whether the SS or the MS it is, at that moment if the retrial occurs immediately service starts for that retriving customer so that if the retrial rate increases which leads to the expected number of customers in the orbit decreases.

8.2 Example illustrative 2

To investigate the consequence of the OA rate (λ_2) on the expected orbit size (E_{orbit}) and total cost of the system TC. We fix λ_1 = 1; δ_1 = 8; δ_2 = 13; σ = 2; θ_1 = 0.8; θ_2 = 0.8; Ψ = 8; τ = 3; c_1 = 0.4; d_1 = 0.6; b_1 = 0.5; b_2 = 0.5; C_H = 18; C_{MMO} = 6; C_{MBI} = 5; C_{MBO} = 5; C_{SI} = 3; C_{SBI} = 4; C_{SBO} = 4; C_1 = 1; C_2 = 2; C_3 = 2; C_4 = 1; C_5 = 1; C_6 = 2; C_7 = 7.

Table 4 Outgoing arrival rate versus E_{orbit} and TC – ERLA

λ_2	*ERLA*					
	ERLS		*EXPS*		*HEXS*	
	E_{orbit}	TC	E_{orbit}	TC	E_{orbit}	TC
4.0	0.0993	86.1515	0.1139	86.4434	0.2041	88.1666
4.5	0.1119	86.4119	0.1275	86.7201	0.2225	88.5275
5.0	0.1247	86.6720	0.1412	86.9965	0.2409	88.8865
5.5	0.1375	86.9320	0.1550	87.2731	0.2594	89.2447
6.0	0.1505	87.1925	0.1690	87.5503	0.2780	89.6027
6.5	0.1636	87.4538	0.1830	87.8282	0.2967	89.9615
7.0	0.1768	87.7161	0.1972	88.1073	0.3155	90.3213
7.5	0.1902	87.9797	0.2116	88.3878	0.3345	90.6828
8.0	0.2037	88.2449	0.2261	88.6699	0.3536	91.0463
8.5	0.2174	88.5118	0.2407	88.9539	0.3729	91.4122
9.0	0.2312	88.7807	0.2555	89.2401	0.3925	91.7809
9.5	0.2452	89.0519	0.2705	89.5286	0.4122	92.1525
10.0	0.2594	89.3254	0.2857	89.8196	0.4322	92.5276
10.5	0.2738	89.6014	0.3011	90.1135	0.4524	92.9063
11.0	0.2884	89.8802	0.3167	90.4102	0.4729	93.2888
11.5	0.3031	90.1619	0.3325	90.7102	0.4936	93.6756
12.0	0.3181	90.4467	0.3486	91.0134	0.5146	94.0668
12.5	0.3333	90.7347	0.3648	91.3202	0.5359	94.4627

Table 5 Outgoing arrival rate versus E_{orbit} and TC – EXPA

λ_2	*EXPA*					
	ERLS		*EXPS*		*HEXS*	
	E_{orbit}	TC	E_{orbit}	TC	E_{orbit}	TC
4.0	0.1555	87.1614	0.1812	87.6507	0.2827	89.5594
4.5	0.1700	87.4555	0.1970	87.9665	0.3054	89.9988
5.0	0.1848	87.7515	0.2130	88.2837	0.3281	90.4352
5.5	0.1997	88.0498	0.2291	88.6025	0.3508	90.8699
6.0	0.2149	88.3504	0.2455	88.9234	0.3735	91.3038
6.5	0.2303	88.6535	0.2620	89.2466	0.3963	91.7379
7.0	0.2460	88.9595	0.2788	89.5724	0.4193	92.1729
7.5	0.2619	89.2684	0.2958	89.9010	0.4423	92.6093
8.0	0.2780	89.5804	0.3131	90.2327	0.4656	93.0479
8.5	0.2943	89.8958	0.3305	90.5678	0.4891	93.4890
9.0	0.3110	90.2148	0.3483	90.9065	0.5128	93.9332
9.5	0.3278	90.5374	0.3663	91.2489	0.5367	94.3810
10.0	0.3450	90.8638	0.3845	91.5953	0.5609	94.8327
10.5	0.3624	91.1943	0.4031	91.9460	0.5854	95.2887
11.0	0.3801	91.5290	0.4219	92.3010	0.6101	95.7495
11.5	0.3980	91.8681	0.4410	92.6606	0.6352	96.2153
12.0	0.4163	92.2116	0.4605	93.0251	0.6607	96.6865
12.5	0.4348	92.5599	0.4802	93.3945	0.6864	97.1634

Table 6 Outgoing arrival rate versus E_{orbit} and TC – HYP-EXPA

λ_2	*HYP-EXPA*					
	ERLS		*EXPS*		*HEXS*	
	E_{orbit}	*TC*	E_{orbit}	*TC*	E_{orbit}	*TC*
4.0	0.6833	96.6541	0.7724	98.2748	0.9747	101.9594
4.5	0.7244	97.4254	0.8196	99.1574	1.0471	103.2961
5.0	0.7672	98.2271	0.8683	100.0646	1.1204	104.6468
5.5	0.8118	99.0585	0.9185	100.9970	1.1947	106.0139
6.0	0.8581	99.9194	0.9702	101.9553	1.2701	107.3992
6.5	0.9061	100.8098	1.0235	102.9400	1.3468	108.8043
7.0	0.9558	101.7296	1.0784	103.9518	1.4247	110.2308
7.5	1.0073	102.6789	1.1348	104.9913	1.5039	111.6800
8.0	1.0604	103.6579	1.1929	106.0590	1.5845	113.1532
8.5	1.1153	104.6669	1.2527	107.1555	1.6667	114.6517
9.0	1.1719	105.7060	1.3141	108.2813	1.7503	116.1766
9.5	1.2303	106.7757	1.3773	109.4369	1.8355	117.7291
10.0	1.2904	107.8763	1.4422	110.6229	1.9224	119.3100
10.5	1.3524	109.0080	1.5088	111.8399	2.0109	120.9206
11.0	1.4161	110.1713	1.5773	113.0884	2.1012	122.5619
11.5	1.4816	111.3665	1.6475	114.3690	2.1933	124.2347
12.0	1.5489	112.5941	1.7197	115.6822	2.2872	125.9402
12.5	1.6182	113.8545	1.7937	117.0286	2.3831	127.6794

