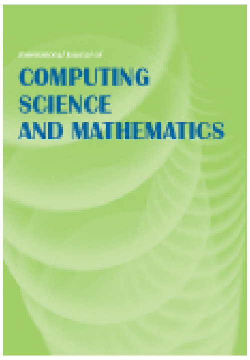

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M.P. Ramachandran, M.P. Ramachandran

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Computational advantage in evaluating oscillatory integral using quadratic spline

M.P. Ramachandran

ISRO Satellite Centre (Retired),
Esteem Classic, Industrial Suburb,
Bangalore, 560022, India
Email: mprama@gmail.com

Abstract: The quadratic spline is used in the conventional Levin's method to evaluate the oscillatory integral. Generally, the Levin method requires $O(n^3)$ computations and can be unstable. Here, the quadratic spline interpolation method requires solving recurrence relations of the derivatives of the given function and needs only $O(n^2)$ computations, where (n) is the number of selected nodes. The recurrence relations for large (n) are shown to be not ill-conditioned. Linear piecewise and cubic interpolation do not offer such advantages. The bound on the solution is obtained in terms of frequency. Numerical examples, including an application to a scattering problem, adequately illustrate the performance of the proposed method. They exhibit stability when the nodes are adequately large, unlike the conventional Levin method.

Keywords: oscillatory integral; Levin method; quadratic spline; stability; recurrence; ill-conditioned; interpolation; convergence.

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Biographical notes: M.P. Ramachandran took his Ph. D in mathematics from Indian Institute of Technology, Bombay. He then served in ISRO and retired as Group Director of Flight Dynamics Group, ISRO Satellite Centre, Bangalore. He has publications in mathematical numerical methods and aerospace.

1 Introduction

The oscillatory integral,

$$I(f) = \int_0^1 f(x)\exp\big(i\omega g(x)\big)\mathrm{d}x \tag{1}$$

given the frequency $\omega > 0$ and smooth functions $f(x)$ and $g(x)$ has wide applications. Stationary point of $g(x)$ of order one at ξ in [0,1] means

$$g^1(\xi) = 0, \tag{2}$$

where $g^1(x)$ denotes the derivative of $g(x)$.

Some applications of the integral in equation (1) are in electromagnetic problems (Delgado et al., 2007), full spectrum inversion of radio occultation of signals (Gorbunov et al., 2004), and in estimating attitude liberations of orbiting satellite (Aidoo and Osei-Frimpong, 2012).

The numerical quadrature given by Filon (1928) is efficient yet it involves the evaluation of integrals called moments. Levin (1982) brought out a quadrature technique for overcoming the evaluation of moments. A good account of other types of methods is referenced in Huybrechs and Olver (2009). We confine only to the Levin-type methods in this paper which is described as follows.

The integrand in (1.1) is expressed in terms of $F_o(x)$ which satisfies:

$$\mathrm{d/dx}\{F_o(x)\exp\left(i\omega g\left(x\right)\right)\} = f\left(x\right)\exp\left(i\omega g\left(x\right)\right) \tag{3}$$

and when expanded is

$$[F_o{}^1(x)+i\omega g^1(x)F_o(x)]\exp\left(i\omega g\left(x\right)\right) = f\left(x\right)\exp\left(i\omega g\left(x\right)\right) \tag{4}$$

Equation (4) is the differential equation

$$F_o{}^1(x)+i\omega g^1(x)F_o(x) = f\left(x\right) \tag{5}$$

Here, it is assumed that $g^1(x)$ does not have stationary point, that is $g^1(x) \neq 0$, for all x in [0,1] as in Levin (1982). The collocation method is applied in solving the non-oscillatory differential equation. Boundary conditions are not present with equation (5) as the non-oscillatory particular solution, $F_o(x)$ alone is sufficient. The function $F_o(x)$ is approximated by polynomial interpolation either over the entire interval [0,1] or piecewise over meshes selected in [0,1]. We apply n-point collocation method at the selected nodes which then leads to a system of (n) linear equations.

$$\boldsymbol{A_0}\,\boldsymbol{\alpha} = \boldsymbol{f} \tag{6}$$

Based on the solution to the linear system in equation (6), the computed values of $F_o(0)$ and $F_o(1)$ are further used to determine the integral in equation (1) by using the relation:

$$I\left(f\right) = F_o(1)\exp(i\omega\, g(1) - F_o(0)\exp(i\omega\, g(1)) \tag{7}$$

The conventional Levin method tends to be ill-conditioned. The solution oscillates as illustrated by Li et al. (2010) and Motygin (2017) when the size of the system of equations increases. It has also been observed Levin method is instable for low frequencies (Olver, 2006).

