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A new heavy-tailed exponentiated generalised-G family of distributions: properties and applications

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Abstract: In this paper, we introduce a new family of heavy-tailed distributions called the type-I heavy-tailed exponentiated generalised-G (TIHTEG-G) family of distributions. A special model of the proposed family, namely the type-I heavy-tailed exponentiated generalised-log-logistic (TIHTEG-LLoG) model is studied in detail. Statistical properties of the new family of distributions are presented. These include, among others, the hazard rate function, quantile function, moments, distribution of order statistics and Rényi entropy. The maximum likelihood method of estimation is conducted to examine the performance of the model. Actuarial measures are also derived and simulation study for these measures is done to show that the proposed TIHTEG-LLoG model is a heavy-tailed model. Real datasets are analysed to illustrate the usefulness of the proposed model.

Keywords: heavy-tailed; exponentiated generalised-G; family of distributions; properties; applications; simulation; actuarial measures.

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Biographical notes: Gomolemo Jacqueline Lekono is currently pursuing her PhD in Statistics at the Botswana International University of Science and Technology. Her main focus is in distribution theory with applications to lifetime data. She completed her Master's degree in 2020 at the University of Botswana. Her Master's degree research was based on modelling the HIV/AIDS transmission knowledge score of Botswana using negative binomial distribution, applied on the Botswana AIDS Impact Survey IV dataset. During her Master's degree time, she was a teaching assistant at the University of Botswana in the Statistics Department, through this she learnt professionalism and team work. She also worked as a research assistant under UBUPENN cancer projects.

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1 Introduction

The use of traditional probability models to estimate real-life occurrences leads to lack of satisfaction among applied practitioners. Most of these models do not fit and predict data in several applied areas, therefore, generalised families of distributions are considered for the improvements and extensions of classical distributions. Many generalised families of distributions have been developed and applied to describe various phenomena. This development is made by addition of one or more parameters to the baseline distribution. Examples of recently generalised distributions include the exponentiated half logistic exponential (EHLE) distribution by Abdullah et al. (2018), the Odd exponentiated half-logistic-G (OEHL-G) distribution by Afify et al. (2017), the Weibull-G family of distribution by Bourguignon et al. (2014), the Gompertz-G (Gom-G) family of distributions by Alizadeh et al. (2017), the Marshall-Olkin inverse Lomax (MO-ILD) distribution by Maxwell et al. (2019), the new power generalised Weibull-G family of distributions by Oluyede et al. (2020), the Topp-Leone odd log-logistic family of distributions by Brito et al. (2017), the Marshall-Olkin extended inverse Weibull distribution by Pakungwati et al. (2018), generalised beta-generated distributions by Alexander et al. (2012), the exponentiated half logistic odd Lindley-G (EHLOL-G) family of distributions by Sengweni et al. (2021) and the Zografos-Balakrishnan-G family of distributions by Nadarajah et al. (2015) just to name a few.

The use of heavy-tailed distributions to model datasets in different areas of research like risk management, financial sciences, reliability, engineering, economic and actuarial science is of tremendous practical importance, since these data are in some cases positive, uni-modal shaped, right-skewed and heavier tailed. Hence, there is need for the development of heavy-tailed distributions. Examples of recently developed heavy-tailed distributions include the Weibull-Lomax distribution by Tahir et al. (2015), heavy-tailed log-logistic distribution by Teamah et al. (2021), a new family of heavy-tailed distributions by Ahmad et al. (2022), heavy-tailed beta-power transformed-Weibull distribution by Zhao et al. (2021), the heavy-tailed exponential distribution by Afify et al. (2020) and the type-I heavy tailed Weibull distribution by Zhao et al. (2020).

The cumulative distribution function (cdf) and probability density function (pdf) of the type-I heavy-tailed distribution introduced by Zhao et al. (2020) are given by

$$F_{HT-G}(x;\theta,\psi) = 1 - \left(\frac{1 - G(x;\psi)}{1 - (1 - \theta)G(x;\psi)}\right)^{\theta},$$
(1)

and

$$f_{HT-G}(x;\theta,\psi) = \frac{\theta^2 g(x;\psi) \left(1 - G(x;\psi)\right)^{\theta-1}}{\left(1 - (1-\theta)G(x;\psi)\right)^{\theta+1}},$$
(2)

respectively, for $\theta > 0$, $x \in \mathbb{R}$ and parameter vector ψ , where $G(x; \psi)$ is the baseline cdf. Cordeiro et al. (2013) introduced the exponentiated generalised-G class of distributions with the cdf and pdf given by

$$F_{EG-G}(x;\alpha,\beta,\psi) = \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta},\tag{3}$$

and

$$f_{EG-G}(x;\alpha,\beta,\psi) = \alpha\beta g(x;\psi)\bar{G}^{\alpha-1}(x;\psi)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta-1},\tag{4}$$

respectively for $\alpha, \beta > 0$, $x \in \mathbb{R}$ and parameter vector ψ , where $\overline{G}(x; \psi) = 1 - G(x; \psi)$.

In this paper, we propose a new family of distributions which has more flexibility when fitted to real life data. Its pdf can be right-skewed, left-skewed, almost symmetrical as well as bathtub shapes. The hazard rate function has shapes that include decreasing, increasing, uni-modal, bathtub, and upside down bathtub, which are applicable in real life situations. Another motivation for this paper is to construct heavy-tailed distributions for modelling different types of real data.

The rest of the paper is organised as follows. In Section 2, we introduce the new family of distributions and its sub-families. In Section 3, mathematical and statistical properties of the new family of distributions are explored including expansion of the pdf, quantile function, moments, generating function and Rényi entropy. The estimation of the parameters are obtained using maximum likelihood estimation method in Section 4. Some special cases of the new family of distributions are given in Section 5. Also, in this section, we plot the density function, hazard rate function and present 3D plots of skewness and kurtosis. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates are presented in Section 7 contains actuarial measures. Applications of a member the new family of distributions to real datasets are given in Section 8 and conclusions are given in Section 9.

2 The new family of distributions

We propose a new family of distributions referred to as the type-I heavy tailed exponentiated generalised-G (TIHTEG-G) family of distributions by using the generalisations given in equations (1), (2), (3) and (4). Combining equations (1) and (2) with equations (3) and (4), we get the cdf, pdf and hazard rate function (hrf) of the proposed family of distributions given by

$$F(x; a, \alpha, \beta, \psi) = 1 - \left[\frac{1 - (1 - \bar{G}^{\alpha}(x; \psi))^{\beta}}{1 - (1 - a)(1 - \bar{G}^{\alpha}(x; \psi))^{\beta}} \right]^{a},$$
(5)
$$f(x; a, \alpha, \beta, \psi) = \frac{a^{2} \alpha \beta g(x; \psi) \bar{G}^{\alpha - 1}(x; \psi) (1 - \bar{G}^{\alpha}(x; \psi))^{\beta - 1}}{(1 - (1 - a)(1 - \bar{G}^{\alpha}(x; \psi))^{\beta})^{2}}$$
(6)
$$\times \left[\frac{1 - (1 - \bar{G}^{\alpha}(x; \psi))^{\beta}}{1 - (1 - a)(1 - \bar{G}^{\alpha}(x; \psi))^{\beta}} \right]^{a - 1}$$
(6)

and

$$h(x; a, \alpha, \beta, \psi) = \frac{a^2 \alpha \beta g(x; \psi) \bar{G}^{\alpha - 1}(x; \psi) \left(1 - \bar{G}^{\alpha}(x; \psi)\right)^{\beta - 1}}{\left(1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x; \psi)\right)^{\beta}\right)^2} \times \left[\frac{1 - (1 - \bar{G}^{\alpha}(x; \psi))^{\beta}}{1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x; \psi)\right)^{\beta}}\right]^{-1},$$
(7)

respectively, for x > 0, $a, \alpha, \beta > 0$ and parameter vector ψ .

2.1 Sub-families

Sub-families of the new family of distributions are presented in this subsection.

• When a = 1, we obtain the exponentiated generalised-G (EG-G) family of distributions by Cordeiro et al. (2013) with cdf given by

$$F(x; \alpha, \beta, \psi) = \left(1 - \overline{G}^{\alpha}(x; \psi)\right)^{\beta},$$

for x > 0, $\alpha, \beta > 0$, and parameter vector ψ .

• When $\alpha = 1$, we obtain a new family of type-I heavy-tailed distributions with cdf given by

$$F(x; a, \beta, \psi) = 1 - \left[\frac{1 - G^{\beta}(x; \psi)}{1 - (1 - a)G^{\beta}(x; \psi)}\right]^{a},$$

for x > 0, $a, \beta > 0$, and parameter vector ψ . This is a new family of distributions.

• When $\beta = 1$, we obtain Type I heavy-tailed generalised-G (TIHTG-G) family of distributions with cdf given by

$$F(x;a,\alpha,\psi) = 1 - \left[\frac{\bar{G}^{\alpha}(x;\psi)}{1 - (1-a)\left(1 - \bar{G}^{\alpha}(x;\psi)\right)}\right]^{a},$$

for x > 0, $a, \alpha > 0$ and parameter vector ψ . This is a new family of distributions.

• When $a = \alpha = 1$, we obtain the exponentiated-G family of distributions with cdf given by

$$F(x;\beta,\psi) = G^{\beta}(x;\psi)$$

for x > 0, $\beta > 0$ and parameter vector ψ .

• When $a = \beta = 1$, we obtain a family of distributions with cdf given by

 $F(x; \alpha, \psi) = 1 - (1 - G(x; \psi))^{\alpha}$

for x > 0, $\alpha > 0$ and parameter vector ψ .

• When $\alpha = \beta = 1$, we obtain the TIHT-G family of distributions with cdf given by

$$F(x;a,\psi) = 1 - \left[\frac{\bar{G}(x;\psi)}{1 - (1 - a)G(x;\psi)}\right]^a$$

for x > 0, a > 0 and parameter vector ψ (Zhao et al., 2020).

• When $a = \alpha = \beta = 1$, we obtain the baseline cdf

 $F(x;\psi) = G(x;\psi)$

for x > 0, and parameter vector ψ .