Table 7 Outgoing arrival rate versus E_{orbit} and TC – MAP-NC

λ_2	*MAP-NC*					
	ERLS		*EXPS*		*HEXS*	
	E_{orbit}	*TC*	E_{orbit}	*TC*	E_{orbit}	*TC*
4.0	0.1613	87.2642	0.1832	87.6826	0.2696	89.3194
4.5	0.1747	87.5381	0.1979	87.9794	0.2909	89.7332
5.0	0.1883	87.8136	0.2127	88.2770	0.3122	90.1441
5.5	0.2021	88.0909	0.2278	88.5758	0.3334	90.5530
6.0	0.2161	88.3703	0.2430	88.8760	0.3547	90.9610
6.5	0.2303	88.6518	0.2583	89.1780	0.3760	91.3688
7.0	0.2447	88.9356	0.2739	89.4819	0.3975	91.7770
7.5	0.2594	89.2220	0.2897	89.7882	0.4190	92.1863
8.0	0.2742	89.5110	0.3056	90.0970	0.4408	92.5973
8.5	0.2893	89.8029	0.3218	90.4085	0.4627	93.0104
9.0	0.3045	90.0977	0.3382	90.7230	0.4847	93.4260
9.5	0.3200	90.3958	0.3548	91.0407	0.5071	93.8447
10.0	0.3358	90.6971	0.3716	91.3617	0.5296	94.2667
10.5	0.3518	91.0020	0.3887	91.6864	0.5524	94.6926
11.0	0.3680	91.3105	0.4061	92.0149	0.5755	95.1225
11.5	0.3845	91.6228	0.4237	92.3473	0.5988	95.5570
12.0	0.4012	91.9390	0.4416	92.6839	0.6224	95.9963
12.5	0.4182	92.2593	0.4597	93.0249	0.6464	96.4407

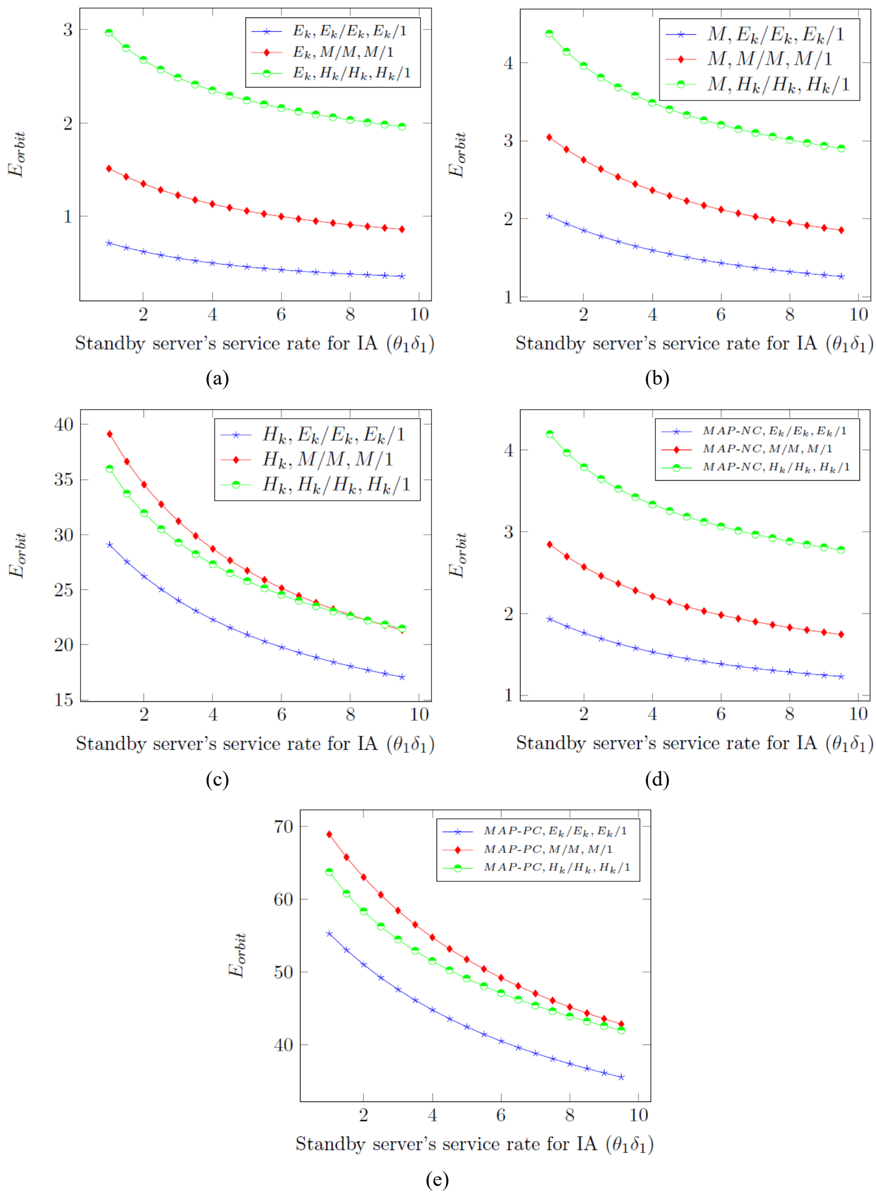
Table 8 Outgoing arrival rate versus E_{orbit} and TC – MAP-PC

λ_2	*MAP-PC*					
	ERLS		*EXPS*		*HEXS*	
	E_{orbit}	TC	E_{orbit}	TC	E_{orbit}	TC
4.0	0.8281	99.2670	1.0688	103.6229	1.6002	113.2623
4.5	0.9314	101.1589	1.1947	105.9216	1.8100	117.0681
5.0	1.0435	103.2080	1.3282	108.3546	2.0257	120.9806
5.5	1.1642	105.4088	1.4690	110.9163	2.2467	124.9855
6.0	1.2930	107.7554	1.6166	113.6011	2.4724	129.0727
6.5	1.4297	110.2411	1.7709	116.4038	2.7023	133.2352
7.0	1.5738	112.8597	1.9316	119.3197	2.9362	137.4688
7.5	1.7251	115.6050	2.0984	122.3448	3.1741	141.7708
8.0	1.8831	118.4717	2.2711	125.4757	3.4157	146.1398
8.5	2.0477	121.4548	2.4497	128.7095	3.6610	150.5757
9.0	2.2185	124.5501	2.6338	132.0439	3.9102	155.0787
9.5	2.3955	127.7540	2.8235	135.4770	4.1632	159.6499
10.0	2.5783	131.0636	3.0187	139.0074	4.4201	164.2906
10.5	2.7670	134.4765	3.2192	142.6343	4.6810	169.0024
11.0	2.9613	137.9905	3.4251	146.3568	4.9459	173.7873
11.5	3.1612	141.6043	3.6364	150.1746	5.2151	178.6473
12.0	3.3666	145.3167	3.8529	154.0876	5.4887	183.5846
12.5	3.5775	149.1270	4.0748	158.0960	5.7667	188.6016