We outline a few Levin-type methods in the literature that overcome instability due to collocation or achieve a reduction in computation. Singular value decomposition applied in Li et al. (2010) and Motygin (2017) using clustered Chebychev points at the ends – increase the stability for large (n). A Levin-type method in Olver (2006) uses multiple collocations along with the derivatives in equation (5) to achieve high asymptotic convergence but it does not offer stability while Olver (2010) enhances stability and convergence. Meshless approaches are seen in Ma and Duan (2019) and in Geng and Wu (2021). Sevastianov et al. (2020) use the Gauss-Lobatto points to reduce the computations whenever $g(x)$ is linear. Here, we shall have a reduction without any such condition.

The direct Levin (1982) and above-mentioned methods (Iserles and Norsett, 2005; Olver, 2010), all need to solve a $(n \times n)$ system of equations and require $O(n^3)$ computations. Matrix inversion is necessitated in all these methods. Transformation and matrix multiplications in these formulations add more computations to mainly overcome instability in conventional Levin methods. Here we shall achieve stability directly and with lesser computations.

This paper uses quadratic spline in interpolating the function $F_o(x)$. When nodes are equally spaced then equation (6) is shown to reduce to a system of equations that is lower diagonal. Evaluation of the integral in equation (1) then requires solving recurrence relation instead of solving a linear system of equations and thus reduces the computations. Next, the stability of the recurrence relations is analysed for larger values of (n) to assure that the method is not ill-conditioned. The proposed method offers two advantages, namely reduction in computation as well as stability. It may be noted that the first advantage is particularly useful when using a limited processor without using matrix inversions.

The paper is organised as follows: Section 2 describes the quadratic spline interpolation. In Section 3, this is used in equation (4) to obtain the recurrence relations. Here, the formulation carefully avoids any further computation and shows that the recurrence relation solution obtains the approximate value of the integral (1). In Section 4, the stability analysis of the recurrence relation is demonstrated to prove that the method is well-conditioned. Also, the bound of the solution in terms of frequency as in any Levin method is obtained. Here, it is noted that the linear and cubic spline are shown not to offer the same advantages. As a remark, the extension of the approach in the evaluation of two-dimensional integral is given. Finally in Section 5, we consider examples that illustrate the efficiency and the computational stability of the proposed method. This ensures trust, which is primary of any numerical scheme.

2 Quadratic spline interpolation

The (n) nodes selected on [0,1] are:

$$0 = x_1 < x_2 < \cdots < x_n = 1 \tag{8}$$

The approximation of F_o on the nodes is denoted as y_i; $i = 1,2,., \text{n}$. The piecewise quadratic spline interpolation $\text{S}(x) = S_j(x)$ over the interval $[x_j, x_{j+1}]$; $j = 1,2, \ldots, (n-1)$ with the derivative $S_j^1(x_j) = m_j$ is given as (Behforooz, 1998):

$$S_j(x) = a_j (x - x_j)^2 + m_j (x - x_j) + y_j \tag{9}$$

where

$$a_j = (y_{j+1} - y_j - h_{j+1} m_j) / h^2_{(j+1)} \tag{10}$$

and the step sizes are given by:

$$h_{j+1} = x_{j+1} - x_j \tag{11}$$

The step size is set equal, that is

$$h_{j+1} = h \tag{12}$$

that is, $h = (1/(n-1))$. This makes the proposal simpler. An explanation regarding unequal step sizes shall be made in Section 4.

The consistency relation on the derivatives m_j at the nodes between the adjacent intervals is the continuity condition and given in Behforooz (1998):

$$m_{k+1} + m_k = (2/h)\,(y_{k+1} - y_k);\ k = 1, 2, \ldots, (n-1) \tag{13}$$

The $(n-1)$ equations in equation (13) are usually associated with an additional end condition. This helps to fully describe the quadratic spline. We usually set m_1, either to a known value available from the boundary value problem or approximate value is assigned, this will be later mentioned in equation (17).