3 Some statistical properties

Statistical properties of the TIHTEG-G family of distributions are explored in this section. The statistical properties considered include expansion of the density function as well as the quantile function, moments, generating function, probability weighted moments, Rényi entropy, and distribution of the order statistics.

3.1 Linear representation of the density function

In this subsection, we demonstrate that the TIHTEG-G density function can be expressed as an infinite linear combination of exponentiated-G (exp-G) densities. Consider the

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generalised series expansion given by $(1-x)^n = \sum_{i=0}^{\infty} (-1)^i {n \choose i} x^i$, |x| < 1. Now, applying the following generalised binomial series expansions:

$$\begin{split} \left[1 - (1 - a)(1 - \bar{G}^{\alpha}(x;\psi))^{\beta}\right]^{-(a+1)} &= \sum_{i=0}^{\infty} \binom{-(a+1)}{i} (-1)^{i} \\ &\times (1 - a)^{i} (1 - \bar{G}^{\alpha}(x;\psi))^{\beta i}, \\ \left[1 - (1 - \bar{G}^{\alpha}(x;\psi))^{\beta}\right]^{a-1} &= \sum_{j=0}^{\infty} (-1)^{j} \binom{a-1}{j} (1 - \bar{G}^{\alpha}(x;\psi))^{\beta j}, \\ (1 - \bar{G}^{\alpha}(x;\psi))^{\beta j + \beta i + \beta - 1} &= \sum_{k=0}^{\infty} (-1)^{k} \binom{\beta j + \beta i + \beta - 1}{k} \bar{G}^{\alpha k}(x;\psi) \end{split}$$

and

$$[1 - G(x; \psi)]^{\alpha k + \alpha - 1} = \sum_{l=0}^{\infty} (-1)^l {\binom{\alpha k + \alpha - 1}{l}} G^l(x; \psi),$$

we can write the pdf of the TIHTEG-G family of distribution as

$$f(x; a, \alpha, \beta, \psi) = a^2 \alpha \beta \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j+k+l} \binom{-(a+1)}{i} \binom{a-1}{j}$$

$$\times \binom{\beta j + \beta i + \beta - 1}{k} \binom{\alpha k + \alpha - 1}{l} (1-a)^i g(x; \psi) G^l(x; \psi)$$

$$= a^2 \alpha \beta \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l}}{l+1} \binom{-(a+1)}{i} \binom{a-1}{j} \tag{8}$$

$$\times \binom{\beta j + \beta i + \beta - 1}{k} \binom{\alpha k + \alpha - 1}{l}$$

$$\times (1-a)^i (l+1) g(x; \psi) G^l(x; \psi)$$

$$= \sum_{l=0}^{\infty} t_{l+1} h_{l+1}(x; \psi),$$

where $h_{l+1}(x;\psi) = (l+1) g(x;\psi)G^l(x;\psi)$ is the exp-G density with power parameter (l+1) and parameter vector ψ , and

$$t_{l+1} = a^2 \alpha \beta \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k+l}}{l+1} \binom{-(a+1)}{i} \binom{a-1}{j} \binom{\beta j + \beta i + \beta - 1}{k} \times \binom{\alpha k + \alpha - 1}{l} (1-a)^i.$$
(9)

The new density can be expressed as an infinite linear combination of exponentiated-g (exp-G) densities. The mathematical and statistical properties of the new family of distributions follow directly from those of exp-G family of distributions.

3.2 Quantile function

The quantile function of the TIHTEG-G family of distributions is obtained by inverting the nonlinear equation

$$F_{TIHTEG-G}(x; a, \alpha, \beta, \psi) = 1 - \left[\frac{1 - (1 - \bar{G}^{\alpha}(x; \psi))^{\beta}}{1 - (1 - a)(1 - \bar{G}^{\alpha}(x; \psi))^{\beta}}\right]^{a} = u$$

for $0 \le u \le 1$. Note that

$$\frac{1 - \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}}{1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}} = (1 - u)^{\frac{1}{a}},$$

which simplifies to

$$G(x;\psi) = 1 - \left[1 - \left[\frac{1 - (1-u)^{\frac{1}{a}}}{1 - (1-u)^{\frac{1}{a}}(1-a)}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}.$$

Consequently, the quantile function of the TIHTEG-G family of distributions is given by

$$Q_G(u; a, \alpha, \beta, \psi) = G^{-1} \left(1 - \left[1 - \left[\frac{1 - (1 - u)^{\frac{1}{a}}}{1 - (1 - u)^{\frac{1}{a}} (1 - a)} \right]^{\frac{1}{\beta}} \right]^{\frac{1}{\alpha}} \right).$$
(10)

Quantiles are obtained using equation (10) via a specified baseline cdf G using R software.

3.3 Moments and probability weighted moments

In this section, we present the moment, moment generating functions and probability weighted moments (PWMs) of the TIHTEG-G family of distributions. Using equation (8), we can obtain the r^{th} moment of the TIHTEG-G family of distributions as follows

$$E(X^{r}) = \int_{-\infty}^{\infty} x^{r} f(x; a, \alpha, \beta, \psi) dx = \sum_{l=0}^{\infty} t_{l+1} E(Y_{l+1}^{r}),$$
(11)

where $E(Y_{l+1}^r)$ is the r^{th} moment of Y_{l+1} which follows exp-G distribution with power parameter l+1 and t_{l+1} is defined as equation (9). The moment generating function is given by

$$M_X(t) = E(e^{tX}) = \sum_{l=0}^{\infty} t_{l+1} E(e^{tY_{l+1}}),$$

where $E(e^{tY_{l+1}})$ is the moment generating function of the exp-G distribution with power parameter l + 1 and t_{l+1} is given by equation (9). The PWMs of a random variable X are defined by

$$\omega_{a,r} = E\left(X^a[F(X)]^r\right) = \int_{-\infty}^{\infty} x^a[F(x)]^r f(x) dx.$$

From equations (5) and (6), we can write

$$f(x)[F(x)]^{r} = \frac{a^{2}\alpha\beta g(x;\psi)\bar{G}^{\alpha-1}(x;\psi)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta-1}}{\left(1-\left(1-a\right)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right)^{2}} \\ \times \left[\frac{1-\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}{1-\left(1-a\right)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}\right]^{a-1} \\ \times \left[1-\left(\frac{1-\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}{1-\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}\right)^{a}\right]^{r}.$$

Using the generalised binomial series expansion,

$$\left(1 - \left[\frac{1 - (1 - \bar{G}^{\alpha}(x;\psi))^{\beta}}{1 - (1 - a)(1 - \bar{G}^{\alpha}(x;\psi))^{\beta}}\right]^{a}\right)^{r}$$

= $\sum_{i=0}^{\infty} (-1)^{i} {r \choose i} \left[\frac{1 - (1 - \bar{G}^{\alpha}(x;\psi))^{\beta}}{1 - (1 - a)(1 - \bar{G}^{\alpha}(x;\psi))^{\beta}}\right]^{ai},$

we can write

$$f(x)[F(x)]^{r} = a^{2} \alpha \beta g(x;\psi) \bar{G}^{\alpha-1}(x;\psi) \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta-1} \sum_{i=0}^{\infty} (-1)^{i} \binom{r}{i}$$
$$\times \frac{\left[1 - \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right]^{ai+a-1}}{\left[1 - \left(1 - a\right) \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right]^{ai+a-1}}.$$

Considering the following generalised binomial series expansions:

$$\begin{split} \left[1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right]^{-(ai+a-1)} &= \sum_{j=0}^{\infty} \binom{-(ai+a-1)}{j} \\ &\times (-1)^{j} \left(1 - a\right)^{j} \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta j}, \\ \left[1 - \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right]^{ai+a-1} &= \sum_{k=0}^{\infty} (-1)^{k} \binom{ai+a-1}{k} \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta k}, \\ \left(1 - \bar{G}^{\alpha}(x;\psi)\right)^{\beta k+\beta j+\beta-1} &= \sum_{l=0}^{\infty} (-1)^{l} \binom{\beta k+\beta j+\beta-1}{l} \bar{G}^{\alpha l}(x;\psi) \end{split}$$

and

$$[1 - G(x; \psi)]^{\alpha l + \alpha - 1} = \sum_{m=0}^{\infty} (-1)^m {\alpha l + \alpha - 1 \choose m} G^m(x; \psi),$$

we have

$$f(x)[F(x)]^{r} = a^{2}\alpha\beta \sum_{j,j,k,l,m=0}^{\infty} (-1)^{i+j+k+l+m} {r \choose i} {-(ai+a-1) \choose j}$$
$$\times {ai+a-1 \choose k} \times {\beta k+\beta j+\beta-1 \choose l} {\alpha l+\alpha-1 \choose m}$$
$$\times (1-a)^{j} g(x;\psi) G^{m}(x;\psi)$$
$$= \sum_{m=0}^{\infty} U_{m+1}h_{m+1}(x;\psi),$$

where $h_{m+1}(x;\psi)=(m+1)g(x;\psi)G^m(x;\psi)$ and

$$U_{m+1} = a^2 \alpha \beta \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l+m}}{m+1} {\binom{r}{i}} {\binom{-(ai+a-1)}{j}} {\binom{ai+a-1}{k}} \times {\binom{\beta k+\beta j+\beta-1}{l}} {\binom{\alpha l+\alpha-1}{m}} (1-a)^j.$$

Consequently, the PWMs of TIHTEG-G family of distributions is given by

$$\omega_{a,r} = \sum_{m=0}^{\infty} U_{m+1} \int_{-\infty}^{\infty} x^a h_{m+1}(x;\psi) dx.$$