A quick observation from Tables 4, 5, 6, 7 and 8 as follows:

- When increasing the OA rate then the E_{orbit} as well as total cost of the system increases simultaneously as well with the different arrival and service time groupings.
- Considering the arrival times, the E_{orbit} and TC increasing correspondingly while increasing the OA rate deliberates the arrival times pairs of (MAP-PC, ERLA), (MAP-PC, EXPA), (MAP-PC, HYP-EXPA) are increases fastly and the arrival times pairs of (ERLA, ERLA), (ERLA, EXPA) and (ERLA, HYP-EXPS) are increases moderately as compared to other arrival times. Then, look into service times of this example, the E_{orbit} and TC increases altogether highly for HYP-EXPS and slowly for ERLS.
- Whenever the MS becomes idle try to making the OAs which may become successful so in that kind of way if the arrival rate of outgoing increases makes the situation the IA finds the server busy will join the orbit leads to the orbit size increases obviously. Nevertheless, consider the service point of view which is offered by the server who starts service when the arrival of outgoing customer due to the availability of the MS impacts the TC increases as well.

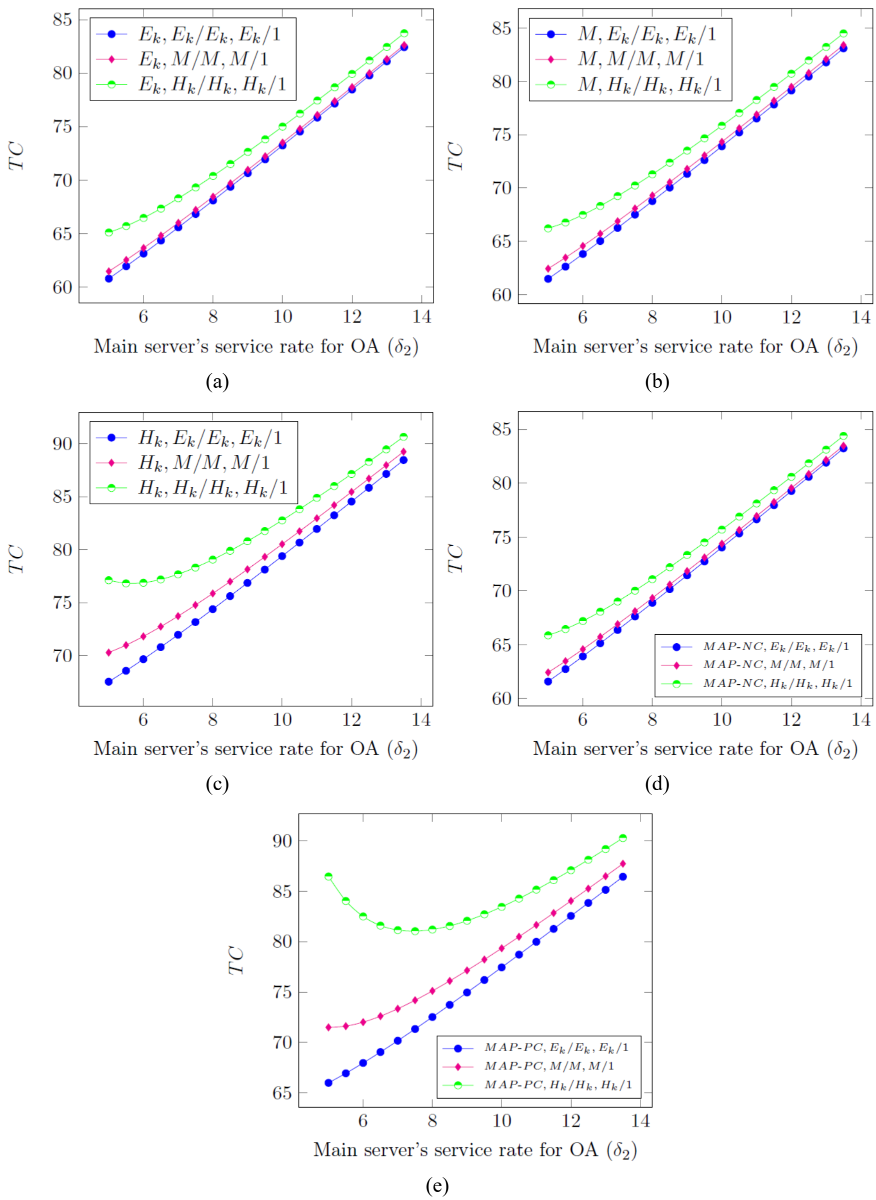
Figure 2 Standby server's service rate for IA vs. E_{orbit} (see online version for colours)



8.3 *Example illustrative 3*

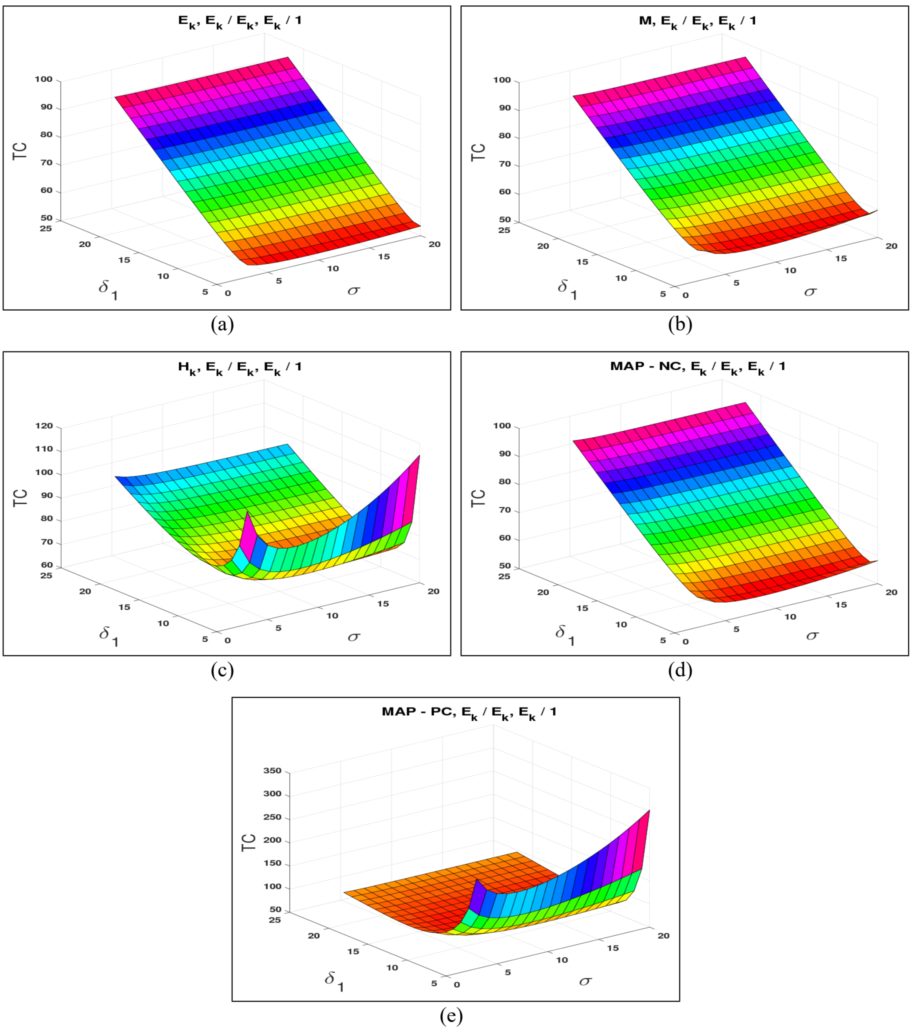
To examine the impact of the SS's service rate for incoming arriving customers ($\theta_1\delta_1$) on the expected orbit size (E_{orbit}). We fix λ_1 = 4; λ_2 = 1; δ_1 = 10; δ_1 = 8; σ = 4; θ_2 = 0.8; Ψ = 6; τ = 1; b_1 = 0.4; b_2 = 0.6; c_1 = 0.3; d_1 = 0.7.