3 Computational advantage

In equation (3), we substitute F_o, with $(F(x) - F(x_1))$ and obtain:

$$d/dx\{(F(x) - F(x_1))\exp(i\omega g(x))\} = f(x)\exp(i\omega g(x)) \tag{14}$$

which is,

$$F^1(x) + \{F(x) - F(x_1)\}\ i\ \omega\ g^1(x) = f(x). \tag{15}$$

The integral $I(f)$ in equation (1) is got by integrating (14) and is:

$$I(f) = \{F(x_n) - F(x_1)\}\ \exp(i\omega g(x_n)) \tag{16}$$

The function $F(x)$ is now approximated by the quadratic spline as in equation (9) over each of the $(n-1)$ intervals $[x_j, x_{j+1}]$; $j = 1,2,\ldots, (n-1)$. Let m_k and y_k denote the approximation of $F^1(x_k)$ and $F(x_k)$ respectively. When setting $x = x_1$ in equation (15) we have:

$$m_1 = f(x_1). \tag{17}$$

As mentioned earlier, this end condition supplements the $(n-1)$ equations in equation (13) to uniquely determine m_k, $k = 1,2, \ldots, (n)$.. The linear system obtained from equations (15) using collocation at $x = x_j$; $j = 2,3, \ldots, (n)$ gives rise to :

$$m_j + (y_j - y_1)\ i\omega\ g^1(x_j) = f(x_j),\ j = 2,3, \ldots, (n). \tag{18}$$

In equation (18), we write,

$$(y_j - y_1) = \sum_{k=1}^{(j-1)} (y_{(k+1)} - y_k). \tag{19}$$

On the other hand, using equation (13) we have:

$$y_{k+1} - y_k = (h/2)\,(m_{k+1} + m_k) \tag{20}$$

First, we substitute the right-hand side of equation (19) with the relation (20) and next use it in equation (18) to eliminate the y's and finally get:

$$\{1 + i\,\alpha_j\}\, m_j + i\,\alpha_j \sum_{k=1}^{(j-2)} (m_{(k+1)} + m_k) + i\,\alpha_j\,(m_{j-1}) = f(x_j);$$

$$j = 2,3, \ldots, n \tag{21}$$

where

$$\alpha_j = h\,\omega\,g^1\,(x_j)/2 \tag{22}$$

We find that equation (17) along with the system of $(n-1)$ equations in equation (21) forms a lower triangular system of equations. The solution m_j; $j = 1,2,3..(n)$ requires only $O(n^2)$ computations. The first mentioned advantage (a) of reducing the computational task while applying the Levin method has been established.

The procedure to solve the system (21)–(22) is as follows. We know m_1 from (17). Using this value of m_1 in equation (21) when in ($j = 2$), the summation vanishes and we easily determine m_2. Subsequently when ($j = 3$), Equation (21) is:

$$\{1 + i\,\alpha_3\}\, m_3 + i\,\alpha_3\,(m_2 + m_1) + i\,\alpha_3\,(m_2) = f(x_3),$$

which helps in determining m_3 using both m_1 and m_2, that are now known besides α_3 and $f(x_3)$ which are respectively known in equations (22) and (1). Similarly, at the jth equation we solve for m_j using the previous values of m_p, $p = 1, 2, \ldots (j-1)$. Finally when $j = n$, all the derivatives m_i; $i = 1, 2, \ldots n$ are computed. This is simple even as the system in equation (21) involves complex numbers.

On the other hand the approximate value of the integral $I(f)$ in (16) is denoted as $Q_n(f)$. Similar to the relation seen in equation (19) we have:

$$Q_n(f) = \sum_{k=1}^{(n-1)} (y_{(k+1)} - y_k)\ \exp\left(i\omega g\left(x_n\right)\right), \tag{23}$$

Substituting the relation in equation (20) in the above equation and then using the solution of equation (21) that is, the derivatives m_i; $i = 1,2,\ldots(n)$, the approximate value of the integral becomes:

$$Q_n(f) = (h/2) \sum_{k=1}^{(n-1)} (m_{(k+1)} + m_k)\ \exp\left(i\omega g\left(x_n\right)\right) \tag{24}$$

Thus, the integral $I(f)$ which is approximated by applying quadratic spline in the Levin approach finally enables in finding $Q_n(f)$. It is interesting to note that the determination of $(m_1, m_2, m_3, \ldots, m_n)$ alone is sufficient and any additional computations in obtaining y_j ($j = 1,.2, \ldots, n$) are not required to evaluate $I(f)$.