3.4 Rényi entropy

Rényi entropy of the TIHTEG-G family of distributions is given in this section. Rényi entropy (Rényi, 1960) is a measure of uncertainty associated to a random variable X and is defined as

$$H_R(v) = \frac{1}{1-v} \log\left(\int_0^\infty f^v(x) dx\right)$$

for $v>0,\,v\neq 1.$ From equation (6), $f^v_{TIHTEG-G}(x;a,\alpha,\beta,\psi)=f^v(x)$ can be written as

$$f^{v}(x) = \frac{\left(a^{2}\alpha\beta\right)^{v}g^{v}(x;\psi)\bar{G}^{v(\alpha-1)}(x;\psi)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{v(\beta-1)}}{\left(1-\left(1-a\right)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}\right)^{2v}} \times \left[\frac{1-\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}{1-\left(1-a\right)\left(1-\bar{G}^{\alpha}(x;\psi)\right)^{\beta}}\right]^{v(a-1)}.$$

Using the generalised binomial series expansions:

$$\left(1 - (1 - a)(1 - \bar{G}^{\alpha}(x;\psi))^{\beta}\right)^{-(va+v)} = \sum_{i=0}^{\infty} \binom{-(va+v)}{i} (-1)^{i} \\ \times (1 - a)^{i} (1 - \bar{G}^{\alpha}(x;\psi))^{\beta i}, \\ \left[1 - (1 - \bar{G}^{\alpha}(x;\psi))^{\beta}\right]^{v(a-1)} = \sum_{j=0}^{\infty} (-1)^{j} \binom{v(a-1)}{j} (1 - \bar{G}^{\alpha}(x;\psi))^{\beta j}, \\ (1 - \bar{G}^{\alpha}(x;\psi))^{\beta j+\beta i+v(\beta-1)} = \sum_{k=0}^{\infty} (-1)^{k} \binom{\beta j+\beta i+v(\beta-1)}{k} \bar{G}^{\alpha k}(x;\psi),$$

and

$$[1 - G(x; \psi)]^{\alpha k + v(\alpha - 1)} = \sum_{l=0}^{\infty} (-1)^l \binom{\alpha k + v(\alpha - 1)}{l} G^l(x; \psi),$$

we can write

$$f^{v}(x) = \left(a^{2}\alpha\beta\right)^{v}\sum_{\substack{i,j,k,l=0\\ k}}^{\infty} (-1)^{i+j+k+l} \binom{-(va+v)}{i} \binom{v(a-1)}{j} \times \binom{\beta j+\beta i+v(\beta-1)}{k} \binom{\alpha k+v(\alpha-1)}{l} (1-a)^{i} g^{v}(x;\psi) G^{l}(x;\psi).$$

Now, we can write Rényi entropy of the TIHTEG-G family of distributions as

$$\begin{split} H_R(v) &= \frac{1}{1-v} \log \left[\int_0^\infty f^v(x) dx \right] \\ &= \frac{1}{1-v} \log \left[\sum_{l=0}^\infty U_l \left(\int_0^\infty g^v(x;\psi) G^l(x;\psi) dx \right) \right], \end{split}$$

where

$$U_{l} = \left(a^{2}\alpha\beta\right)^{v}\sum_{i,j,k=0}^{\infty} (-1)^{i+j+k+l} \binom{-(va+v)}{i} \binom{v(a-1)}{j} \times \binom{\beta j + \beta i + v(\beta-1)}{k} \binom{\alpha k + v(\alpha-1)}{l} (1-a)^{i}.$$

Note that, $\int_0^\infty g^v(x;\psi)G^l(x;\psi)dx$ can be obtained numerically. Also, Rényi entropy of the TIHTEG-G family of distributions can be obtained directly from that of the exponentiated-G distribution as follows:

$$H_R(v) = \frac{1}{1-v} \log \left[\sum_{l=0}^{\infty} \phi_l e^{(1-v)I_{REG}} \right],$$
(12)

where

$$\phi_l = \left(a^2 \alpha \beta\right)^v \sum_{i,j,k=0}^\infty \frac{(-1)^{i+j+k+l}}{\left(\frac{l}{v}+1\right)^v} \binom{-(va+v)}{i} \binom{v(a-1)}{j} \\ \times \binom{\beta j + \beta i + v(\beta-1)}{k} \binom{\alpha k + v(\alpha-1)}{l} (1-a)^i$$

and $I_{REG} = \frac{1}{1-v} \log \left(\int_0^\infty \left[\left(\frac{l}{v} + 1 \right) g(x; \psi) (G(x; \psi))^{\frac{l}{v}} \right]^v dx \right)$ is the Rényi entropy of the exp-G distribution with power parameter $\frac{l}{v}$.

3.5 Distribution of order statistics

Let $X_1, X_2, ..., X_n$ be independent and identically distributed TIHTEG-G random variables. The pdf of the i^{th} order statistic, $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{j+i-1}.$$

Using the result form the PWMs, with r = i + j - 1, the pdf of the i^{th} order statistic for the TIHTEG-G family of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{p=0}^{\infty} (-1)^j \binom{n-i}{j} U_{p+1}h_{p+1}(x;\psi),$$

where $h_{p+1}(x;\psi) = (p+1)g(x;\psi)G^p(x;\psi)$ and

$$U_{p+1} = a^2 \alpha \beta \sum_{k,l,m,n=0}^{\infty} \frac{(-1)^{k+m+p}}{p+1} {i+j-1 \choose k} {l+ka+a \choose ka+a} {ka+a-1 \choose m} \times {m\beta+l\beta+\beta-1 \choose n} {n\alpha+\alpha-1 \choose p} (1-a)^l.$$

4 Maximum likelihood estimation

Suppose $X_1, X_2, ..., X_n$ is a random sample obtained from the TIHTEG-G family with pdf given by equation (6). Let $X \sim TIHTEG - G(a, \alpha, \beta, \psi)$ and $\Delta = (a, \alpha, \beta, \psi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n is given by

$$\ell_n(\Delta) = 2n \log(a) + n \log(\alpha\beta) + \sum_{i=1}^n \log [g(x_i; \psi)] + (\alpha - 1) \sum_{i=1}^n \log [1 - G(x_i; \psi)] + (\beta - 1) \sum_{i=1}^n \log [1 - \bar{G}^{\alpha}(x_i; \psi)]$$

$$-2\sum_{i=1}^{n} \log \left[1 - (1 - a)(1 - \bar{G}^{\alpha}(x_{i};\psi))^{\beta}\right] \\+ (a - 1)\sum_{i=1}^{n} \log \left[1 - (1 - \bar{G}^{\alpha}(x_{i};\psi))^{\beta}\right] \\- (a - 1)\sum_{i=1}^{n} \log \left[1 - (1 - a)(1 - \bar{G}^{\alpha}(x_{i};\psi))^{\beta}\right]$$

The maximum likelihood estimates of the parameters, denote by $\hat{\Delta}$ is obtained by solving the nonlinear equation $\left(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \psi}\right)^T = \mathbf{0}$ using a numerical method such as Newton-Raphson procedure. Elements of the score vector are given in the appendix.

5 Some special models

We present some special cases of the TIHTEG-G family of distributions, when the baseline cdf $G(x; \psi)$ is specified to be log-logistic, Weibull, Power and Burr III distributions.

5.1 Type-I heavy tailed exponentiated generalised-log-logistic distribution

Suppose the baseline distribution is the log-logistic distribution with the cdf and pdf given by $G(x;c) = 1 - (1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, respectively, for c, x > 0. From equations (5) and (6), we obtain the cdf and pdf of the type-I heavy tailed exponentiated generalised-log-logistic (TIHTEG-LLoG) distribution as

$$F(x; a, \alpha, \beta, c) = 1 - \left[\frac{1 - \left(1 - [1 + x^c]^{-\alpha}\right)^{\beta}}{1 - (1 - a)\left(1 - [1 + x^c]^{-\alpha}\right)^{\beta}} \right]^a,$$

and

$$f(x; a, \alpha, \beta, c) = \frac{a^2 \alpha \beta c x^{c-1} \left[1 + x^c\right]^{-2} \left[1 + x^c\right]^{-(\alpha - 1)} \left(1 - \left[1 + x^c\right]^{-\alpha}\right)^{\beta - 1}}{\left(1 - \left(1 - a\right) \left(1 - \left[1 + x^c\right]^{-\alpha}\right)^{\beta}\right)^2} \\ \times \left[\frac{1 - \left(1 - \left[1 + x^c\right]^{-\alpha}\right)^{\beta}}{1 - \left(1 - a\right) \left(1 - \left[1 + x^c\right]^{-\alpha}\right)^{\beta}}\right]^{a-1},$$

respectively, for $x, a, \alpha, \beta, c > 0$. When $\beta = 1$, we obtain the Type I heavy-tailed generalised log-logistic (TIHTG-LLoG) distribution.

Tables 1 and 2 give the quantiles and moments for TIHTEG-LLoG distribution for selected parameter values.

21			(a, α, β, c)		
u	(1, 1.5, 1.3, 3.5)	(1.7, 1.1, 1.8, 2.5)	(1, 0.5, 2.5, 4.5)	(2, 0.3, 2.9, 5.4)	(9.9, 1, 1.5, 6.5)
0.1	0.5587	0.4934	1.1337	1.1445	0.4965
0.2	0.6763	0.6015	1.3145	1.2709	0.5357
0.3	0.7686	0.6926	1.4790	1.3809	0.5644
0.4	0.8546	0.7772	1.6552	1.4862	0.5870
0.5	0.9407	0.8648	1.8519	1.6029	0.6086
0.6	1.0331	0.9622	2.0996	1.7399	0.6259
0.7	1.1396	1.0800	2.4428	1.9158	0.6459
0.8	1.2890	1.2408	2.9820	2.1759	0.6656
0.9	1.5352	1.5186	4.1240	2.6795	0.6992

Table 1 Quantiles for TIHTEG-LLoG distribution

 Table 2
 Moments for TIHTEG-LLoG distribution

21			(a, α, β, c)		
u	(2, 1.5, 1.3,	3.5) (2.7, 1.1, 1.8,	2.5) (3.4, 0.5, 2.5,	4.5) (4, 1.3, 2.9,	5.4) (6.9, 1, 1.5, 6.5)
E(X)	0.6702	0.6845	1.0707	0.8372	0.6515
$E(X^2)$	0.4995	0.5385	1.1985	0.7106	0.4323
$E(X^3)$	0.4113	0.4877	1.4080	0.6111	0.2916
$E(X^4)$	0.3737	0.5176	1.7494	0.5324	0.1997
$E(X^5)$	0.3764	0.6723	2.3351	0.4698	0.1387
$E(X^6)$	0.4251	1.1963	3.4849	0.4197	0.0976
SD	0.2245	0.2644	0.2282	0.0983	0.0886
CV	0.3349	0.3863	0.2131	0.1174	0.1360
CS	0.7898	1.2689	1.1124	0.0620	-0.3350
CK	4.8041	7.6365	7.4602	3.4783	3.3300

Figure 1 Plots of the density and hazard rate function for the TIHTEG-LLoG distribution (see online version for colours)



Figure 1 shows flexibility of the TIHTEG-LLoG distribution for selected parameter values. The density adopts various shapes including reverse-J, left-skewed, right-skewed and almost symmetric shapes. Moreover, plots of the hrf of the TIHTEG-LLoG

distribution reveal J-shape, decreasing, upside down bathtub, bathtub followed by upside down bathtub shapes.

