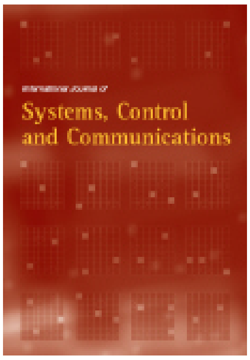


International Journal of Systems, Control and Communications

ISSN online: 1755-9359 - ISSN print: 1755-9340
https://www.inderscience.com/ijscc

Synchronisation scheme for cluster-based interconnected network of nonlinear systems

Akshay Kumar Jaiswal, Bharat Bhushan Sharma

DOI: 10.1504/IJSCC.2023.10057873

Article History:

Received:	23 December 2022
Last revised:	22 March 2023
Accepted:	31 May 2023
Published online:	01 December 2023

Synchronisation scheme for cluster-based interconnected network of nonlinear systems

Akshay Kumar Jaiswal* and
Bharat Bhushan Sharma

Department of Electrical Engineering,
National Institute of Technology,
Hamirpur, Himachal Pradesh, India
Email: akshayjswl09@gmail.com
Email: bhushan@nith.ac.in
*Corresponding author

Abstract: The work presented here addresses cluster synchronisation for a class of nonlinear systems using Lyapunov stability theory with nonlinearities satisfying Lipschitz condition. Generally, Lyapunov stability-based adaptive nonlinear control techniques are used to design the controller for nonlinear systems. These techniques are also utilised to address synchronisation in complex interconnected systems. The cluster synchronisation in a complex network of different nonlinear systems is achieved when each state of system of one cluster is synchronised to every corresponding state system of other cluster. Here, using Lyapunov stability theory, a general criterion for cluster synchronisation is obtained. For meeting the goal of synchronisation, bidirectional connections within a cluster and across the clusters are considered. To achieve the results, nonlinearities are assumed to satisfy Lipschitz conditions. The appropriate design of gains for within cluster and across the cluster coupling using Lyapunove stability theory, along with application of Barbalat's lemma, ensure synchronisation of an overall network consisting clusters of dissimilar nonlinear systems. Numerical simulation are presented further for example systems belonging to the considered class of nonlinear systems to verify the efficacy of the proposed approach. For this purpose, cluster of chaotic Lorenz and Lu systems are considered as part of complex network.

Keywords: cluster synchronisation; complex dynamic network; Lyapunov stability; Lipschitz condition.

Reference to this paper should be made as follows: Jaiswal, A.K. and Sharma, B.B. (2024) 'Synchronisation scheme for cluster-based interconnected network of nonlinear systems', *Int. J. Systems, Control and Communications*, Vol. 15, No. 1, pp.1–16.

Biographical notes: Akshay Kumar Jaiswal received his BE in Electrical Engineering from the Rewa Engineering College, Rewa, Madhay Pradesh, India, in 2020. He is presently enrolled in Master's in Signal Processing and Control at the NIT Hamirpur, India, commencing in 2021. His research focuses on areas such as control, nonlinear systems, cluster synchronisation of nonlinear systems, chaotic systems, and the development of nonlinear observers. These research interests highlight his inclination towards exploring the dynamics and synchronisation of complex systems.

Bharat Bhushan Sharma received his BTech in Electrical Engineering from the Regional Engineering College Hamirpur, India, in 1997. Further, he received his Master's in Engineering in Electronics and Control from the BITS Pilani, India, in 1999. He received his PhD in Electrical Engineering from the Indian Institute of Technology Delhi, India, in 2010. His research interests mainly include control and synchronisation of nonlinear systems, chaotic dynamics, design of nonlinear observers, etc. He has published more than 100 papers in various international journals and conferences of repute in these areas. He is a member of IEEE Control System Society, Institution of Engineers, India and Robotic Society of India. He is presently working as an Associate Professor with the Department of Electrical Engineering, National Institute of Technology Hamirpur (Himachal Pradesh), India.

1 Introduction

Understanding the dynamics of nonlinear systems and their control have been the primary focus of research community over the years. The design of appropriate nonlinear control schemes to and synchronise stabilise the behaviour of such systems with varied nonlinearities and complicated input-output interaction has been widely explored in Krstic et al. (1995), Yeh and Kokotovic (1995), Ge et al. (2000), Sharma and Kar (2011), Chen et al. (2015), Precup et al. (2017), Liu and Zhu (2021, 2022) and Bora and Sharma (2022). Along with control, synchronisation of complex networks of nonlinear systems has also been widely expanded and has become much studied topic in control literature in last few decades. Cluster synchronisation of complex networks find applications in areas including biology, physics, electronics, computer graphics, etc. A complex network may consist of many clusters which involve connections of nodes of one cluster to nodes of other clusters. There are many research contributions showing close linkage between the cluster synchronisation for different network topologies. Synchronisation of complex networks is approached by employing various nonlinear control techniques over the years. For example, adaptive feedback control used in Wu et al. (2012) and Pham et al. (2019), backstepping-based control explained in Changder and Sharma (2022), pinning control adopted in Liu et al. (2007) and Wang and Chen (2002), intermittent control-based synchronisation as elaborated in Xia and Cao (2009), contraction-based synchronisation in chain networks given in work of Chauhan et al. (2021), and many more. Furthermore, pinning synchronisation is an effective method for managing the collective dynamic behaviours of networked systems as elaborated in Wang et al. (2020). This method has been used for global synchronisation in regular networks, including ring structured networks, with global connectivity. In real time scenario, it is a challenging task to achieve synchronisation in complex networks involving non-identical systems in different topologies. Adaptive control method given in work of Chen et al. (2016) for complete synchronisation of multiple nonlinear systems consists of many controllers which results in increased cost and size of the system. So, in order to reduce the number of controllers, the pinning control methodology is one of the best and efficient method as compared to adaptive control or any other method used for cluster synchronisation as given by Wu et al. (2008). The idea of pinning control was developed to promote and implement the control approach to address synchronisation for complex networks as explored in Sun et al. (2015), DeLellis et al. (2018), Nian and Wang (2011), Wang and Shen (2013), Cai et al. (2015) and Sun et al. (2019). For instance, by using Schur complement and Lyapunov stability theory, pinning control method was designed to realise the lag synchronisation between two nonlinear coupled networks as given by Sun et al. (2015). Pinning control method is also used in meeting out synchronisation in directed networks as elaborated in Nian and Wang (2011). A directed dynamical network with non-identical nodes was driven to cluster synchronisation using the pinning control mechanism in Sun et al. (2015). Similarly, for given directed heterogeneous dynamical network, intermittent pinning control problem has been described in work of Cai et al. (2015). Keeping in mind that many real networks cannot have arbitrarily huge coupling strength, Lyapunov stability-based method using Lipschitz condition as given by Nian and Wang (2011) is considered as a popular method for achieving cluster synchronisation. In Wang et al. (2012), cluster synchronisation of nonlinearly coupled complex networks with non-identical nodes and asymmetrical unidirectional coupling matrix has been discussed. Cluster synchronisation in case of complex networks involving non-identical clusters with one-way coupling is explored in Sun et al. (2019).