4 Stability, well condition and boundedness

In this section, the transformation that was applied on equation (21) to obtain the solution $\boldsymbol{m}$, is used to examine the stability and the condition number as the dimension (n) increases. Note that (17) and (21) are represented as:

$$\boldsymbol{A}\,\boldsymbol{m} = \boldsymbol{f} \tag{25}$$

where **A** ($n \times n$) is a lower triangular matrix. In the jth equation we derive m_j using previously computed m_r; $r = (j-1), \ldots, 2$ and m_1. We adopt the algebraic substitution or elementary transformation on (25). This retains m_1 and diagonal elements. The reduced form is:

$$\boldsymbol{B}\,\boldsymbol{m} = \boldsymbol{p} \tag{26}$$

Here, $\boldsymbol{m} = [m_1, m_2, \ldots, m_n]$ and the elements of $B = [b_{ij}]$ deduced by algebraic manipulation and given by:

$b_{11} = 1$, $b_{1j} = 0$; $j = 2, \ldots, n$; $p(x_1) = f(x_1)$

$b_{21} = i\,\alpha_2$, $b_{22} = (1 + i\,\alpha_2)$, $b_{2j} = 0 \quad \forall j \neq 1$ and 2; $\mathrm{p}(x_2) = f(x_2)$

$b_{31} = i\,\alpha_3\,(1 - i\,\alpha_2)/(1 + i\,\alpha_2)$, $b_{33} = (1 + i\,\alpha_3)$,

$b_{3j} = 0$; $\forall \mathrm{j} \neq 1$ and 3; $\mathrm{p}(x_3) = f(x_3) - (2i\,f(x_2)\,\alpha_3)/(1 + i\,\alpha_2)$

$b_{41} = i\,\alpha_4\,(1 - i\,\alpha_2)(1 - i\,\alpha_3)/(1 + i\,\alpha_2)(1 + i\,\alpha_3)$, $b_{44} = (1 + i\,\alpha_4)$, $b_{4j} = 0$; $\forall j \neq 1$ and 4;

$p(x_4) = f(x_4) - (2i\,f(x_3)\,\alpha_4)/(1 + i\,\alpha_3) - 2i\,f(x_2)\,\alpha_4\,(1 - i\,\alpha_3)/$
$((1 + i\,\alpha_2)(1 + i\,\alpha_3))$

$b_{51} = i\,\alpha_5\,(1 - i\,\alpha_2)(1 - i\,\alpha_3)(1 - i\,\alpha_4)/((1 + i\,\alpha_2)(1 + i\,\alpha_3)(1 + i\,\alpha_4))$,

$b_{55} = (1 + i\,\alpha_5)$

$b_{5j} = 0$; $\forall j \neq 1$ and 5;

$p(x_5) = f(x_5) - (2i\,f(x_4)\,\alpha_5)/(1 + i\,\alpha_4) - 2i\,f(x_3)\,\alpha_5\,(1 - i\alpha_4)/((1 + i\,\alpha_3)$
$(1 + i\,\alpha_4)) - 2i\,f(x_2)\,\alpha_5\,(1 - i\,\alpha_4)(1 - i\,\alpha_3)/((1 + i\,\alpha_2)\,(1 + i\,\alpha_3)\,(1 + i\alpha_4))$

Based on inductive reasoning the above algebraic relations allow us to write the square matrix $\boldsymbol{B}$ in a more general form as:

$$\left.\begin{array}{l} m_1 = p(1) \\ i\,\beta_j \alpha_j m_1 + (1 + i\,\alpha_j)\,m_j = p(j);\; j = 2, \ldots, (n) \text{ where } |\beta_j| \le 1 \end{array}\right\} \tag{27}$$

We have transformed the linear equations (17) and (21) into (27). This is equivalent to the actual computational steps of elimination that we had already described while determining (**m**). The system in equation (27) is evidently diagonally dominant and hence stable. It is easy to notice in equation (12) that for a selected frequency, as (n) increases the value of (h) diminishes. If $g^1(x)$ is bounded, then we find that α_j in equation (22) also tends to zero. However, the diagonal element in equation (27) tends to unity. Hence the Levin method using quadratic spline for approximation is not ill-conditioned when (n) is large. This establishes the second advantage namely (b) that the method of applying quadratic spline in Levin method is not ill-conditioned.