Figure 2 shows the 3D-plots of skewness and kurtosis for TIHTEG-LLoG distribution. The plots show that when we fix the parameter β and c, the skewness and kurtosis of the TIHTEG-LLoG distribution increases as a and α increase.





5.2 Type-I heavy tailed exponentiated generalised-Weibull distribution

Suppose we take the baseline distribution to be the one parameter Weibull distribution with the cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^{\lambda}}$ and $g(x; \lambda) = \lambda x^{\lambda - 1} e^{-x^{\lambda}}$, respectively, for $\lambda > 0$. From equations (5) and (6), we obtain the cdf and pdf of the type-I heavy tailed exponentiated generalised-Weibull (TIHTEG-W) distribution as

$$F(x; a, \alpha, \beta, \lambda) = 1 - \left[\frac{1 - \left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta}}{1 - (1 - a)\left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta}}\right]^{a},$$

and

$$f(x; a, \alpha, \beta, \lambda) = \frac{a^2 \alpha \beta \lambda x^{\lambda - 1} e^{-\alpha x^{\lambda}} \left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta - 1}}{\left(1 - (1 - a) \left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta}\right)^2} \times \left[\frac{1 - \left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta}}{1 - (1 - a) \left(1 - e^{-\alpha x^{\lambda}}\right)^{\beta}}\right]^{a - 1},$$

respectively, for $x, a, \alpha, \beta, \lambda > 0$.

21			$(a, \alpha, \beta, \lambda)$		
u	(1, 1.5, 1.3, 3.5)	(1.7, 1.1, 1.8, 2.5)	(1, 0.5, 2.5, 4.5)	(2, 0.3, 2.9, 5.4)	(9.9, 1, 1.5, 6.5)
0.1	0.5495	0.4760	1.0050	1.0203	0.4961
0.2	0.6553	0.5744	1.0927	1.0807	0.5349
0.3	0.7321	0.6458	1.1550	1.1254	0.5632
0.4	0.7980	0.7114	1.2096	1.1619	0.5855
0.5	0.8575	0.7741	1.2626	1.1954	0.6067
0.6	0.9190	0.8400	1.3107	1.2264	0.6231
0.7	0.9861	0.9108	1.3644	1.2625	0.6431
0.8	1.0593	1.0000	1.4238	1.3036	0.6621
0.9	1.1661	1.1280	1.5091	1.3624	0.6940

Table 3 Quantiles for TIHTEG-W distribution

Table 4 Moments for TIHTEG-W distribution

			$(a, \alpha, \beta, \lambda)$		
u	(2, 1.5, 7.3, 1.9)	(1.3, 1.2, 0.6, 2.3)	(2.1, 2.9, 2.7, 1.8)	(1.1, 1.5, 2.5, 1.6)	(2.1, 1.5, 1.7, 1.9)
E(X)	1.0287	0.5072	0.4697	0.9585	0.5318
$E(X^2)$	1.1006	0.3671	0.2464	1.0848	0.3346
$E(X^3)$	1.2234	0.3276	0.1426	1.4091	0.2414
$E(X^4)$	1.4120	0.3378	0.0901	2.0598	0.1958
$E(X^5)$	1.6915	0.3887	0.0618	3.3374	0.1759
$E(X^6)$	2.1025	0.4880	0.0457	5.9236	0.1730
SD	0.2057	0.3315	0.1606	0.4075	0.2275
CV	0.2000	0.6537	0.3418	0.4251	0.4278
CS	0.4758	0.8204	0.6186	0.7549	0.7146
CK	3.5782	3.4294	3.6892	3.7916	3.7545

Figure 3 Plots of the density and hazard rate function for the TIHTEG-W distribution (see online version for colours)



Tables 3 and 4 give the quantiles and moments for TIHTEG-W distribution for selected parameter values. Figure 3 shows flexibility of the TIHTEG-W distribution for selected parameter values. The density adopts various shapes including reverse-J, left-skewed,

right-skewed and almost symmetric shapes. Also, the plots of the hrf of the TIHTEG-W distribution reveal J-shape, decreasing, upside down bathtub and bathtub shapes.

Figures 4 and 5 show the 3D-plots of skewness and kurtosis for TIHTEG-W distribution. The plots show that when we fix the parameter β and λ , the skewness and kurtosis of the TIHTEG-W distribution increases as a and α increase. Furthermore, when we fix the parameter a and α , the skewness and kurtosis of the TIHTEG-W distribution decreases then increase as β and λ increase.

Figure 4 Plots of skewness and kurtosis for TIHTEG-W distribution (see online version for colours)



Figure 5 Plots of skewness and kurtosis for TIHTEG-W distribution (see online version for colours)



5.3 Type-I heavy tailed exponentiated generalised-Burr III distribution

Suppose we take the baseline distribution to be the Burr III distribution with the cdf and pdf given by $G(x;\lambda,\gamma) = (1+x^{-\lambda})^{-\gamma}$ and $g(x;\lambda,\gamma) = \lambda \gamma x^{-(\lambda+1)} (1+x^{-\lambda})^{-(\gamma+1)}$,

respectively, for $\lambda, \gamma > 0$. From equations (5) and (6) we obtain the cdf and pdf of the type-I heavy tailed exponentiated generalised-Burr III (TIHTEG-BIII) distribution as

$$F(x; a, \alpha, \beta, \lambda, \gamma) = 1 - \left[\frac{1 - \left(1 - \left(1 - \left(1 + x^{-\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{\beta}}{1 - (1 - a)\left(1 - \left(1 - (1 + x^{-\lambda})^{-\gamma}\right]^{\alpha}\right)^{\beta}}\right]^{a}$$

and

$$f(x; a, \alpha, \beta, \lambda, \gamma) = a^{2} \alpha \beta \lambda \gamma x^{-(\lambda+1)} \left(1 + x^{-\lambda}\right)^{-(\gamma+1)} \left[1 - \left(1 + x^{-\lambda}\right)^{-\gamma}\right]^{\alpha-1} \\ \times \left(1 - \left[1 - \left(1 + x^{-\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{\beta-1} \\ \times \left[1 - (1 - a) \left(1 - \left[1 - \left(1 + x^{-\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{\beta}\right]^{-2} \\ \times \left[\frac{1 - \left(1 - \left[1 - \left(1 - (1 + x^{-\lambda})^{-\gamma}\right]^{\alpha}\right)^{\beta}\right]^{a-1}}{1 - (1 - a) \left(1 - \left[1 - (1 + x^{-\lambda})^{-\gamma}\right]^{\alpha}\right)^{\beta}}\right]^{a-1},$$

respectively, for $x, a, \alpha, \beta, \lambda, \gamma > 0$.

			$(a, \alpha, \beta, \lambda, \gamma)$		
u	(1, 1.5, 3.3, 2.5, 1.2)	(1.7, 1.1, 1.8, 0.5, 2.1)	(1, 0.5, 2.5, 2.5, 1.1)	(1, 0.3, 2.9, 3.4, 2)	(1.9, 1, 1.5, 2.5, 1.1)
0.1	0.7306	0.0223	1.0050	1.2278	0.3970
0.2	0.8334	0.0617	1.1728	1.3597	0.4921
0.3	0.9086	0.1110	1.2990	1.4585	0.5625
0.4	0.9770	0.1827	1.4084	1.5437	0.6289
0.5	1.0431	0.2789	1.5200	1.6204	0.6918
0.6	1.1096	0.4175	1.6272	1.7003	0.7577
0.7	1.1834	0.6264	1.7475	1.7883	0.8296
0.8	1.2685	0.9950	1.8935	1.8902	0.9202
0.9	1.3977	1.8117	2.0968	2.0308	1.0448

Table 5 Quantiles for TIHTEG-BIII distribution

Table 5 gives the table of quantiles for TIHTEG-BIII distribution for selected parameter values. Figure 6 demonstrates flexibility of the TIHTEG-BIII distribution for selected parameter values. The density shows various shapes including reverse-J, left skewed, right skewed and almost symmetric shapes. The plots of the hrf of the TIHTEG-BIII distribution reveal J-shape, decreasing shape as well as upside down bathtub shape.

Figure 6 Plots of the density and hazard rate function for the TIHTEG-BIII distribution (see online version for colours)



5.4 Type-I heavy tailed exponentiated generalised-power distribution

Suppose we take the baseline distribution to be the power distribution with the cdf and pdf given by $G(x; \theta, k) = (\theta x)^k$ and $g(x; \theta, k) = k\theta^k x^{k-1}$, respectively, for $\theta, k > 0$. From equations (5) and (6), we obtain the cdf and pdf of the type-I heavy tailed exponentiated generalised-power (TIHTEG-P) distribution as

$$F(x; a, \alpha, \beta, \theta, k) = 1 - \left[\frac{1 - \left(1 - \left(1 - \left(\theta x\right)^k\right]^\alpha\right)^\beta}{1 - (1 - a)\left(1 - \left[1 - \left(\theta x\right)^k\right]^\alpha\right)^\beta}\right]^\alpha$$

and

$$f(x; a, \alpha, \beta, \theta, k) = \frac{a^2 \alpha \beta k \theta^k x^{k-1} \left[1 - (\theta x)^k\right]^{\alpha - 1} \left(1 - \left[1 - (\theta x)^k\right]^{\alpha}\right)^{\beta - 1}}{\left[1 - (1 - a) \left(1 - \left[1 - (\theta x)^k\right]^{\alpha}\right)^{\beta}\right]^2} \\ \times \left[\frac{1 - \left(1 - \left[1 - (\theta x)^k\right]^{\alpha}\right)^{\beta}}{1 - (1 - a) \left(1 - \left[1 - (\theta x)^k\right]^{\alpha}\right)^{\beta}}\right]^{a - 1},$$

respectively, for $x, a, \alpha, \beta, \theta, k > 0$.