Figure 3 Main server's service rate for OA vs. TC (see online version for colours)



The scrutiny from Figure 2 as follows: while maximising the service rate of the SS for IA of customers leads to the expected orbit size decreases accordingly with the different groupings of arrival and service times. If the IA of customers' arrival or retrial of IA customers from the orbit occurs while the SS being idle, the service will start immediately so that the E_{orbit} decreases. In aspects of arrival times, the arrival pairs of (MAP-PC, ERLA), (MAP-PC, EXPA), (MAP-PC, HYP-EXPA) decreases fastly and the arrival pairs of (ERLA, ERLA), (ERLA, EXPA), (ERLA, HYP-EXPA) decreases slowly. In the case of service times, the ERLS decreases slowly for all the pairs of arrival times, consider the EXPS decreases highly for the combination of HYP-EXPA, MAP-PC with three distinct PH arrival times, and the EXPS slowly decreases for the combination of ERLA, EXPA, MAP-NC with three distinct PH arrival times. Take a look into HYP-EXPS decreases highly for the combination of ERLA, EXPA, MAP-NC with three distinct PH arrival times and it decreases much slowly than the EXPS for the combination of HYP-EXPA, MAP-PC with three distinct PH arrival times.

Figure 4 Retrial rate of the IA customers and service rate of the MS for IA customers vs. TC – Erlang service (see online version for colours)

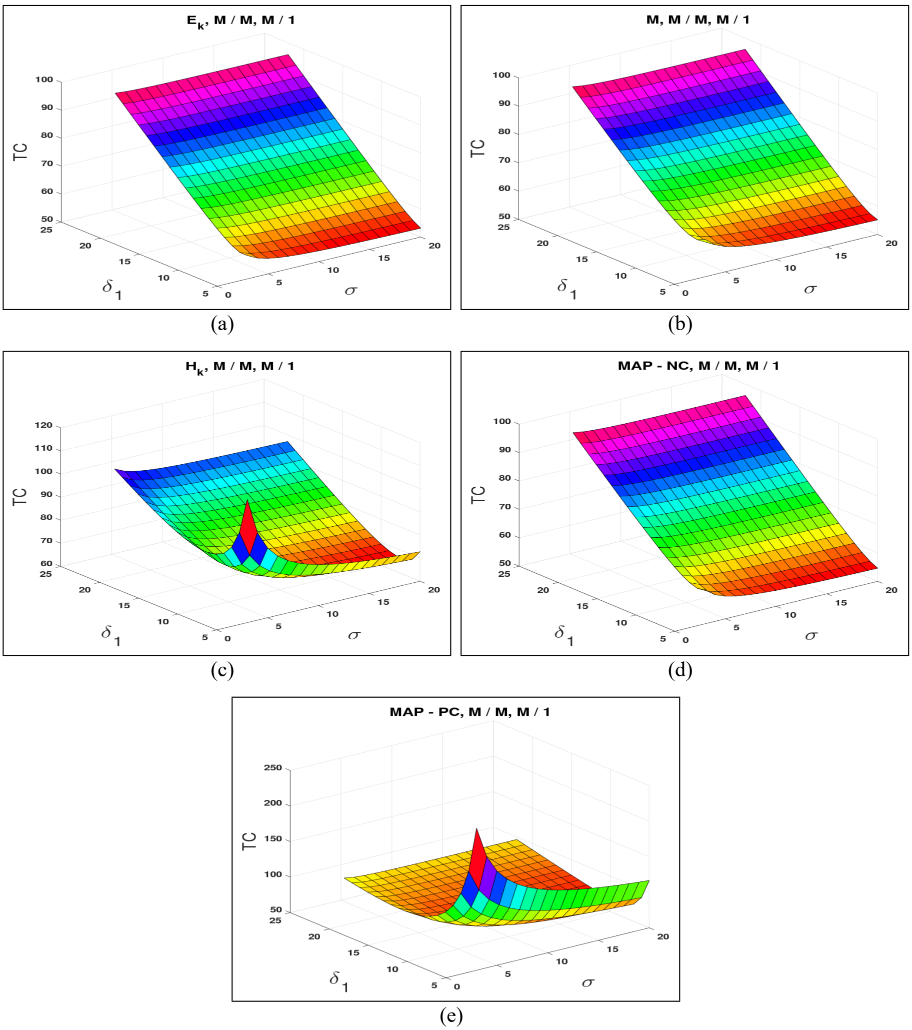


8.4 *Example illustrative 4*

To study the consequence of the MS's service rate for outgoing arriving customers (δ_2) on the total cost of the system (TC). We fix $\lambda_1 = 1$; $\lambda_2 = 2$; $\delta_1 = 7$; $\sigma = 3$; $\theta_1 = 0.7$;