Cluster synchronisation involving the synchronisation of sub-network in a particular network has drawn lots of interest in recent years as a unique synchronisation phenomenon described as in Belykh et al. (2008), Zhang et al. (2001), Belykh et al. (2001), Kouomou and Woafo (2003), Kouomou and Woafo (2003) and Jalan et al. (2005). In fact, the concept of cluster (or partial) synchronisation is relatively new. The first study of cluster synchronisation was based on a natural physical phenomena that occurred in coupled oscillators and coupled map lattices. It refers to oscillator cluster synchronisation with one another but the entire group is not synchronised as elaborated in work of Belykh et al. (2008). Apart from cluster synchronisation, evidently the linked systems can exhibit the well known spatiotemporal chaos and partial synchronisation states as explored in Zhang et al. (2001). Amritkar et al. (2005) and Belykh et al. (2005) studies about coupled map network synchronisation and persistent synchronisation in lattices of coupled nonidentical chaotic system are discussed in detail. With the advancement in modern science and technology, cluster synchronisation is applied to biological engineering and secure communication application as well as highlighted in Qin and Chen (2004) and Ma et al. (2006).

Till date, the contributions towards cluster synchronisation are mainly based on complex networks of identical systems and very limited exploration has been done towards network with non-identical systems in different clusters. However, several real-world networks are based on multiple oscillators with non-identical dynamical properties. Motivated by these, in the present work, cluster synchronisation in dynamical networks with non-identical clusters is formulated. Here, a complex network with two non-identical clusters is considered in which each cluster contains identical systems, but systems in second cluster differ from first cluster, i.e., first cluster is different from second cluster in terms of nonlinear systems involved in it. Further, to derive above results, nonlinearities of systems are considered to be bounded with Lipschitz condition. In general, we present cluster synchronisation in a network having connections within cluster and across the cluster, i.e., the case with inner connection and outer connections with all connections assumed to have bidirectional coupling. Using Lyapunov stability theory blended with Barbalat's lemma, gains for coupling of the systems within cluster and across the clusters are appropriately selected to achieve the synchronisation with such gain design. The desired analytical results ensure, matching of corresponding states of all the systems across the clusters. The proposed results are successfully validated for example network consisting of clusters of Lorenz and Lu chaotic systems, respectively.

The manuscript is outlined as follows: in Section 2, main results for network synchronisation of proposed class of nonlinear systems are discussed. In Section 3, bidirectional all-to-all synchronisation of network of chaotic systems, across and within clusters, is discussed in detail. In Section 4, simulation result are shown and the proposed work is concluded in Section 5.

2 Main results

For meeting out the objective of cluster synchronisation, two groups of non-identical systems are considered to constitute a network:

First group of nonlinear systems is given as

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{\phi}\mathbf{x}_i(t) + \mathbf{f}(\mathbf{x}_i, t); \tag{1}$$

where $i = 1, 2, 3, ..., N_1$; and second group of nonlinear systems is described as

$$\dot{\mathbf{x}}_j(t) = \boldsymbol{\theta}\mathbf{x}_j(t) + \mathbf{g}(\mathbf{x}_j, t); \tag{2}$$

where $j = (N_1 + 1), (N_1 + 2), (N_1 + 3), ..., (N_1 + N_2)$.

For nonlinear systems in equations (1) and (2), $\mathbf{x}_i \in \Re^n$, $\mathbf{x}_j \in \Re^n$ are state vectors of systems with associated matrices involving parameters given by $\boldsymbol{\phi} \in \Re^{n\times n}$ and $\boldsymbol{\theta} \in \Re^{n\times n}$, respectively. The respective nonlinear functions of these groups are represented as vector functions given by, $\mathbf{f}(\mathbf{x}_i, t) \in \Re^n$ and $\mathbf{g}(\mathbf{x}_j, t) \in \Re^n$, respectively.

Remark 1: The complex network may be assumed to be consisting of M nodes which are divided into m-clusters C_1, C_2, ..., C_m, with each cluster having same or different numbers of nodes.

Now, consider two clusters, each consisting of N numbers of nodes and each node is an n-dimensional dynamical system. The local dynamics of individual nodes in each group are identical, while both clusters are different from each other. For example, we may take first cluster having chaotic Lorenz system as its members and second cluster as chaotic Lu systems. Accordingly, in these groups, the coupling equation of systems in first cluster can be written in the following general form:

$$\begin{aligned}\dot{\mathbf{x}}_i(t) = \boldsymbol{\phi}\mathbf{x}_i(t) + \mathbf{f}(\mathbf{x}_i, t) + k_1\left[\sum_{m=1,m\neq i}^{N_1} \mathbf{x}_m(t) - \Gamma\mathbf{x}_i(t)\right] \\ + k_2\sum_{h=1+N_1}^{N_1+N_2}(\mathbf{x}_h(t) - \mathbf{x}_i(t))\end{aligned} \tag{3}$$

where index $i = 1, 2, 3, ..., N_1$.