Table 6 gives the quantiles for TIHTEG-P distribution for selected parameter values. Figure 7 shows flexibility of the TIHTEG-P distribution for selected parameter values. The density adopts various shapes including reverse-J, J, left-skewed, right-skewed and almost symmetric shapes. Also, the plots of the hrf of the TIHTEG-P distribution reveal J-shape, decreasing, increasing, bathtub, upside down bathtub followed by bathtub shapes.

			$(a, \alpha, \beta, \theta, k)$		
u	(2.1, 1.5, 1.3, 0.1, 1.6)	(1.7, 1.1, 1.8, 0.5, 2.1)	(2, 0.5, 0.8, 0.8, 1.1)	(1.9, 1.3, 0.9, 0.4, 2)	(1.9, 1, 0.5, 0.3, 1.1)
0.1	1.2950	0.7996	0.0387	0.3059	0.0067
0.2	1.8703	0.9711	0.0872	0.4674	0.0213
0.3	2.3558	1.1033	0.1460	0.6061	0.0472
0.4	2.8159	1.2113	0.2148	0.7417	0.0937
0.5	3.2760	1.3081	0.3010	0.8758	0.1600
0.6	3.7662	1.4039	0.4021	1.0193	0.2590
0.7	4.3198	1.4993	0.5218	1.1805	0.4145
0.8	4.9915	1.6051	0.6768	1.3751	0.6635
0.9	5.9410	1.7324	0.8886	1.6379	1.1389

 Table 6
 Quantiles for TIHTEG-P distribution

Figure 7 Plots of the density and hazard rate function for the TIHTEG-P distribution (see online version for colours)





Table 7 Simulation results

Parameter	Sample size .	(1.1	, 3.5, 1.1,	1.5)	(5.0	, 5.0, 2.1,	1.5)
1 <i>ur unicici</i>	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
a	50	1.2112	0.7028	0.1112	8.3738	8.3500	3.3738
	100	1.2105	0.6948	0.1105	7.6220	7.9664	2.6220
	200	1.1804	0.6272	0.0804	6.6696	6.0300	1.6696
	400	1.1894	0.5710	0.0894	5.9271	3.4488	0.9271
	800	1.1724	0.4993	0.0724	5.4459	2.1879	0.4459
	1,600	1.1408	0.3958	0.0408	5.2962	1.6724	0.2962
α	50	4.7744	3.6471	1.2744	8.4037	12.5111	3.4037
	100	4.3653	2.8225	0.8653	7.1306	5.9244	2.1306
	200	4.2434	2.6250	0.7434	6.5487	5.2381	1.5487
	400	3.9138	2.0478	0.4138	6.0988	4.3823	1.0988
	800	3.8317	1.8205	0.3317	5.7402	3.6867	0.7402
	1,600	3.7923	1.6123	0.2923	5.2968	2.2241	0.2968

Parameter	Sampla siza	(1.1	1, 3.5, 1.1,	1.5)	(5.	0, 5.0, 2.1,	1.5)
1 urumeter	Sumple size	Mean	RMSE	Bias	Mean	RMSE	Bias
β	50	3.1812	12.2667	2.0812	8.2279	21.5558	6.1279
	100	1.6755	3.5844	0.5755	4.6672	8.1359	2.5672
	200	1.2814	1.1787	0.1814	3.3710	4.9560	1.2710
	400	1.1577	0.3667	0.0577	2.6017	2.7516	0.5017
	800	1.1302	0.2603	0.0302	2.2778	1.6127	0.1778
	1,600	1.1215	0.2013	0.0215	2.1979	1.1430	0.0979
c	50	1.7023	1.2136	0.2023	3.2421	3.1986	1.7421
	100	1.5875	0.7798	0.0875	2.5633	2.3603	1.0633
	200	1.5427	0.5121	0.0427	2.1007	1.6406	0.6007
	400	1.5463	0.3843	0.0463	1.8178	1.0453	0.3178
	800	1.5352	0.2879	0.0352	1.6486	0.5551	0.1486
	1,600	1.5217	0.2339	0.0217	1.5623	0.3128	0.0623

 Table 7
 Simulation results (continued)

6 Simulation study

The performance of the TIHTEG-LLoG distribution is examined by conducting various simulations for different sizes (n = 100, 200, 400, 800 and 1,600) via the R package. We simulate N = 3,000 samples for the true parameter values given in Table 7. The tables list the mean MLE estimates of the model parameters along with the respective average bias (ABias) and root mean square errors (RMSEs). The ABias and RMSE for the estimated parameter, say $\hat{\theta}$ are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}}, \tag{13}$$

respectively.

7 Actuarial measures

Actuaries are mostly concerned with evaluating the exposure of market risk in a portfolio of instruments. Risk measures are calculated in this section, these includes value at risk (VaRq), tail value at risk ($TVaR_q$), tail variance (TV_q) and tail variance premium (TVP_q) for the TIHTEG-G family of distributions. These risk measures play a very important role in portfolio optimisation under uncertainty.

7.1 Risk measures

• Value at risk measure

Let X follow the TIHTEG-G family of distributions with pdf (6), then the VaR_q , where q is a specified level of significance, is given by

$$VaR_q = x_q = G^{-1} \left(1 - \left[1 - \left[\frac{1 - (1 - q)^{\frac{1}{a}}}{1 - (1 - q)^{\frac{1}{a}} (1 - a)} \right]^{\frac{1}{\beta}} \right]^{\frac{1}{\alpha}} \right),$$

for $0 \le q \le 1$.

• Tail value at risk measure

This measure is used to determine the expected value of loss given that an event outside a given probability level has occurred. Let X has TIHTEG-G pdf, then using equations (8) and (9), $TVaR_q$ of X is computed as

$$TVaR_q(X) = \frac{1}{1-q} \int_{VaR_q}^{\infty} xf(x)dx$$
$$= \frac{1}{1-q} \sum_{l=0}^{\infty} \int_{VaR_q}^{\infty} xt_{l+1}h_{l+1}(x;\psi)dx,$$

where t_{l+1} is defined as equation (9) and $h_{l+1}(x; \psi) (l+1) g(x; \psi) G^l(x; \psi)$ is the exp-G density with power parameter (l+1) and parameter vector ψ .

• Tail variance measure

Tail variance (TV_q) is an actuarial measure that pays attention to the tail variance beyond VaR_q . TV_q is given as

$$TV_q = E(X^2|X > x_q) - (TVaR_q)^2$$

= $\frac{1}{1-q} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2$
= $\frac{1}{1-q} \sum_{l=0}^{\infty} \int_{VaR_q}^{\infty} x^2 t_{l+1} h_{l+1}(x;\psi) dx - (TVaR_q)^2$,

where t_{l+1} is defined as equation (9) and $h_{l+1}(x; \psi) (l+1) g(x; \psi) G^l(x; \psi)$ is the exp-G density with power parameter (l+1) and parameter vector ψ .

• Tail variance premium measure

 TVP_q is one of the important actuarial measures and is given by

$$TVP_q = TVaR_q + \delta\left(TV_q\right),$$

for $0 < \delta < 1$.

Distribution	Level of significance	VaR	TVaR	TV	TVP
TIHTEG-LLoG	0.7	1.4159	1.7973	0.1895	1.9300
<i>a</i> = 1.9	0.75	1.4818	1.8672	0.1981	2.0157
$\alpha = 1.0$	0.8	1.5616	1.9539	0.2099	2.1218
$\beta = 3.8$	0.85	1.6644	2.0683	0.2271	2.2614
<i>c</i> = 3.0	0.9	1.8117	2.2361	0.2553	2.4659
	0.95	2.0770	2.5448	0.3144	2.8435
TIHTG-LLoG	0.7	0.7462	1.0070	0.0754	1.0599
<i>a</i> = 1.9	0.75	0.7953	1.0544	0.0770	1.1122
$\alpha = 1.1$	0.8	0.8533	1.1122	0.0795	1.1758
<i>c</i> = 3.0	0.85	0.9261	1.1869	0.0835	1.2579
	0.9	1.0270	1.2938	0.0906	1.3753
	0.95	1.2009	1.4839	0.1065	1.5851
TIHT-LLoG	0.7	0.7750	1.0567	0.0931	1.1219
<i>a</i> = 1.9	0.75	0.8269	1.1080	0.0959	1.1799
<i>c</i> = 3.0	0.8	0.8885	1.1708	0.1001	1.2509
	0.85	0.9663	1.2525	0.1065	1.3431
	0.9	1.0749	1.3704	0.1177	1.4763
	0.95	1.2642	1.5826	0.1424	1.7179
APTLW	0.7	1.3067	1.5633	0.0767	1.6170
$\alpha = 1.9$	0.75	1.3533	1.6101	0.0789	1.6692
$\beta = 1.0$	0.8	1.4090	1.6676	0.0820	1.7332
<i>c</i> = 3.8	0.85	1.4798	1.7425	0.0867	1.8162
$\lambda = 3.9$	0.9	1.5795	1.8507	0.0946	1.9358
	0.95	1.7546	2.0450	0.1111	2.1505
LLoG	0.7	1.1311	1.3638	0.0664	1.4102
<i>c</i> = 3.0	0.75	1.1732	1.4062	0.0687	1.4577
	0.8	1.2234	1.4584	0.0720	1.5160
	0.85	1.2870	1.5266	0.0771	1.5921
	0.9	1.3767	1.6254	0.0855	1.7024
	0.95	1.5351	1.8048	0.1034	1.9030

Table 8 Simulation results of VaR, TVaR, TV and TVP

7.2 Numerical study of actuarial measures

In this section, numerical study of actuarial measures is done for the TIHTEG-LLoG, type-I heavy-tailed generalised log-logistic (TIHTG-LLoG), type-I heavy tailed log-logistic (TIHT-LLoG), APTLW and log-logistic (LLoG) distributions for different sets of parameters. Firstly, a random sample of size n = 100 is generated from these distributions, and the maximum likelihood method of estimation is used to estimate the model parameters. Secondly, a repetition of 1,000 iterations is made in order to find the values of the risk measures for the distributions. The model with the highest values of risk measures has the heavier tail.