The above analysis applies when the nodes are equally spaced and as set in Equation (12). However, when the step sizes in equation (11) are unequal, then with mathematical manipulations it can be shown that the system of Equations in equation (26) will not be ill-conditioned if:

$$h_2 \le h_3 \le h_4 \le \cdots \le h_n$$

The purpose of the method here is in obtaining a simpler method and hence the choice of equal step size is adopted for convenience.

Next in this section we analyse the bounds of the solution of the equation (27), namely ($\boldsymbol{m}$), in terms of the frequency ω.

$$\| \boldsymbol{m} \|_\infty = \| \sum_j (b_{ij})^{-1} p(j) \|_\infty \text{ denotes maximal absolute row sum)}$$

$$\le \max_{1\le i\le n}\{ \sum_j |(b_{ij}^{-1})||p (j)|\}$$

$$\le \|\mathbf{B}^{-1}\|_\infty \|\mathbf{p}\|_\infty \tag{28}$$

We use the result from (Varah, 1975) to get a bound on $\|\mathbf{B}^{-1}\|_\infty$.

$$\|\boldsymbol{B}^{-1}\|_\infty \le \max_j \{1/(|(1 + i\, \alpha_j)| -| i\, \alpha_j |)\} \tag{29}$$

The bounds of the matrix in terms of ω is

$$= O\,(\alpha_j)^{-1} \tag{30}$$

We notice in equation (22) that when $g^1(x)$ is bounded over [0,1], then the right hand side of equation (27) is $O(1)$. Hence the bound in equation (30), in terms of the frequency ω, is $O(\alpha_j)^{-1}$, which is $O(\omega)^{-1}$ in equation (28). This result is agreeable to that in Levin (1997) and shall be illustrated too.

Finally, we note that as the solution **m** in equation (26) is bounded, the approximation Q_n(f) in equation (24) is also bounded whenever $g^1(x)$ is smooth in [0,1]. Convergence depends on the quadratic spline but the interesting advantage is the reduction in computations.

Now we shall use linear polynomial $L(x) = L_j(x)$ on $[x_j, x_{j+1}]$ applied to approximate $F(x)$ instead of quadratic spline.

$$L_j(x) = m_j\,(x - x_j) + y_j, j = 1, 2, \ldots, (n - 1) \tag{31}$$

where the slope $m_j, j = 1,2, \ldots, (n - 1)$ is given by:

$$m_j = (y_{j+1} - y_j)/h; j = 1,2, \ldots, (n - 1) \tag{32}$$

The formulation is similar to that of quadratic spline:

$$F^1(x_j) = (y_{j+1} - y_j)/h; j = 1,2,\ldots(n - 1)$$

When we set $x = x_1$ in equation (15) and then use (32) we have:

$$(y_2 - y_1)/h = f(x_1) \tag{33}$$

Again choosing the collocation nodes as $x = x_j; j = 2,3, \ldots, (n - 1)$ we have the system of $(n-2)$ equations:

$$((y_{j+1} - y_j)/h) + i\,(y_j - y_1)\,\omega\, g^1(x_j) = f(x_j); j = 2,3,\ldots, (n - 1) \tag{34}$$

Applying the summation (19) in equation (34), we have finally have

$$((y_{j+1}-y_j)/h) + \sum_{k=1}^{(j-1)} (y_{(k+1)} - y_k)\ i\ \omega\ g^1(x_j) = f(x_j);\ j = 2,3,\ \ldots,\ (n) \tag{35}$$