By adding and subtracting $\boldsymbol{\theta}\mathbf{x}_i(t)$ in equation (3), we get

$$\dot{\mathbf{x}}_i(t) = \boldsymbol{\phi}\mathbf{x}_i(t) + \boldsymbol{\theta}\mathbf{x}_i(t) - \boldsymbol{\theta}\mathbf{x}_i(t) + \mathbf{f}(\mathbf{x}_i, t) + k_1\left[\sum_{m=1, m\neq i}^{N_1} \mathbf{x}_m(t) - \Gamma\mathbf{x}_i(t)\right] + k_2 \sum_{h=N_1+1}^{N_1+N_2} (\mathbf{x}_h(t) - \mathbf{x}_i(t)) \tag{4}$$

Similarly, the description of 2^{nd} cluster systems can be written as

$$\dot{\mathbf{x}}_j(t) = \boldsymbol{\theta}\mathbf{x}_j(t) + \mathbf{g}(x_j, t) + k_1\left[\sum_{k=N_1+1, k\neq j}^{N_1+N_2} \mathbf{x}_k(t) - \Gamma\mathbf{x}_j(t)\right] + k_2\left[\sum_{l=1}^{N_1}(\mathbf{x}_l(t) - \mathbf{x}_j(t))\right]; \tag{5}$$

where index $j = (N_1 + 1), (N_1 + 2), (N_1 + 3), ..., (N_1 + N_2)$.

Also $\boldsymbol{\phi}\mathbf{x}_i(t)$ represents linear part of dynamics of individual nodes where vector $\mathbf{x}_i$ has elements $[\mathbf{x}_{i1}, \mathbf{x}_{i2}, ..., \mathbf{x}_{in}]$ and $\mathbf{f}(\mathbf{x}_i, t)$ is nonlinear part of dynamical system of cluster one. On the other hand, $\boldsymbol{\theta}\mathbf{x}_j(t)$ represents linear part of individual nodes where vector $\mathbf{x}_j$ has elements $[\mathbf{x}_{j1}, \mathbf{x}_{j2}, ..., \mathbf{x}_{jn}]$ and $\mathbf{g}(\mathbf{x}_j, t)$ is nonlinear part of dynamical system of cluster two and Γ is a constant matrix of size $n \times n$. For simplicity, the constants $k_1 \geq 0$ and $k_2 \geq 0$ are used to represent coupling strength of within cluster and across the cluster connections, respectively.

Synchronisation error vector is defined as:

$$\mathbf{e}_{ij}(t) = \mathbf{x}_j(t) - \mathbf{x}_i(t) \tag{6}$$

where index $i = 1, 2, 3, ..., N_1$; and index $j = 1, 2, 3, ..., (N_1 + N_2)$.

Here, $\mathbf{x}_j(t)$ denotes cluster-2 and $\mathbf{x}_i(t)$ represents cluster-1.

The proposed class of nonlinear systems, which are considered member of each cluster, satisfy the following assumption:

Assumption 1: The nonlinearities associated with system of cluster-1, i.e., $\mathbf{f}(\mathbf{x})$ and that of cluster-2, i.e., $\mathbf{g}(\mathbf{x})$ are bounded by Lipschitz constant (γ), given as:

$$||\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{x})|| \leq \gamma||\mathbf{y} - \mathbf{x}|| \tag{7}$$

$$\Rightarrow ||\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{x})||^2 \leq \gamma^2||\mathbf{y} - \mathbf{x}||^2 \tag{8}$$

$||.||$ is considered to be the Euclidean norm in $\Re^n$.

Assumption 2: The state vector **x**(t) of systems involved in cluster-1 and cluster-2 is also norm bounded.

Remark 2: The class of systems described in equations (1) and (2) are considered to have underlying nonlinearities as smooth functions. Majority of nonlinear systems qualify the above requirement and, thus, either locally or globally Lipschitz in nature. Moreover, certain category of systems, like chaotic or hyperchaotic systems, have bounded state vector for given initial conditions or certain range of parameter values. Thus, Assumptions 1 and 2 are quite reasonable for the proposed class of systems depicting cluster synchronisation.

Remark 3: In case of cluster synchronisation, the Lipschitz condition is often used to bound the coupling strength between the individual systems in the cluster. This ensures that the systems remain sufficiently close to each other in phase space, and therefore synchronise over time. The choice of the Lipschitz constant L depends on the specific system being studied, and may require some analysis to determine a reasonable value. However, in general, a larger value of L (Lipschitz constant) corresponds to a stronger coupling strength, which can lead to faster convergence to synchronisation but may also increase the risk of instability or chaotic behaviour. Overall, the Lipschitz condition is useful for analysing the stability and convergence of cluster synchronisation in coupled chaotic systems. However, it is important to carefully choose and validate the value of the Lipschitz constant to ensure that it is appropriate for the specific system being studied.

Lemma 1: Barbalat's lemma (Farkas and Wegner, 2016), if f is differentiable function with finite limit, and $\dot{f}$ is continuous, then $\dot{f} \rightarrow 0$ as $t \rightarrow \infty$. In term of Lyapunov stability like formulation, let a function $V(x(t))$ be differentiable scalar function such that

1 $V(\mathbf{x}(t))$ is bounded

2 $\dot{V}(\mathbf{x}(t)) \leq 0$

3 $\dot{V}(\mathbf{x}(t))$ is uniformly continuous, i.e., $\ddot{V}(\mathbf{x}(t))$ is to be bounded; then $\dot{V}(\mathbf{x}(t)) \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 2: We know that for any real number a and b, the following statement holds:

$$(b-a)^2 \geq 0; \Rightarrow 2ab \leq (b^2 + a^2) \tag{9}$$

For the systems in equations (1) and (2), by incorporating above inequality, one can write:

$$2[\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{x})]P\mathbf{e} \leq [\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{x})]^T[\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{x})] + \mathbf{e}^T P^T P\mathbf{e} \tag{10}$$

where P is positive definite symmetric square matrix of size $(n \times n)$, which satisfies the following:

1 All the ordered principal minor determinants of P being positive.

2 P is symmetric and $X^T PX > 0$; $\forall X \neq 0$.

By using equation (6), the network structure given in equations (4) and (5) are said to be realising cluster synchronisation, if the synchronisation errors satisfy the following condition:

$$\lim_{t\rightarrow\infty} \mathbf{e}_{ij}(t) = 0 \tag{11}$$

Using equations (4) and (5), error dynamics can be given as:

$$\begin{aligned}
\dot{\mathbf{e}}_{ij}(t) &= \boldsymbol{\theta}\mathbf{x}_j(t) + \mathbf{g}(x_j, t) + k_1\left[\sum_{k=N_1+1, k\neq j}^{N_1+N_2} \mathbf{x}_k(t) - \Gamma\mathbf{x}_j(t)\right] \\
&+ k_2\sum_{l=1}^{N_1}(\mathbf{x}_l(t) - \mathbf{x}_j(t)) - \boldsymbol{\phi}\mathbf{x}_i(t) - \boldsymbol{\theta}\mathbf{x}_i(t) + \boldsymbol{\theta}\mathbf{x}_i(t) \\
&- \mathbf{f}(\mathbf{x}_i, t) - k_1\left[\sum_{m=1, m\neq i}^{N_1} \mathbf{x}_m(t) - \Gamma\mathbf{x}_i(t)\right] \\
&- k_2\sum_{h=N_1+1}^{N_1+N_2}(\mathbf{x}_h(t) - \mathbf{x}_i(t))
\end{aligned} \tag{12}$$

Here, index $i = 1, 2, ..., N_1$; and $j = N_1 + 1, N_1 + 1, ..., N_1 + N_2$.