Simulated results of VaR, TVaR, TV and TVP are shown in Tables 8 and 9. These results show that the TIHTEG-LLoG distribution has higher values of risk measures as compared to the TIHTG-LLoG, TIHT-LLoG, APTLW and log-logistic distributions

respectively, therefore, it is evident that TIHTEG-LLoG has a heavier tail and it can be used effectively to model heavy-tailed data.

Distribution	Level of significance	VaR	TVaR	TV	TVP
TIHTEG-LLoG	0.7	1.9172	3.1769	2.4241	4.8737
<i>a</i> = 0.9	0.75	2.1301	3.4083	2.5867	5.3484
$\alpha = 1.4$	0.8	2.4047	3.6952	2.8205	5.9515
$\beta = 2.1$	0.85	2.7865	4.0657	3.2075	6.7920
<i>c</i> = 1.9	0.9	3.3905	4.5679	4.0394	8.2034
	0.95	4.6539	5.2132	7.1179	11.9752
TIHTG-LLoG	0.7	1.3427	2.4426	1.8734	3.7540
a = 0.9	0.75	1.5107	2.6464	1.9984	4.1452
$\alpha = 1.4$	0.8	1.7261	2.9047	2.1635	4.6355
<i>c</i> = 1.9	0.85	2.0238	3.2510	2.4026	5.2931
	0.9	2.4917	3.7580	2.8235	6.2992
	0.95	3.4630	4.6146	4.1039	8.5133
TIHT-LLoG	0.7	0.5914	0.9708	0.2187	1.1238
<i>a</i> = 2.1	0.75	0.6532	1.0406	0.2331	1.2154
<i>c</i> = 1.9	0.8	0.7291	1.1284	0.2528	1.3306
	0.85	0.8282	1.2458	0.2816	1.4851
	0.9	0.7930	1.4211	0.3293	1.7175
	0.95	1.2408	1.7531	0.4324	2.1638
APTLW	0.7	0.5313	1.3496	0.0797	1.4054
$\alpha = 0.9$	0.75	0.5771	1.4117	0.1784	1.5455
$\beta = 1.4$	0.8	0.6294	1.4842	0.3477	1.7623
<i>c</i> = 2.1	0.85	0.6919	1.5630	0.6890	2.1487
$\lambda = 1.9$	0.9	0.7724	1.6152	1.5618	3.0209
	0.95	0.8950	1.7338	4.9846	6.0691
LLoG	0.7	1.0997	1.2671	0.0311	1.2888
<i>c</i> = 1.9	0.75	1.1312	1.2975	0.0317	1.3212
	0.8	1.1683	1.3345	0.0327	1.3607
	0.85	1.2150	1.3825	0.0342	1.4115
	0.9	1.2798	1.4512	0.0368	1.4842
	0.95	1.3919	1.5732	0.0423	1.6134

Table 9 Simulation results of VaR, TVaR, TV and TVP

8 Applications

In this section, we present examples to illustrate the flexibility and usefulness of the TIHTEG-LLoG distribution for data modelling. Several goodness-of-fit statistics are used to compare TIHTEG-LLoG distribution to other equi-parameter models. These include: -2log-likelihood statistic (-2ln(L)), Akaike information criterion (AIC), Bayesian information criterion (BIC), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. We also used the Cramér-von Mises (W^*) , Anderson-Darling (A^*) , the Kolmogorov-Smirnov (K-S) statistic as well as its associated p-value, and sum of squares (SS) from the probability plots to assess goodness-of-fit. The SS described by Chen and Balakrishnan (1995) is given by SS =

 $\sum_{j=1}^{n} \left[F_{TIHTEG-LLoG}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right) \right]^2, \text{ where } j = 1, 2, ..., n \text{ and } x_{(j)} \text{ are ordered values of the observed data. In general, the smaller the values of goodness-of-fit statistics and the highest p-value for the K-S statistic, the better the fit.$

The TIHTEG-LLoG distribution is fitted to datasets and these fits are compared to several models including the Kumaraswamy Weibull (KumW) distribution by Cordeiro et al. (2010), the Alpha power exponentiated log-logistic (APExLLD) distribution by Mead et al. (2019), the type II exponentiated half-logistic-Weibull (TIIEHLW) distribution by Al-Mofleh et al. (2020), Topp-Leone Marshall-Olkin-Weibull (TLMOW) distribution by Chipepa et al. (2020) and the type II generalised inverse exponentiated Lomax (TIIGIELx) distribution by Jamal et al. (2020). The pdfs are

$$f_{KumW}(x) = abc\lambda^{c}x^{c-1}e^{-(\lambda x)^{c}} \left[1 - e^{-(\lambda x)^{c}}\right]^{a-1} \left[1 - \left(1 - e^{-(\lambda x)^{c}}\right)^{a-1}\right]^{b-1}$$

for $a, b, c, \lambda > 0$, and x > 0,

$$f_{APEXLLD}(x) = \frac{ac\log(\alpha)}{b(\alpha-1)} \left(\frac{x}{b}\right)^{-a-1} \left[\left(\frac{x}{b}\right)^{-a} + 1 \right]^{-c-1} \alpha^{\left(\left(\frac{x}{b}\right)^{-a} + 1\right)^{-c}}$$

for $\alpha, a, b, c > 0$ and x > 0,

$$f_{TIIEHLW}(x) = 2a\lambda\delta\gamma x^{\gamma-1}e^{-\delta x^{\gamma}} \left[1 - e^{-\delta x^{\gamma}}\right]^{\lambda-1} \frac{\left[1 - \left(1 - e^{-\delta x^{\gamma}}\right)^{\lambda}\right]^{a-1}}{\left[1 + \left(1 - e^{-\delta x^{\gamma}}\right)^{\lambda}\right]^{a+1}}$$

for $a, \lambda, \delta, \gamma > 0$, and x > 0,

$$f_{TLMOW}(x) = \frac{2b\delta^2 \gamma \lambda^{\gamma} x^{\gamma-1} e^{-2(\lambda x)^{\gamma}}}{\left(1 - \bar{\delta} e^{-(\lambda x)^{\gamma}}\right)^3} \left(1 - \left(1 - \bar{\delta} e^{-(\lambda x)^{\gamma}}\right)^2\right)^{b-1}$$

for $b, \delta, \lambda, \gamma > 0$ and x > 0, and

$$f_{TIIGIELx}(x) = \lambda \alpha \left(\frac{a}{b}\right) \left[1 + \frac{x}{b}\right]^{-(a+1)} \left(1 + \frac{x}{b}\right)^{a(\alpha+1)} e^{\left[\lambda(1 - \left[1 + \frac{x}{b}\right]^{a\alpha})\right]}$$

for $\lambda, \alpha, a, b > 0$, and x > 0.

8.1 Waiting times data

The first dataset represents the waiting times (in minutes) before service of 100 bank customers. The data was used and analysed by Ghitany et al. (2011). The data are: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7,

6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Figure 8 Fitted densities and probability plots for waiting times data (see online version for colours)



Figure 9 Estimated cdf plot, Kaplan-Meier survival plot and estimated hazard rate function plot of the TIHTEG-LLoG distribution for waiting times data (see online version for colours)



From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 8 and 9, we can conclude that the TIHTEG-LLoG distribution provide a better fit compared to the other models. The estimated variance-covariance matrix for the waiting times data is given by,

$$\begin{bmatrix} 2.4505 \times 10^{-7} & 2.7046 \times 10^{-5} & -4.6395 \times 10^{-7} & -4.8485 \times 10^{-6} \\ 2.7046 \times 10^{-5} & 2.9854 \times 10^{-3} & -5.1213 \times 10^{-5} & -5.3720 \times 10^{-4} \\ -4.6395 \times 10^{-7} & -5.1213 \times 10^{-5} & 8.7855 \times 10^{-7} & 9.2237 \times 10^{-6} \\ -4.8485 \times 10^{-6} & -5.3720 \times 10^{-4} & 9.2237 \times 10^{-6} & 1.1021 \times 10^{-4} \end{bmatrix}$$

and the 95% asymptotic confidence intervals for the parameters a, α , β , and c are: 2.3338 × 10¹ ± 0.0010, 2.9937 ± 0.1071, 7.8826 × 10¹ ± 0.0018 and 1.3107 × 10⁻¹ ± 0.0206, respectively.

8.2 Agriculture data

This dataset represents the total factor productivity (TFP) growth agricultural production for 37 African countries from 2001–2010, see https://dataverse.harvard.edu/dataset. xhtml?persistentId=doi:10.7910/DVN/9IOAKR. The data are: 4.6, 0.9, 1.8, 1.4, 0.2, 3.9, 1.8, 0.8, 2.0, 0.8, 1.6, 0.8, 2.0, 1.6, 0.5, 0.1, 2.5, 2.4, 0.6, 1.1, 0.7, 1.7, 1.0, 1.7, 2.5, 3.5, 0.3, 0.9, 2.3, 0.5, 1.5, 5.1, 0.2, 1.5, 3.3, 1.4, 3.3.