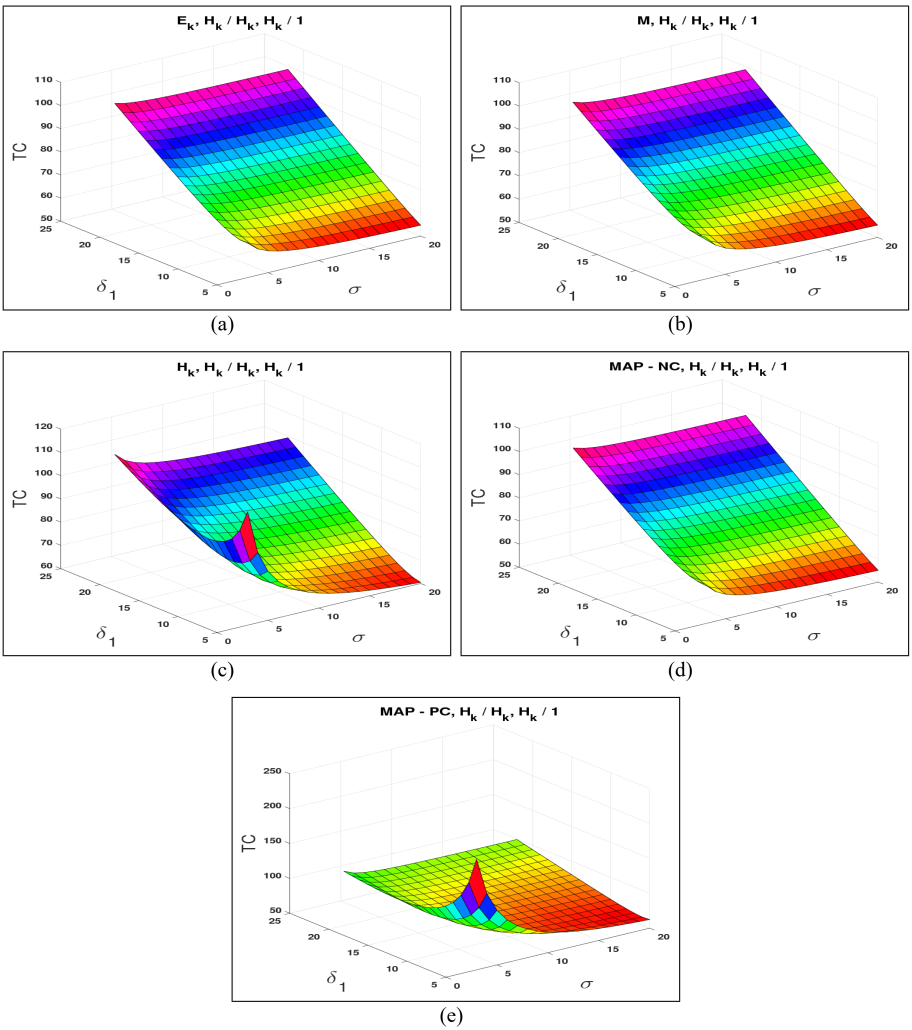
θ_2 = 0.7; Ψ = 9; τ = 2; b_1 = 0.5; b_2 = 0.5; c_1 = 0.2; d_1 = 0.8; C_H = 18; C_{MMO} = 6; C_{MBI} = 5; C_{MBO} = 5; C_{SI} = 3; C_{SBI} = 4; C_{SBO} = 4; C_1 = 1; C_2 = 2; C_3 = 2; C_4 = 1; C_5 = 1; C_6 = 2; C_7 = 7.

Figure 5 Retrial rate of the IA customers and service rate of the MS for IA customers vs. TC – exponential service (see online version for colours)



The scrutiny from Figures 3 as follows: when increasing the MS's service rate for OA of customers reveals that the TC increases as well with the distinct arrangements of service and arrival times. When the service completion epoch or the repair completion epoch may become idle so at that moment the MS try to making OAs, if the OA occurs successfully while increasing the MS's service rate which leads to the TC increases simultaneously due to the service offering in the service station. Concern the service times of the ERLS and HYP-EXPS, the arrival pairs of HYP-EXPA with three different PH arrival times increases fastly than the all others. However, consider the service times of EXPS deliberates the arrival pairs of MAP-PC with three distinct PH arrival times increases fastly than the other combinations of arrival and service times.

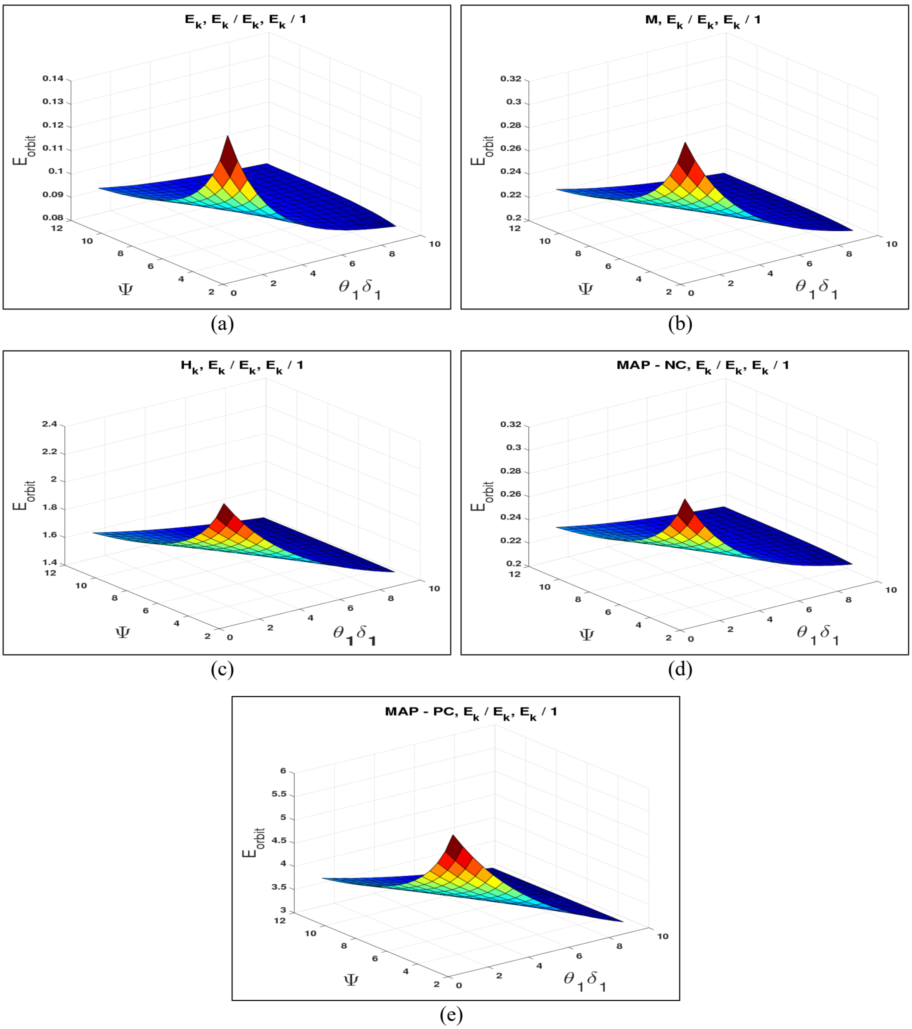
Figure 6 Retrial rate of the IA customers and service rate of the MS for IA customers vs. TC – hyper-exponential service (see online version for colours)