Further, equation (12) can be rearranged as:

$$\begin{aligned}
\dot{\mathbf{e}}_{ij} &= \boldsymbol{\theta}[\mathbf{x}_j(t) - \mathbf{x}_i(t)] + [\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))] \\
&+ k_1\left[\sum_{k=N_1+1, k\neq j}^{N_1+N_2} \mathbf{x}_k(t) - \sum_{m=1, m\neq i}^{N_1} \mathbf{x}_m(t)\right] \\
&- k_1\boldsymbol{\Gamma}[\mathbf{x}_j(t) - \mathbf{x}_i(t)] - k_2\left[\sum_{l=1}^{N_1}(\mathbf{x}_j(t) - \mathbf{x}_l(t))\right] \\
&- k_2\left[\sum_{h=N_1+1}^{N_1+N_2}(\mathbf{x}_h(t) - \mathbf{x}_i(t))\right] + \boldsymbol{\theta}\mathbf{x}_i(t) - \boldsymbol{\phi}\mathbf{x}_i(t)
\end{aligned} \tag{13}$$

Equation (13) can be further be expressed as follows:

$$\begin{aligned}
\dot{\mathbf{e}}_{ij} &= \boldsymbol{\theta}\mathbf{e}_{ij}(t) + [\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))] - k_1\Gamma\mathbf{e}_{ij}(t) \\
&+ k_1\left[\sum_{k=N_1+1, k\neq j}^{N_1+N_2} \sum_{m=1, m\neq i}^{N_1} \mathbf{e}_{mk}(t)\right] - k_2\sum_{l=1}^{N_1}\mathbf{e}_{lj}(t) \\
&- k_2\sum_{h=N_1+1}^{N_1+N_2}\mathbf{e}_{ih}(t) + \boldsymbol{\theta}\mathbf{x}_i(t) - \boldsymbol{\phi}\mathbf{x}_i(t)
\end{aligned} \tag{14}$$

Theorem 1: For a complex system with clusters given in equations (4) and (5), with error dynamics as in (14), if coupling gain k_1 is selected such that $[\boldsymbol{\theta}^T P + P\boldsymbol{\theta} + \gamma^2 I + P^2 - 2k_1 P\Gamma] \leq 0$ and the coupling gain k_2 is selected such that $2k_1\sum_{k=N_1+1, k\neq j}^{N_1+N_2}\sum_{m=1, m\neq i}^{N_1}\mathbf{e}_{mk}(t)P\mathbf{e}_{ij}(t) - 2k_2\sum_{l=1}^{N_1}\mathbf{e}_{ij}(t)P\mathbf{e}_{lj}(t) - 2k_2\sum_{h=N_1+1}^{N_1+N_2}\mathbf{e}_{ij}(t)P\mathbf{e}_{ih}(t) \leq 0$ then both the clusters achieve the complete synchronisation condition.

Proof: To drive the results for cluster synchronisation, let us consider the Lyapunov function candidate as follows:

$$V(t) = \mathbf{e}_{ij}^T(t)P\mathbf{e}_{ij}(t) \tag{15}$$

Time derivative of the equation (15) is given as follows:

$$\dot{V}(t) = \dot{\mathbf{e}}_{ij}^T(t)P\mathbf{e}_{ij}(t) + \mathbf{e}_{ij}^T(t)P\dot{\mathbf{e}}_{ij}(t) \tag{16}$$

Using error dynamics from equation (14), the above equation can be re-written as:

$$\begin{aligned}
\dot{V}(t) = {} & [\boldsymbol{\theta}\mathbf{e}_{ij}(t) + [\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))] - k_1\Gamma\mathbf{e}_{ij}(t) \\
& + k_1 \sum_{k=N_1+1,k\neq j}^{N_1+N_2} \sum_{m=1,m\neq i}^{N_1} \mathbf{e}_{mk}(t) - k_2 \sum_{l=1}^{N_1} \mathbf{e}_{lj}(t) \\
& - k_2 \sum_{h=N_1+1}^{N_1+N_2} \mathbf{e}_{ih}(t) + \boldsymbol{\theta}\mathbf{x}_i(t) - \boldsymbol{\phi}\mathbf{x}_i(t)]^T P\mathbf{e}_{ij}(t) \\
& + \mathbf{e}_{ij}^T(t)P[\boldsymbol{\theta}\mathbf{e}_{ij}(t) + [\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))] \\
& - k_1\Gamma\mathbf{e}_{ij}(t) + k_1 \sum_{k=N_1+1,k\neq j}^{N_1+N_2} \sum_{m=1,m\neq i}^{N_1} \mathbf{e}_{mk}(t) \\
& - \left. k_2 \sum_{l=1}^{N_1} \mathbf{e}_{lj}(t) - k_2 \sum_{h=N_1+1}^{N_1+N_2} \mathbf{e}_{ih}(t) + \boldsymbol{\theta}\mathbf{x}_i(t) - \boldsymbol{\phi}\mathbf{x}_i(t) \right]
\end{aligned} \tag{17}$$

where index $i = 1, 2, ..., N_1$; and index $j = N_1 + 1, N_1 + 1, ..., N_1 + N_2$.