Figure 11 Estimated cdf plot, Kaplan-Meier survival plot and estimated hazard rate function plot of the TIHTEG-LLoG distribution for agriculture data (see online version for colours)



Distribution		Estim	iates		-2100(1.)	AIC	CAIC	BIC	*M	*7	K - S	P-value	33
	a	α	β	с	+108(H)	2007	2002	277	:		4		2
TIHTEG-LLoG	$\begin{array}{c} 2.3338 \times 10^{1} \\ (4.9502 \times 10^{-4}) \end{array}$	$\begin{array}{c} 2.9937 \\ (5.4639 \times \ 10^{-2}) \end{array}$	$\begin{array}{c} 7.8826 \times \ 10^{1} \\ (9.3731 \times \ 10^{-4}) \end{array}$	$\begin{array}{c} 1.3107 \times \ 10^{-1} \\ (1.0498 \times \ 10^{-2}) \end{array}$	634.0978	642.0978	642.5188	652.5185	0.0187	0.1331	0.0372	0.9991	0.0181
	a	q	с	Y									
KumW	$\begin{array}{r} 4.0992 \times \ 10^{-1} \\ (2.9356 \times \ 10^{-2}) \end{array}$	3.6333×10^2 (9.7083 $\times 10^{-7}$)	$\begin{array}{ccc} 1.3107 \times & 10^{-3} \\ (3.8444 \times & 10^{-4}) \end{array}$	$\begin{array}{c} 3.3894 \\ (3.5865 \times \ 10^{-3}) \end{array}$	637.9178	645.9178	646.3388	656.3385	0.0564	0.3564	0.0586	0.0.8826	0.0990
	σ	a	p	с									
APEXLLD	$\begin{array}{c} 2.0820 \times \ 10^4 \\ (1.3025 \times \ 10^{-7}) \end{array}$	$\frac{1.3617}{(1.0795 \times 10^{-1})}$	$\begin{array}{rrr} 1.1703 \times & 10^{-2} \\ (5.6843 \times & 10^{-3}) \end{array}$	$\begin{array}{c} 4.5063 \times \ 10^2 \\ (1.9991 \times \ 10^{-7}) \end{array}$	658.9610	6096.9609	667.3820	677.3816	0.2825	1.8842	0.0956	0.3202	0.2084
	a	Y	δ	Å									
TIIEHLW	85.9947 (0.0132)	5.1096 (1.6824)	0.1909 (0.1152)	0.3638 (0.0802)	635.4997	643.4997	643.9207	653.9204	0.0391	0.2496	0.0543	0.9301	0.0476
	q	δ	λ	γ									
TLMOW	$\begin{array}{c} 2.2474 \times \ 10^{6} \\ (9.0718 \times \ 10^{-9}) \end{array}$	$\frac{1.6928 \times 10^{-3}}{(1.3055 \times 10^{-3})}$	$\begin{array}{ccc} 3.5750 \times & 10^{-1} \\ (7.8796 \times & 10^{-2}) \end{array}$	$\begin{array}{l} 6.9768 \times \ 10^{-1} \\ (3.9044 \times \ 10^{-1}) \end{array}$	651.7877	659.7878	660.2088	670.2084	0.2026	1.3904	0.0875	0.4287	0.0476
	γ	α	a	b									
TIIGIELx	$\begin{array}{c} 3.5426 \times \ 10^2 \\ (1.8547 \times \ 10^{-12}) \end{array}$	$\begin{array}{c} 1.3743 \times \ 10^{4} \\ (4.7883 \times \ 10^{-14}) \end{array}$	$\begin{array}{r} 8.1232 \times \ 10^{-5} \\ (8.1057 \times \ 10^{-6}) \end{array}$	$\begin{array}{c} 3.9069 \times \ 10^3 \\ (1.6810 \times \ 10^{-13}) \end{array}$	658.0280	666.0280	666.4491	676.4487	0.0271	0.1792	0.1730	0.0050	0.7053

 Table 10
 Parameter estimates and goodness-of-fit statistics for various models for waiting times data

Distribution		Estim	ates		-210a(I)	JIF	CAIC	BIC	*/11	*7	$K_{-}c$	D_vidua	33
Distribution	a	σ	β	с	(7)2017-	210	7410	777	4	۲,	C V	- vuuuc	00
TIHTEG-LLoG	$\begin{array}{c} 2.6851 \times \ 10^2 \\ (2.4195 \times \ 10^{-5}) \end{array}$	$5.9306 (2.7213 \times 10^{-2})$	$\begin{array}{c} 7.2720 \times \ 10^2 \\ (5.3599 \times \ 10^{-5}) \end{array}$	$\begin{array}{r} 4.0750 \times \ 10^{-2} \\ (5.1098 \times \ 10^{-3}) \end{array}$	107.2680	115.2679	116.5179	121.7116	0.0333	0.2044	0.0995	0.8573	0.0312
	a	q	С	Y									
KumW	0.1294 (0.0447)	1.8135 (1.0046)	7.4198 (2.5729)	0.2016 (0.0385)	109.6151	117.6158	118.8658	124.0595	0.0746	0.4570	0.1044	0.8151	0.0691
	σ	a	p	с									
APEXLLD	3.5133×10^2 (0.3781)	1.2271 (1.6704)	$\begin{array}{c} 1.2917 \times \ 10^{-3} \\ (10.0873) \end{array}$	5.1436×10^2 (0.7151)	120.6588	128.6586	129.9086	135.1023	0.2154	1.3397	0.1749	0.2078	0.2114
	ø	X	δ	λ									
TIIEHLW	$\frac{1.4815 \times 10^3}{(1.9743 \times 10^{-6})}$	$\begin{array}{c} 4.7980 \times \ 10^3 \\ (5.3436 \times \ 10^{-6}) \end{array}$	$\begin{array}{c} 6.2998 \\ (2.5647 \times \ 10^{-2}) \end{array}$	$\begin{array}{ccc} 2.6519 \times & 10^{-2} \\ (3.2667 \times & 10^{-3}) \end{array}$	107.5327	115.5327	116.7827	121.9763	0.0370	0.2278	0.1032	0.8258	0.2326
	q	δ	λ	Y									
TLMOW	$\begin{array}{c} 5.7211 \times \ 10^{1} \\ (3.2395 \times \ 10^{-4}) \end{array}$	$\begin{array}{c} 1.9838 \times \ 10^{-2} \\ (5.3186 \times \ 10^{-2}) \end{array}$	$\begin{array}{l} 4.9053 \times \ 10^{-1} \\ (7.7166 \times \ 10^{-2}) \end{array}$	$\begin{array}{c} 1.3305 \times \ 10^{-1} \\ (3.3296 \times \ 10^{-1}) \end{array}$	122.6588	130.6968	131.9468	137.1405	0.2115	1.3120	0.1564	0.3260	0.0334
	γ	σ	a	p									
TIIGIELx	$\begin{array}{c} 2.9037 \times \ 10^4 \\ (8.9027 \times \ 10^{-10}) \end{array}$	$\begin{array}{c} 1.0180 \times \ 10^{1} \\ (2.5395 \times \ 10^{-6}) \end{array}$	$\begin{array}{c} 1.2540 \times \ 10^{-2} \\ (2.0615 \times \ 10^{-3}) \end{array}$	$\begin{array}{c} 6.2899 \times \ 10^3 \\ (4.1091 \times \ 10^{-9}) \end{array}$	113.1530	121.1530	122.4030	127.5967	0.0335	0.2054	0.1597	0.3021	0.2195

 Table 11
 Parameter estimates and goodness-of-fit statistics for various models for agriculture data

		Esti	nates		1.00	01		UIU	*111	*	ت 4	-	22
DIStribution	a	σ	β	υ	(7)B017-	AIC	CAIC	BIC	4	Y	C-N	P-value	<i>CC</i>
TIHTEG-LLoC	$\begin{array}{cccc} \mathbf{j} & 1.4603 \times & 10^1 \\ (1.0481 \times & 10^{-3}) \end{array}$	$\begin{array}{c} 4.5478 \\ (4.7474 \times \ 10^{-2}) \end{array}$	$\begin{array}{c} 1.8292 \times \ 10^2 \\ (3.0808 \times \ 10^{-4}) \end{array}$	$\begin{array}{c} 1.4698 \times \ 10^{-1} \\ (1.0959 \times \ 10^{-2}) \end{array}$	376.8533	384.8533	385.2493	395.5070	0.0562	0.3182	0.0702	0.6727	0.0574
	a	q	с	Y									
KumW	23.7408 (7.8797)	527.5472 (0.0196)	0.1833 (0.0289)	2.4473 (3.3830)	378.2826	386.2826	386.6786	396.9364	0.0704	0.4323	0.0732	0.6214	0.0677
	σ	a	q	с									
APEXLLD	$\begin{array}{c} 9.7068 \times \ 10^3 \\ (4.3139 \times \ 10^{-7}) \end{array}$	$\begin{array}{c} 1.9158 \\ (1.4613 \times \ 10^{-1}) \end{array}$	$\begin{array}{rrr} 3.0738 \times & 10^{-2} \\ (9.6251 \times & 10^{-3}) \end{array}$	$\begin{array}{c} 3.2031 \times \ 10^2 \\ (7.6348 \times \ 10^{-7}) \end{array}$	393.2826	401.3900	401.7860	412.0438	0.2237	1.4003	0.0842	0.4403	0.1650
	a	γ	δ	λ									
TIIEHLW	889.4855 (0.0141)	40.0413 (31.9200)	1.5163 (0.7006)	0.1317 (0.0450)	378.4339	386.4339	386.8300	397.0877	0.0718	0.4428	0.0710	0.6598	0.1644
	p	δ	λ	ү									
TLMOW	$\begin{array}{c} 6.4731 \times \ 10^{6} \\ (1.7831 \times \ 10^{-8}) \end{array}$	$\begin{array}{r} 1.2899 \times \ 10^{-3} \\ (8.3886 \times \ 10^{-4}) \end{array}$	$\begin{array}{rrr} 4.6173 \times & 10^{-1} \\ (9.2108 \times & 10^{-2}8) \end{array}$	$\frac{1.0778}{(4.2957 \times \ 10^{-1})}$	386.1004	394.1007	394.4967	404.7544	0.1441	0.9082	0.09096	0.3450	0.0643
	Y	α	a	p									
TIIGIELx	$\begin{array}{rrr} 4.6772 \times & 10^{-1} \\ (1.4520 \times & 10^{-1}) \end{array}$	$\begin{array}{c} 7.4115 \times \ 10^{6} \\ (3.7159 \times \ 10^{-11}) \end{array}$	$\begin{array}{r} 4.2682 \times 10^{-2} \\ (6.3844 \times 10^{-3}) \end{array}$	$\begin{array}{c} 9.4824 \times \ 10^{5} \\ (2.8981 \times \ 10^{-10}) \end{array}$	399.9445	407.9445	408.3406	418.5983	0.2418	1.5731	0.1228	0.0819	0.2848