In the above equations we treat the term $(y_{j+1}-y_j)$; $j = 2,3, \ldots, (n-1)$, as unknowns. Later this solution is used in equations (16) and (23) to directly obtain the approximate value of the integral $I(f)$. It is easy to observe (33) and (35) are lower diagonal systems. It can be solved as in the case of quadratic spline by recurrence relations. Yet we notice that the diagonal element is $(1/h)$ whereas the sum of absolute values of the (j) non-diagonal elements is j (α_j/h). We observe in equation (34) that the (especially when $j = (n-1)$) need not be row-diagonally dominant. It may not be computationally stable and can be ill-conditioned. On the other hand, if cubic spline were to be used then we notice that the continuity conditions in cubic spline (Ahleberg, 1967), which is similar to (13), does not offer a reduction in computational reduction in solving $(n\ x\ n)$ equations, that was seen in the case of the quadratic spline.

Remark: Consider evaluating a two-dimensional integral defined over a rectangle:

$$J(f) = \int_a^b \int_c^d f(x,y) \exp\left(i\omega g(x,y)\right) \mathrm{d}x\mathrm{d}y$$

Applying generalised Stokes theorem, the integration has been reduced to one dimension integral with the suitable function G(x, y), as in equation (1):

$$I[G_x + i\omega\ g_x\ G - G_y - i\omega\ g_y\ G]$$

The subscript denotes the partial derivatives and this has been discussed in detail in the thesis (Olver, 2008). The quadratic spline as described here is applied when there are no stationary points. The reduction in computation becomes more significant when using quadratic spline because we need to evaluate four integrals (Olver, 2008) of the type as in equation (1).

5 Examples

Appropriate examples are taken from literature that do not have stationary points.

Example 1: In this example the integral value is known analytically and hence is used here to illustrate the order of error bounds. This case helps us to understand the stability of the proposed method. The following integral is known in closed form (Iserles and Norsett; 2005):

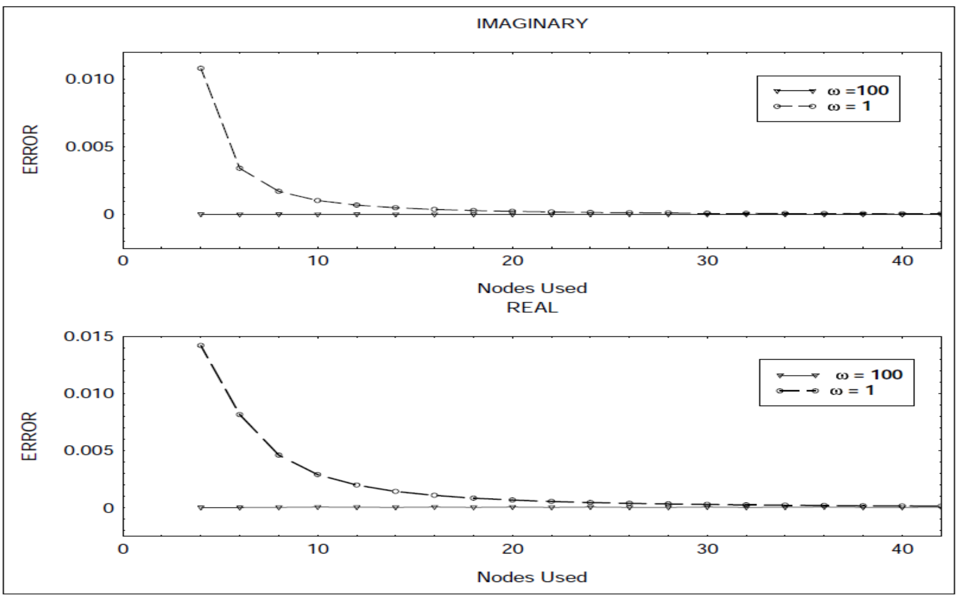
$$\int_0^1 x^3 \ \exp\left(i\omega x^2\right) \mathrm{d}x =$$

$$(-1/2\ \omega^2)\{(-\omega \sin(\omega) - \cos(\omega) + 1),\ i(\omega \cos(\omega) - \sin(\omega)\}$$

We note that $g(x)$ has a stationary point at $x = 0$, however the value of the $f(x) = x^3$ also vanishes there. Here, the node x_1 in equation (8) is $x = 0$. We examine two cases, $\omega = 1$ and 100 to understand the dependence of the convergence on the frequency. In Figure 1, the absolute difference or error between the present approach and the exact value, that is