Further, equation (17) can be re-written as:

$$\begin{aligned}
\dot{V}(t) = {} & \mathbf{e}_{ij}^T(t)[\boldsymbol{\theta}^T P + P\boldsymbol{\theta}]\mathbf{e}_{ij}(t) \\
& + [\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))]^T P\mathbf{e}_{ij}(t) \\
& + \mathbf{e}_{ij}^T(t)P[\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{f}(\mathbf{x}_i(t))] - 2k_1\mathbf{e}_{ij}^T(t)\Gamma P\mathbf{e}_{ij}(t) \\
& + 2k_1 \sum_{k=N_1+1,k\neq j}^{N_1+N_2} \sum_{m=1,m\neq i}^{N_1} \mathbf{e}_{mk}(t)P\mathbf{e}_{ij}(t) \\
& - 2k_2 \sum_{l=1}^{N_1} \mathbf{e}_{ij}(t)P\mathbf{e}_{lj}(t) - 2k_2 \sum_{h=N_1+1}^{N_1+N_2} \mathbf{e}_{ij}(t)P\mathbf{e}_{ih}(t) \\
& + 2\mathbf{e}_{ij}(t)\boldsymbol{\theta}P\mathbf{x}_i(t) - 2\mathbf{e}_{ij}(t)\boldsymbol{\phi}P\mathbf{x}_i(t)
\end{aligned} \tag{18}$$

Now, from equations (8) and (9), we can rewrite the above equation as follows:

$$\begin{aligned}
\dot{V}(t) = {} & \mathbf{e}_{ij}(t)[\boldsymbol{\theta}^T P + P\boldsymbol{\theta} + \gamma^2 I + P^2 - 2k_1\Gamma P]\mathbf{e}_{ij}(t) \\
& + 2k_1 \sum_{k=N_1+1,k\neq j}^{N_1+N_2} \sum_{m=1,m\neq i}^{N_1} \mathbf{e}_{mk}(t)P\mathbf{e}_{ij}(t) \\
& - 2k_2 \sum_{l=1}^{N_1} \mathbf{e}_{ij}(t)P\mathbf{e}_{lj}(t) - 2k_2 \sum_{h=N_1+1}^{N_1+N_2} \mathbf{e}_{ij}(t)P\mathbf{e}_{ih}(t) \\
& + 2\mathbf{e}_{ij}(t)\boldsymbol{\theta}\mathbf{x}_i(t) - 2\mathbf{e}_{ij}(t)\boldsymbol{\phi}\mathbf{x}_i(t)
\end{aligned} \tag{19}$$

To show the stability result for present case, the equation (19) is segregated into three parts, i.e., $\dot{V}_1$, $\dot{V}_2$ and $\dot{V}_3$; such that $\dot{V}_1$ is given as:

$$\dot{V}_1 = \mathbf{e}_{ij}(t)[\boldsymbol{\theta}^T P + P\boldsymbol{\theta} + \gamma^2 I + P^2 - 2k_1 P\Gamma]\mathbf{e}_{ij}(t), \tag{20}$$

$\dot{V}_2$ is given as:

$$\begin{aligned}\dot{V}_2 = {} & 2k_1 \sum_{k=N_1+1,k\neq j}^{N_1+N_2} \sum_{m=1,m\neq i}^{N_1} \mathbf{e}_{mk}(t)P\mathbf{e}_{ij}(t) \\ & - 2k_2 \sum_{l=1}^{N_1} \mathbf{e}_{ij}(t)P\mathbf{e}_{lj}(t) - 2k_2 \sum_{h=N_1+1}^{N_1+N_2} \mathbf{e}_{ij}(t)P\mathbf{e}_{ih}(t)\end{aligned} \tag{21}$$

and $\dot{V}_3$ is given as:

$$\dot{V}_3 = 2\mathbf{e}_{ij}(t)\boldsymbol{\theta}P\mathbf{x}_i(t) - 2\mathbf{e}_{ij}(t)\boldsymbol{\phi}P\mathbf{x}_i(t) \tag{22}$$

Equation (20) involves a matrix identity $[\boldsymbol{\theta}^T P + P\boldsymbol{\theta} + \gamma^2 I + P^2 - 2k_1 P\Gamma]$ which can be shown to be negative definite by appropriate selection of gain k_1, thus, ensuring $\dot{V}_1 \leq 0$. Here, γ is Lipschitz constant as per equation (7). Thus it leads to

$$\dot{V}_1 \leq 0 \tag{23}$$

Equation (21) forms a matrix of dimension $(nN \times nN)$ with diagonal elements involving coupling strength gain k_2 and off-diagonal elements having entries in term of coupling gains k_1 and k_2. By selecting suitable value of coupling gain k_2 in equation (21) the underlying matrix can be shown to be negative definite, leading to $\dot{V}_2 \leq 0$.

Further, in equation (22), the underlying function satisfies all requirements of Lemma 1, i.e., $\ddot{V}_3$ remain bounded as systems involved are chaotic in nature and have bounded states, thus leading to $\dot{V}_3 = 0$ as $t \to \infty$.

The above interpretation of stability ensures that the time derivative of Lyapunov function candidate $\dot{V}(t)$ is uniformly negative definite (UND). It also implies that synchronisation error is stabilised at origin. Stabilisation of synchronising error signifies that the cluster synchronisation goal is achieved. This completes the proof.

3 Synchronisation of bidirectional all-to-all coupled nonlinear networks of a class of chaotic systems

In this section, the theoretical results derived in previous section are verified with a suitable example belonging to the proposed class of nonlinear systems.

Consider two nonidentical clusters with first cluster as a set of chaotic Lorenz systems and second cluster as a set of chaotic Lu systems. Let both the clusters have two systems each.

The dynamics of Lorenz systems involved in first cluster is given as

$$\begin{aligned}
\dot{x}_{11} &= a(x_{12} - x_{11}) \\
\dot{x}_{12} &= bx_{11} - x_{11}x_{13} - x_{12} \\
\dot{x}_{13} &= x_{11}x_{12} - cx_{13}
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\dot{x}_{21} &= a(x_{22} - x_{21}) \\
\dot{x}_{22} &= bx_{21} - x_{21}x_{23} - x_{22} \\
\dot{x}_{23} &= x_{21}x_{22} - cx_{23}
\end{aligned} \tag{25}$$

The dynamics of Lu systems which are part of second cluster is given as

$$\begin{aligned}
\dot{x}_{31} &= p(x_{32} - x_{31}) \\
\dot{x}_{32} &= -x_{31}x_{33} + rx_{32} \\
\dot{x}_{33} &= x_{31}x_{32} - sx_{33}
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\dot{x}_{41} &= p(x_{42} - x_{41}) \\
\dot{x}_{42} &= -x_{41}x_{43} + rx_{42} \\
\dot{x}_{43} &= x_{41}x_{42} - sx_{43}
\end{aligned} \tag{27}$$

The system parameters are given as a = 10, b = 28 and c = –8/3, respectively, for Lorenz attractor and p = 36, r = 20 and s = 3, respectively, for Lu attractor. The phase portrait of Lorenz attractor is shown in Figure 1 and that of Lu attractor is shown in Figure 2.