 Table 12
 Parameter estimates and goodness-of-fit statistics for various models for COVID-19 data

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 10 and 11, we can conclude that the TIHTEG-LLoG distribution provide a better fit compared to the other models. This is evident because the TIHTEG-LLoG distribution has smaller values of goodness-of-fit statistics and the highest p-value compared to other models. Moreover, the TIHTEG-LLoG distribution follows the empirical cdf and Kaplan-Meier survival curves closely. The estimated variance-covariance matrix for the agriculture data is given by,

$$\begin{bmatrix} 5.8540 \times 10^{-10} & 6.5842 \times 10^{-7} & -1.2968 \times 10^{-9} & -7.7059 \times 10^{-8} \\ 6.5842 \times 10^{-7} & 7.4057 \times 10^{-4} & -1.4586 \times 10^{-6} & -8.7036 \times 10^{-5} \\ -1.2968 \times 10^{-9} & -1.4586 \times 10^{-6} & 2.8728 \times 10^{-9} & 1.7163 \times 10^{-7} \\ -7.7059 \times 10^{-8} & -8.7036 \times 10^{-5} & 1.7163 \times 10^{-7} & 2.6110 \times 10^{-5} \end{bmatrix}$$

and the 95% asymptotic confidence intervals for the parameters *a*, α , β , and *c* are: 2.6851 × 10² ± 4.7422 × 10⁻⁵, 5.9306 ± 5.3338 × 10⁻², 7.2720 × 10² ± 1.0505 × 10⁻⁴ and 4.0750 × 10⁻² ± 1.0015 × 10⁻², respectively.

8.3 COVID-19 data

This dataset shows mortality rates of the patients infected by the COVID-19 pandemic in Mexico. The dataset was recorded from 4 March 2020, to 20 July 2020. The data was analysed by Almongy et al. (2021) The data are: 4.4130, 3.0525, 4.6955, 7.4810, 5.1915, 3.6335, 6.6100, 8.2490, 5.8325, 3.0075, 5.4275, 3.0610, 3.3280, 1.7200, 2.9270, 5.3425, 5.0175, 2.6210, 2.1720, 2.5715, 3.8150, 7.3020, 3.9515, 3.1850, 1.7685, 3.1635, 2.3650, 1.6075, 4.6420, 6.4390, 4.4065, 5.0215, 3.6300, 2.9925, 3.2060, 1.6975, 2.2120, 4.9675, 3.9200, 4.7750, 1.7495, 1.8755, 3.4840, 1.6430, 5.0790, 4.0540, 3.3485, 3.5755, 3.2800, 1.0385, 1.8890, 1.4940, 1.6680, 3.4070, 4.1625, 3.9270, 4.2755, 1.6140, 3.7430, 3.3125, 3.0700, 2.4545, 2.3305, 2.6960, 6.0210, 4.3480, 0.9075, 1.6635, 2.7030, 3.0910, 0.5205, 0.9000, 2.4745, 2.0445, 1.6795, 1.0350, 1.6490, 2.6585, 2.7210, 2.2785, 2.1460, 1.2500, 3.2675, 2.3240, 2.3485, 2.7295, 2.0600, 1.9610, 1.6095, 0.7010, 1.2190, 1.6285, 1.8160, 1.6165, 1.5135, 1.1760, 0.6025, 1.6090, 1.4630, 1.3005, 1.0325, 1.5145, 1.0290, 1.1630, 1.2530, 0.9615.

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 12 and 13, we can conclude that the TIHTEG-LLoG distribution provide a better fit compared to the other models. This is evident because the TIHTEG-LLoG distribution has smaller values of goodness-of-fit statistics and the highest p-value compared to other models. Moreover, the TIHTEG-LLoG distribution follows the empirical cdf and Kaplan-Meier survival curves closely. The estimated variance-covariance matrix for the COVID-19 data is given by,

1.0985×10^{-6}	4.9752×10^{-5}	-3.8571×10^{-7}	-9.6300×10^{-6}
4.9752×10^{-5}	0.0023	-1.7474×10^{-5}	-4.4089×10^{-4}
-3.8571×10^{-7}	$-1.7474 imes 10^{-5}$	1.3548×10^{-7}	$3.4259 imes 10^{-6}$
-9.6300×10^{-6}	$-4.4089 imes 10^{-4}$	3.4259×10^{-6}	1.2011×10^{-4}

and the 95% asymptotic confidence intervals for the parameters a, α , β , and c are: 1.4603 × 10¹ ± 0.0021, 1.8292 × 10² ± 0.0930, 4.1209 × 10³ ± 0.0007 and 1.4698 × 10⁻¹ ± 0.0215, respectively.





Figure 13 Estimated cdf plot, Kaplan-Meier survival plot and estimated hazard rate function plot of the TIHTEG-LLoG distribution for COVID-19 data (see online version for colours)



9 Conclusions

A new generalised family of distributions called the type-I heavy-tailed exponentiated generalised-G (TIHTEG-G) distribution was developed and presented. The density of the new family of distributions can be expressed as an infinite linear combination of exponentiated-G densities. We obtained closed form expressions for the moments, distribution of order statistics, probability weighted moments and Rényi entropy. The method of maximum likelihood estimation (MLE) was used to estimate the model parameters. Performance of the TIHTEG-LLoG distribution was examined by conducting various simulations for different sizes. The special case TIHTEG-LLoG distribution was fitted to real datasets to illustrate the applicability and usefulness of the proposed family of distributions. Risk measures were used to illustrate that the new family of distributions is heavy-tailed.

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Appendix

Partial derivatives of log-likelihod function

The first derivatives of the log-likelihood function with respect to each of the parameters in $\Delta = (a, \alpha, \beta, \psi)^T$ are given by

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial a} &= \frac{n}{a} - 2\sum_{i=1}^n \frac{\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}}{\left[1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]} \\ &+ \sum_{i=1}^n \left[1 - \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right] \\ &- \sum_{i=1}^n \left[1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right] \\ &- a\sum_{i=1}^n \frac{\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}}{\left[1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]},\end{aligned}$$

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$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log\left[1 - G(x_i;\psi)\right] - (\beta - 1) \sum_{i=1}^n \frac{\bar{G}^{\alpha}(x_i;\psi) ln \left(1 - G(x_i;\psi)\right)}{\left[1 - \bar{G}^{\alpha}(x_i;\psi)\right]} \\ &- 2 \sum_{i=1}^n \frac{\beta \left(1 - a\right) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta - 1} \bar{G}^{\alpha}(x_i;\psi) ln \left(1 - G(x_i;\psi)\right)}{\left[1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]} \\ &+ (a - 1) \sum_{i=1}^n \frac{\beta \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta - 1} \bar{G}^{\alpha}(x_i;\psi) ln \left(1 - G(x_i;\psi)\right)}{\left[1 - (1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]} \\ &- (a - 1) \sum_{i=1}^n \frac{\beta \left(1 - a\right) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta - 1} \bar{G}^{\alpha}(x_i;\psi) ln \left(1 - G(x_i;\psi)\right)}{\left[1 - (1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]}, \end{aligned}$$

$$\begin{split} \frac{\partial \ell_n(\Delta)}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log\left[1 - \bar{G}^{\alpha}(x_i;\psi)\right] \\ &+ 2\sum_{i=1}^n \frac{\left(1 - a\right)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta} \ln\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)}{\left[1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]} \\ &- (a - 1)\sum_{i=1}^n \frac{\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta} \ln\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)}{\left[1 - (1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]} \\ &+ (a - 1)\sum_{i=1}^n \frac{\left(1 - a\right)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta} \ln\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)}{\left[1 - (1 - a)\left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}\right]}, \end{split}$$

and

$$\begin{aligned} \frac{\partial \ell_n(\Delta)}{\partial \psi_k} &= \sum_{i=1}^n \frac{\left[g(x_i;\psi)\right]'}{g(x_i;\psi)} - (\alpha-1) \sum_{i=1}^n \frac{\left[G(x_i;\psi)\right]'}{1 - G(x_i;\psi)} \\ &- (\beta-1) \sum_{i=1}^n \frac{\alpha \bar{G}^{\alpha-1}(x_i;\psi) \left[\bar{G}(x_i;\psi)\right]'}{1 - \bar{G}^{\alpha}(x_i;\psi)} \\ &- 2 \sum_{i=1}^n \frac{\beta \alpha \left(1-a\right) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta-1} \bar{G}^{\alpha-1}(x_i;\psi) \left[\bar{G}(x_i;\psi)\right]'}{1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}} \\ &+ (a-1) \sum_{i=1}^n \frac{\beta \alpha \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta-1} \bar{G}^{\alpha-1}(x_i;\psi) \left[\bar{G}(x_i;\psi)\right]'}{1 - (1 - \bar{G}^{\alpha}(x_i;\psi))^{\beta}} \\ &- (a-1) \sum_{i=1}^n \frac{\beta \alpha \left(1 - a\right) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta-1} \bar{G}^{\alpha-1}(x_i;\psi) \left[\bar{G}(x_i;\psi)\right]'}{1 - (1 - a) \left(1 - \bar{G}^{\alpha}(x_i;\psi)\right)^{\beta}}, \end{aligned}$$

where $[g(x_i;\psi)]' = \frac{\partial g(x_i;\psi)}{\partial \psi_k}$ and $[\bar{G}(x_i;\psi)]' = \frac{\partial \bar{G}(x_i;\psi)}{\partial \psi_k}$.