8.5 *Example illustrative 5*

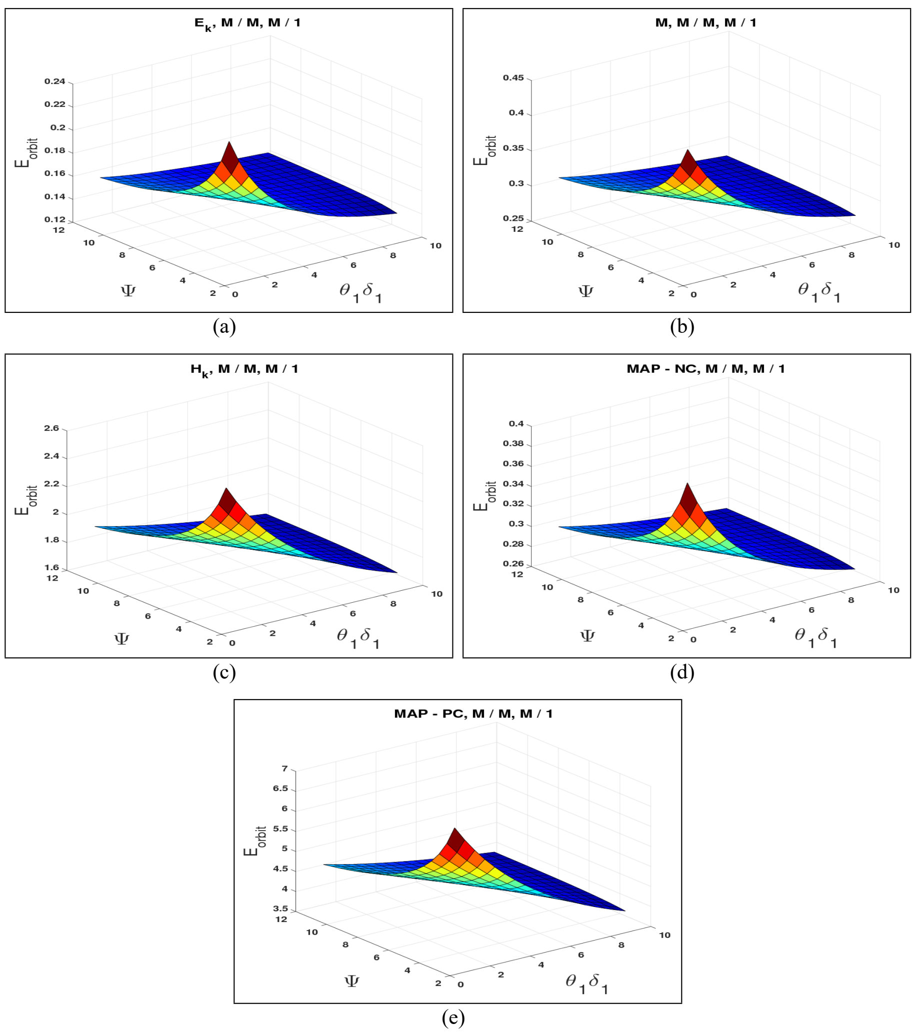
To observe the consequence of both the retrial rate of the IA of customers (σ) and service rate of the MS for IA customers (δ_1) on the total cost of the system (TC). We prefer λ_1 = 2; λ_2 = 1; δ_2 = 5; θ_1 = 0.7; θ_2 = 0.6; Ψ = 7; τ = 1; b_1 = 0.6; b_2 = 0.4; c_1 = 0.4; d_1 = 0.6; C_H = 18; C_{MMO} = 6; C_{MBI} = 5; C_{MBO} = 5; C_{SI} = 3; C_{SBI} = 4; C_{SBO} = 4; C_1 = 1; C_2 = 2; C_3 = 2; C_4 = 1; C_5 = 1; C_6 = 2; C_7 = 7.

Figure 7 Repair rate of the MS and service rate of the SS for IA customers vs. E_{orbit} – Erlang service (see online version for colours)



A quick scrutiny from Figures 4, 5 and 6 as follows: while enhancing both the retrial rate of IA customers from the orbit and the MS's service rate for IA customers, the TC also considerably increases for the different combinations of arrival and service times and also some of the case of arrival and service the TC decreases and then as well as increases. Let take a look at those cases: consider the arrival times, the arrival pairs of MAP-PC with three different PH arrival times highly decreases as well as increases and the pairs of arrival combinations of ERLA with three distinct PH arrival times increases minimally compared to other arrival times. However, now consider the service times as follows: for ERLA with distinct three PH arrivals, the HYP-EXPS increases fastly and the ERLS increases slowly. For EXPA with different three PH arrivals, the ERLS increases slowly as compared to other service times. For the combination of HYP-EXPA with three distinct PH arrivals, the ERLS highly decreases first and then increases slowly, the EXPS decreases moderately as well as increases gradually and in the case of HYP-EXPS decreases slowly as well as increases highly compared to other service times. For the MAP-NC with three different PH arrival times, the HYP-EXPS increases highly as compared to all other service times. For MAP-PC with distinct combination of three types of arrival times, the ERLS highly decreases as well as increases, the EXPS decreases as well as increases moderately and in the case of HYP-EXPS decreases as well as increases slowly as compared to other service times. While enhancing retrial rate of orbital customers as well as service rate of MS deliberates the availability of the MS offers service immediately if the server idle or making OA, hence the TC enhances with all kind of service and arrival groupings.