Here, the case of bidirectional all-to-all, two-way coupled network configuration is considered as shown in Figure 3. As shown this figure, the two clusters, i.e., cluster-1 contains two system $\mathbf{x}_1$ and $\mathbf{x}_2$ and cluster-2 have different systems then cluster-1 as $\mathbf{x}_3$ and $\mathbf{x}_4$. All the systems are coupled bidirectionally to each other with all-to-all to configuration.

The error dynamics for above systems can be obtained for both the clusters as:

$$\begin{aligned}
\dot{e}_{1i} &= f(x_{3i}) - f(x_{1i}) + k_1[(x_{4i} - x_{2i}) - (x_{3i} - x_{1i})] \\
&\quad + k_2[-3(x_{3i} - x_{1i}) - (x_{4i} - x_{2i})] \\
\dot{e}_{2i} &= f(x_{3i}) - f(x_{2i}) + k_1[(x_{4i} - x_{1i}) - (x_{3i} - x_{2i})] \\
&\quad + k_2[-3(x_{3i} - x_{2i}) - (x_{4i} - x_{1i})] \\
\dot{e}_{3i} &= f(x_{4i}) - f(x_{1i}) + k_1[(x_{3i} - x_{2i}) - (x_{4i} - x_{1i})] \\
&\quad + k_2[-3(x_{4i} - x_{1i}) - (x_{3i} - x_{2i})] \\
\dot{e}_{4i} &= f(x_{4i}) - f(x_{2i}) + k_1[(x_{4i} - x_{2i}) + (x_{3i} - x_{1i})] \\
&\quad + k_2[-3(x_{4i} - x_{2i}) - 3(x_{3i} - x_{1i})]
\end{aligned} \tag{28}$$

where index i = 1, 2, 3.

By selecting gains k_1 and k_2 suitably as per results presented in Theorem 1 leads to convergence of above error dynamics.

Figure 1 3D phase portrait of Lorenz attractor (see online version for colours)

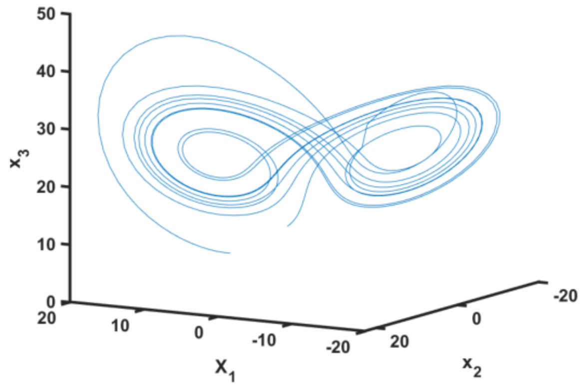


Figure 2 3D phase portrait of Lu attractor

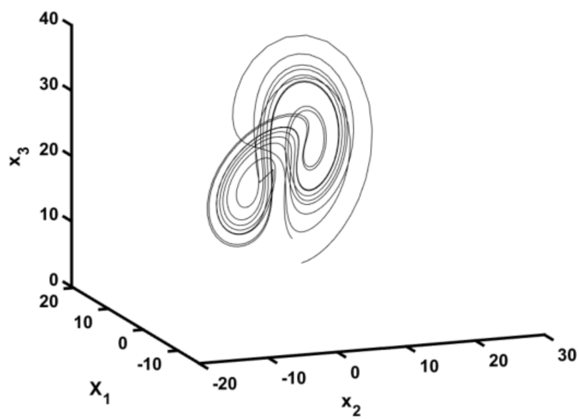


Figure 3 Block diagram of bidirectional all-to-all coupled network

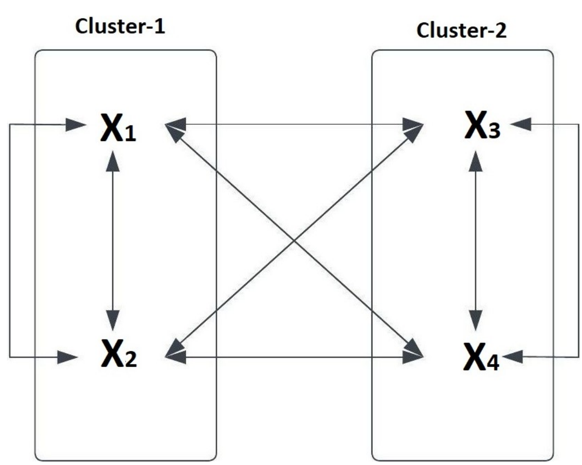


Figure 4 Synchronisation of first state of each system in network (see online version for colours)

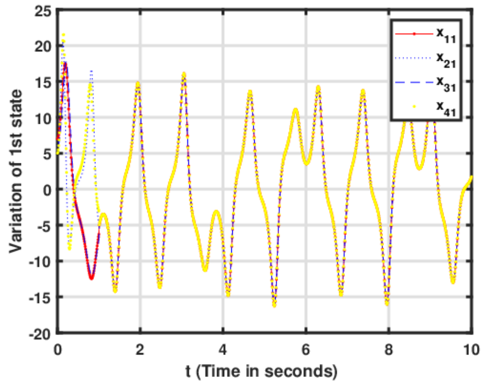


Figure 5 Synchronisation of second state of each system in network (see online version for colours)

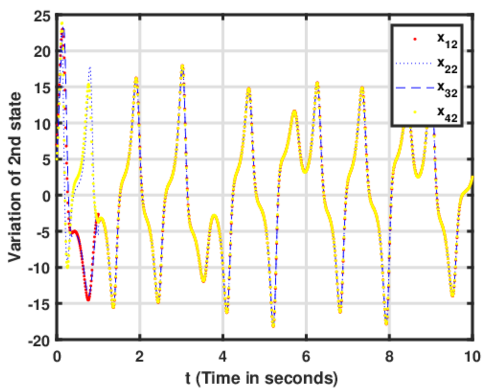


Figure 6 Synchronisation of third state of each system in network (see online version for colours)