$|I(f) - Q_n(f)|$ is depicted. The error for $\omega = 100$, is shown by thick line and agrees in four decimal places from $n = 4$. As (n) is increased, the convergence to the exact values is found to be slow when $\omega = 1$ and is better for the case when, $\omega = 100$. This is as expected from our observations related to the bounds in equation (30). The results here clearly assure that the quadratic spline as Levin method is well condition irrespective of the frequency ω. It is unlike the observation made about the Levin method in Olver (2006) for the lower frequency. There was no instability seen even when (n) is increased up to a large value as 4000 for both the choice of frequencies. This showcases the trustworthiness of the method.

Figure 1 Error in quadratic spline approach with increasing nodes and frequency



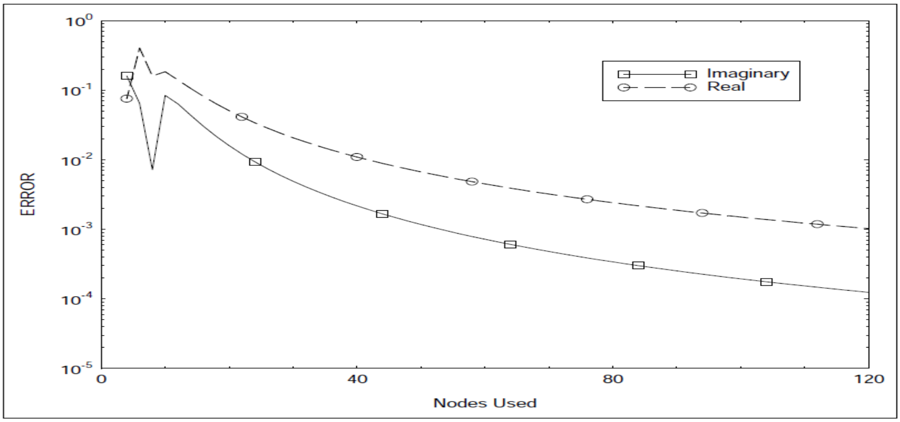
Example 2: Finally consider an application of the scattering problem (Ishimaru, 1978). This example is seen in Li et al. (2010) to remarkably illustrate that Levin's method (Levin, 1997) with the increasing number of nodes (n) does not have stability whereas the method proposed by Li has. Here, we show that the proposed Levin method is stable when we apply the quadratic spline.

$$\int_1^2 (\cos(10x^2) + (10/(1+10x)) \exp(i(10^7 + 10^4 x^2)^{1/2})\, dx = U + i\,V$$

where $U = 0.020332995$ and $V = -0.2160716948$. However, the performance of the Levin method given by Li et al. (2010) can be seen to clearly exhibit certain oscillations when the number of nodes used is close to 40. To describe, the error in the conventional Levin method given in Li et al. (2010) improves when the nodes (n) increase from ($n = 34$) to (n = 36). However, when ($n = 38$), the error is about 10 times worse than it is when ($n = 36$). This reflects that the method cannot be trusted. Whereas the results from the proposed method, as seen in Figure 2, using quadratic spline show no such behaviour. The accuracy gradually improves as the number of nodes increases. The nodes in

Figure 2 are in the steps of $n = 2$. This amply demonstrates the advantages of using quadratic spline as a Levin's method and consistent to the stability analysis.

Figure 2 The behaviour of quadratic spline in Levin method with increasing nodes



These examples amply demonstrate the stability and convergence of the proposed methods for a higher choice of (n) and that without requiring matrix inversion.

6 Conclusions

Quadratic spline is used in the Levin method to evaluate one-dimensional oscillatory integral. The proposed approach requires the evaluation of recurrence relations and demands one order lesser computation than the conventional Levin approach. This advantage is particularly useful when the processor has limitations and matrix inversion is not feasible. The stability of the recurrence relations is then analysed and the system is shown to be not ill-conditioned. The advantage is outlined in extending the approach while evaluating the two-dimensional integral. The paper obtains the error bounds in terms of frequency values. Numerical examples clearly illustrate the advantages namely stability and convergence, as the nodes are increased. This paper focuses on a Levin method though other emerging methods with additional computations have achieved both stability and better convergence.

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