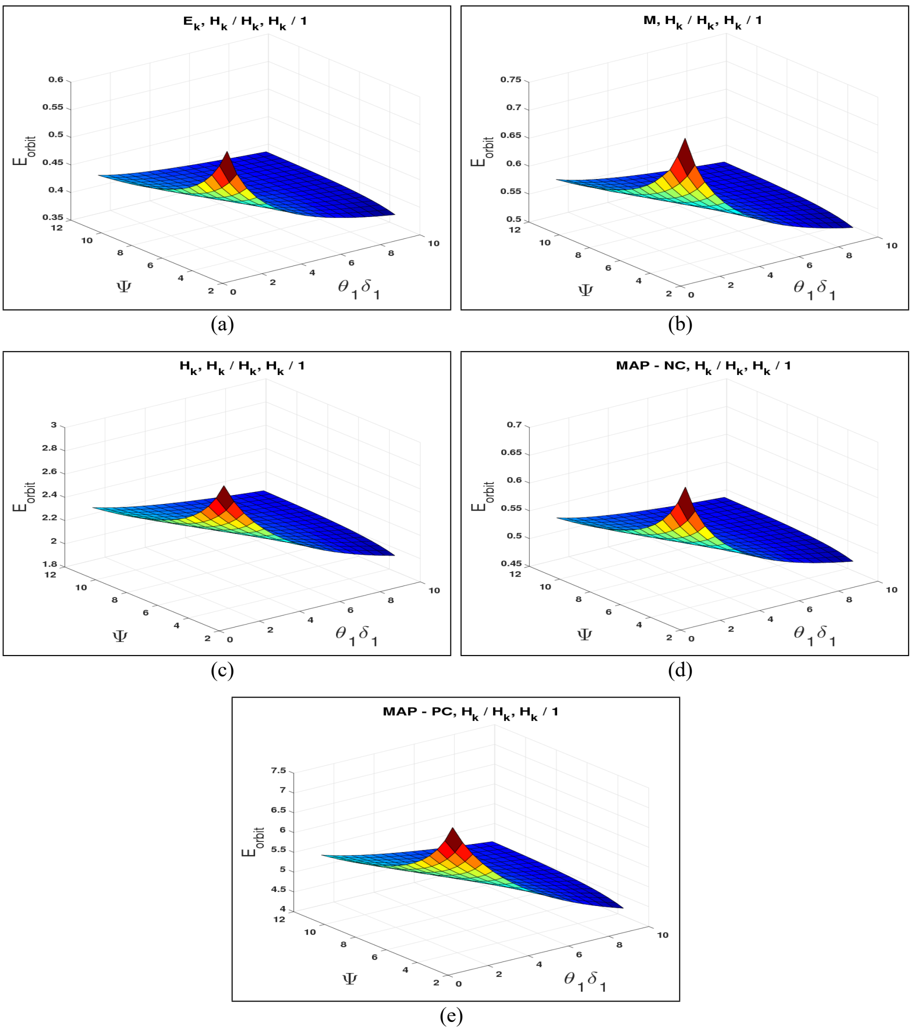
Figure 8 Repair rate of the MS and service rate of the SS for IA customers vs. E_{orbit} – exponential service (see online version for colours)



8.6 Example illustrative 6

To examine the impact of the SS's service rate for IA of customers ($\theta_1\delta_1$) and MS's repair rate (Ψ) on the expected orbit size (E_{orbit}). We fix λ_1 = 2; λ_2 = 2; δ_1 = 10; δ_2 = 10; σ = 4; θ_2 = 0.7; τ = 1; b_1 = 0.4; b_2 = 0.6; c_1 = 0.3; d_1 = 0.7.

Figure 9 Repair rate of the MS and service rate of the SS for IA customers vs. E_{orbit} – hyper-exponential service (see online version for colours)



A quick scrutiny from Figures 7, 8 and 9 as follows: when maximising both the SS's service rate for incoming arriving customers and repair rate of the MS, the expected number of incoming arriving customers in the orbit decreases accordingly with different service and arrival time groupings. For considering the service times, the HYP-EXPS decreases highly and the ERLS decreases gradually as compared to the EXPS and

HYP-EXPS. Take a look into arrival times of combining pairs of correlation having as well as zero correlation arrivals with three distinct PH arrival times, the arrival pairs (MAP-PC, ERLA), (MAP-PC, EXPA), (MAP-PC, HYP-EXPA) are decreases highly and the arrival pairs (ERLA, ERLA), (ERLA, EXPA), (ERLA, HYP-EXPA) are decreases slowly. Enhancing the MS's repair rate as well as SS's service rate simultaneously deliberates during the repair period of the MS, the SS rendering service to the customers and also when the repair completion of the MS will obviously interrupt the SS and take over the service process so both the cases the availability of the server represent through the three dimensional graphs that the E_{orbit} decreases correspondingly.

Figure 10 E_{orbit} vs. service rate of MS and SS for IA of $ERLA$ (see online version for colours)

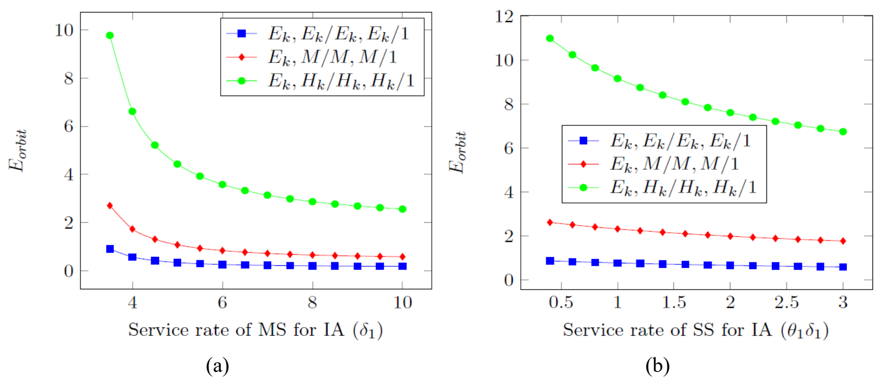
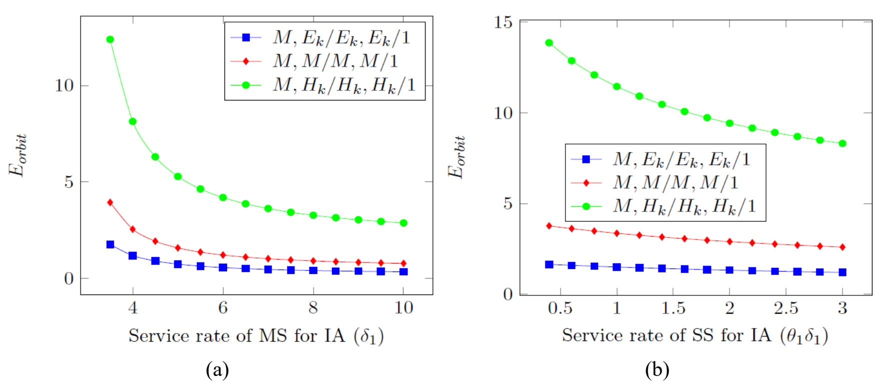