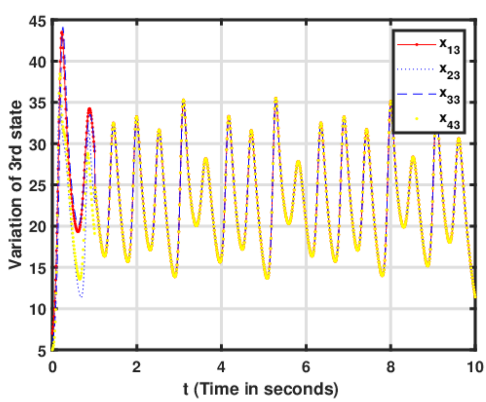


Figure 7 Error convergence of 1st system and 3rd system of the network (see online version for colours)

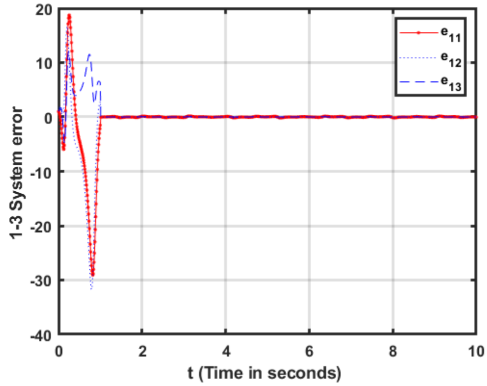


Figure 8 Error convergence of 1st system and 4th system of the network (see online version for colours)

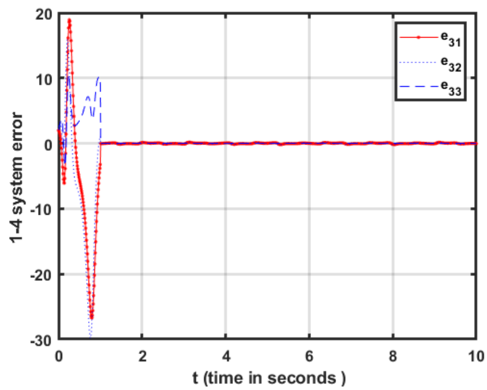


Figure 9 Error convergence of 2nd system and 3rd system of the network (see online version for colours)

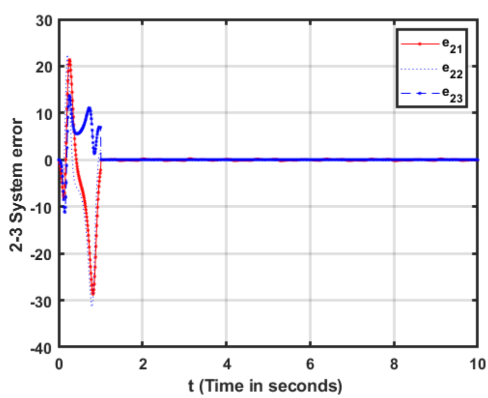
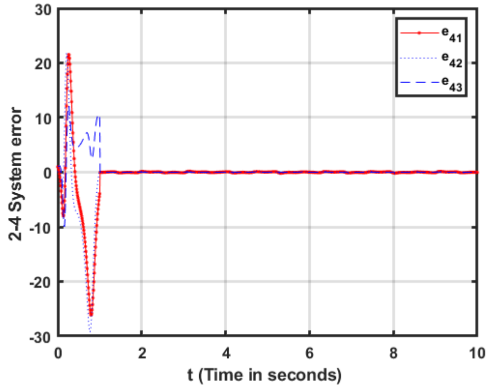


Figure 10 Error convergence of 2nd system and 4th system of the network (see online version for colours)



4 Simulation results

In order to apply analytical results to the synchronisation problem, numerical simulations are performed. To examine the cluster synchronisation behaviour of non-identical complex networks, simulation tests are performed using MATLAB for ten seconds. Initial conditions for Lorenz and Lu systems are considered as $[x_{11}(0), x_{12}(0), x_{13}(0)]$ = [8, 8, 8]; $[x_{21}(0), x_{22}(0), x_{23}(0)]$ = [7, 7, 7]; $[x_{31}(0), x_{32}(0), x_{33}(0)]$ = [5, 5, 5]; and $[x_{41}(0), x_{42}(0), x_{43}(0)]$ = [5, 5, 5], respectively, and Lipschitz constant γ is selected as 1.0. Selection of suitable gains as k_1 = 80 and k_2 = 100 ensure convergence of corresponding states of systems. These gain settings ensure complete synchronisation of complex network. This results in cluster synchronisation of subsystem states of non-identical complex network with each other. The variations of all three states of systems involved in both the clusters are shown in Figures 4 to 6, respectively. Error convergence of bidirectional all-to-all coupled nonlinear systems in complex network is shown in Figures 7 to 10 and the plots clearly indicate that error dynamics converge towards zero with time. Here, controller function is switched on at t = 1 seconds.

5 Conclusions

The presented work is focused towards deriving synchronisation scheme for complex dynamical network with non-identical clusters. Synchronisation controller is derived in term of coupling strengths of network. The appropriate coupling gains are designed while assuming underlying nonlinearities to be following Lipschitz condition. The synchronisation condition are established on the basis of Lyapunove stability theory along with Barbalat lemma by exploiting the bounded nature of system states. Detailed numerical simulations are performed, which verify the efficacy of derived theoretical results. The proposed scheme can be further tested for networks subjected to parametric uncertainty and external disturbance.

References

Amritkar, R.E., Jalan, S. and Hu, C.K. (2005) 'Synchronized clusters in coupled map networks. II. Stability analysis', *Physical Review E*, Vol. 72, No. 1, p.16212.

Belykh, I., Belykh, V., Nevidin, K. and Hasler, M. (2003) 'Persistent clusters in lattices of coupled nonidentical chaotic systems', *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 13, No. 1, pp.165–178.

Belykh, V.N., Belykh, I.V. and Mosekilde, E. (2001) 'Cluster synchronization modes in an ensemble of coupled chaotic oscillators', *Physical Review E*, Vol. 63, No. 3, p.36216.

Belykh, V.N., Osipov, G.V., Petrov, V.S., Suykens, J.A. and Vandewalle, J. (2008) 'Cluster synchronization in oscillatory networks', *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 18, No. 3, p.37106.