Figure 11 E_{orbit} vs. service rate of MS and SS for IA of $EXPA$ (see online version for colours)

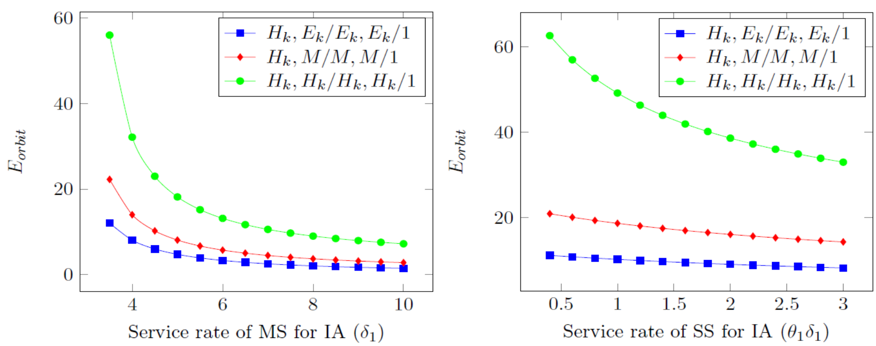


8.7 Illustrating the service scenarios of the MS and the SS

For MS's service, we examine the consequence of the MS's service rate for IA of customers (δ_1) versus E_{orbit}. We fix $\lambda_1 = 2$; $\lambda_2 = 1$; $\delta_2 = 2$; $\theta_1 = 0.8$; $\theta_2 = 0.7$; $\Psi = 6$; $\sigma = 3$; $\tau = 1$; $b_1 = 0.5$; $b_2 = 0.5$; $c_1 = 0.2$; $d_1 = 0.8$. For SS's service, we examine the impact of the SS's service rate for IA of customers ($\theta_1\delta_1$) versus E_{orbit}. We fix

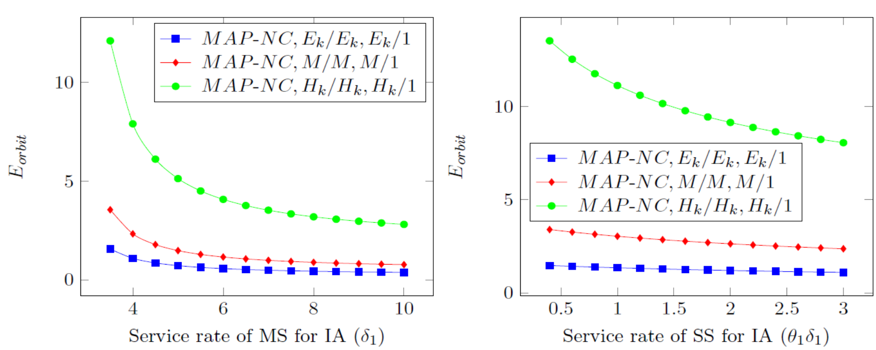
$\lambda_1 = 2$; $\lambda_2 = 1$; $\delta_1 = 4$; $\delta_2 = 2$; $\theta_2 = 0.7$; $\Psi = 6$; $\sigma = 3$; $\tau = 1$; $b_1 = 0.5$; $b_2 = 0.5$; $c_1 = 0.2$; $d_1 = 0.8$. We have observed from Figures 10, 11, 12, 13 and 14 which reveals that when the MS rendering service E_{orbit} decreases rapidly as compared to the service rendering speed of SS. While the SS offering service the E_{orbit} decreases slowly than the service speed of MS. Nonetheless, take a look into the arrival times whether the SS or the MS rendering service, the arrival pairs of (MAP-PC, ERLA), (MAP-PC, EXPA), (MAP-PC, HYP-EXPA) are decreases rapidly and the combining pairs of arrivals (ERLA, ERLA), (ERLA, EXPA), (ERLA, HYP-EXPA) are decreases slowly. Similarly, concern the service times, the ERLS slowly decreases than the EXPS and HYP-EXPS.

Figure 12 E_{orbit} vs. service rate of MS and SS for IA of $HYP\text{-}EXPA$ (see online version for colours)



(a) (b)

Figure 13 E_{orbit} vs. service rate of MS and SS for IA of $MAP\text{-}NC$ (see online version for colours)

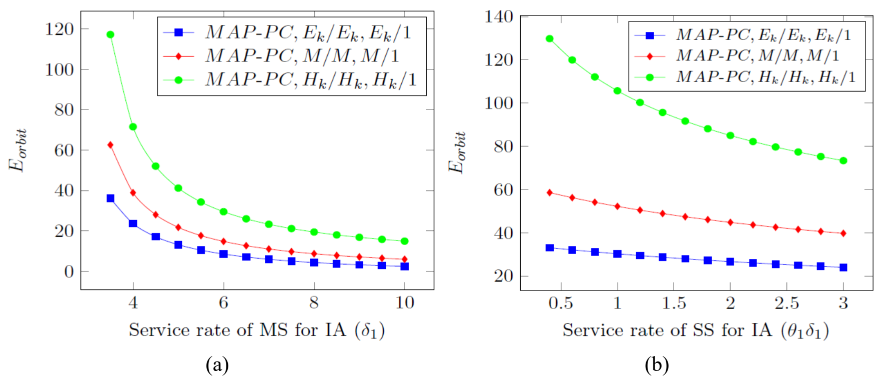


(a) (b)

9 Conclusions

In this paper, we considered the IA follows the MAP is a suitable way of approaching the correlated and non-correlated arrival concepts. The OAs, service for both incoming and OAs as well as repair times for MS are all follows PH distribution with the concepts of two way communication and collision of orbital customers. We also execute a system's busy period and cost analysis for our proposed model. We obtained tabulated numerical values and examined the pictorial representations of two dimensional as well as three dimensional graphs utilising the numerical inputs of arrival and service times. Hence, it precisely communicated the impact of various parameters related to our model on system performance measures, and cost analysis. The overall motivation for our model is that we studied the situations faced by the consumers in the online shopping from any one of the web portals. We are currently considering the MAP and PH for arrivals; however, in the future, we would like to expand our model to incorporate the BMAP for arrivals, which is best suited for modelling batch arrivals using the simulation technique.

Figure 14 E_{orbit} vs. service rate of MS and SS for IA of $MAP\text{-}PC$ (see online version for colours)



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