Bora, R.M. and Sharma, B.B. (2022) 'LMI based adaptive robust control scheme for reduced order synchronization (ROS) for a class of chaotic systems', *IFAC-PapersOnLine*, Vol. 55, No. 1, pp.253–258.

Cai, S., Jia, Q. and Liu, Z. (2015) 'Cluster synchronization for directed heterogeneous dynamical networks via decentralized adaptive intermittent pinning control', *Nonlinear Dynamics*, Vol. 82, No. 1, pp.689–702.

Changder, S. and Sharma, B.B. (2022) 'Synchronisation among multiple chaotic systems connected in chain and star configuration using backstepping strategy', *International Journal of Systems, Control and Communications*, Vol. 14, No. 1, pp.60–80.

Chauhan, Y., Sharma, B.B. and Ranjan, R.K. (2021) 'Synchronization of nonlinear systems in chain network configuration with parametric uncertainty', *2021 IEEE 18th India Council International Conference (INDICON)*, IEEE, pp.1–6.

Chen, M., Shi, P. and Lim, C.C. (2015) 'Robust constrained control for MIMO nonlinear systems based on disturbance observer', *IEEE Transactions on Automatic Control*, Vol. 60, No. 12, pp.3281–3286.

Chen, X., Cao, J., Qiu, J., Alsaedi, A. and Alsaadi, F.E. (2016) 'Adaptive control of multiple chaotic systems with unknown parameters in two different synchronization modes', *Advances in Difference Equations*, Vol. 2016, No. 1, pp.1–17.

DeLellis, P., Garofalo, F. and Iudice, F.L. (2018) 'The partial pinning control strategy for large complex networks', *Automatica*, Vol. 89, pp.111–116.

Farkas, B. and Wegner, S.A. (2016) 'Variations on Barbălat's lemma', *The American Mathematical Monthly*, Vol. 123, No. 8, pp.825–830.

Ge, S.S., Wang, C. and Lee, T.H. (2000) 'Adaptive backstepping control of a class of chaotic systems', *International Journal of Bifurcation and Chaos*, Vol. 10, No. 5, pp.1149–1156.

Jalan, S., Amritkar, R.E. and Hu, C.K. (2005) 'Synchronized clusters in coupled map networks', *Physical Review E*, Vol. 72, No. 1, p.16211.

Kouomou, Y.C. and Woafo, P. (2003) 'Transitions from spatiotemporal chaos to cluster and complete synchronization states in a shift-invariant set of coupled nonlinear oscillators', *Physical Review E*, Vol. 67, No. 4, p.46205.

Kouomou, Y.C. and Woafo, P. (2003) 'Cluster synchronization in coupled chaotic semiconductor lasers and application to switching in chaos-secured communication networks', *Optics Communications*, Vol. 223, Nos. 4–6, pp.283–293.

Krstic, M., Kokotovic, P.V. and Kanellakopoulos, I. (1995) *Nonlinear and Adaptive Control Design*, John Wiley & Sons, Inc.

Liu, Y. and Zhu, Q. (2021) 'Adaptive fuzzy finite-time control for nonstrict-feedback nonlinear systems', *IEEE Transactions on Cybernetics*, Vol. 52, No. 10, pp.10420–10429.

Liu, Y. and Zhu, Q. (2022) 'Adaptive neural network asymptotic control design for MIMO nonlinear systems based on event-triggered mechanism', *Information Sciences*, Vol. 603, pp.91–105.

Liu, Z.X., Chen, Z.Q. and Yuan, Z.Z. (2007) 'Pinning control of weighted general complex dynamical networks with time delay', *Physica A: Statistical Mechanics and its Applications*, Vol. 375, No. 1, pp.345–354.

Ma, Z., Liu, Z. and Zhang, G. (2006) 'A new method to realize cluster synchronization in connected chaotic networks', *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 16, No. 2, p.23103.

Nian, F. and Wang, X. (2011) 'Optimal pinning synchronization on directed complex network', *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 21, No. 4, p.43131.

Pham, V.T., Vaidyanathan, S., Volos, C., Jafari, S., Alsaadi, F.E. and Alsaadi, F.E. (2019) 'Chaos in a simple snap system with only one nonlinearity, its adaptive control and real circuit design', *Archives of Control Sciences*, Vol. 29.

Precup, R.E., Radac, M.B., Roman, R.C.and Petriu, E.M. (2017) 'Model-free sliding mode control of nonlinear systems: algorithms and experiments', *Information Sciences*, Vol. 381, pp.176–192.

Qin, W.X. and Chen, G. (2004) 'Coupling schemes for cluster synchronization in coupled Josephson equations', *Physica D: Nonlinear Phenomen*, Vol. 197, Nos. 3–4, pp.375–391, p.23.

Sharma, B.B. and Kar, I.N. (2011) 'Stabilization and tracking controller for a class of nonlinear discrete-time systems', *Chaos, Solitons & Fractals*, Vol. 44, No. 10, pp.902–913.

Sun, W., Wang, S., Wang, G. and Wu, Y. (2015) 'Lag synchronization via pinning control between two coupled networks', *Nonlinear Dynamics*, Vol. 79, No. 4, pp.2659–2666.

Sun, W., Sun, M., Guan, J. and Jia, Q. (2019) 'Robustness of coherence in noisy scale-free networks and applications to identification of influential spreaders', *IEEE Transactions on Circuits and Systems II: Express Briefs*, Vol. 67, No. 7, pp.1274–1278.

Wang, X.F. and Chen, G. (2002) 'Pinning control of scale-free dynamical networks', *Physica A: Statistical Mechanics and its Applications*, Vol. 310, Nos. 3–4, pp.521–531.

Wang, G. and Shen, Y. (2013) 'Cluster synchronisation of directed complex dynamical networks with nonidentical nodes via pinning control', *International Journal of Systems Science*, Vol. 44, No. 9, pp.1577–1586.

Wang, J., Feng, J., Xu, C. and Zhao, Y. (2012) 'Cluster synchronization of nonlinearly-coupled complex networks with nonidentical nodes and asymmetrical coupling matrix', *Nonlinear Dynamics*, Vol. 67, No. 2, pp.1635–1646.

Wang, L., Guo, Y., Wang, Y., Fan, H. and Wang, X. (2020) 'Pinning control of cluster synchronization in regular networks', *Physical Review Research*, Vol. 2, No. 2, p.23084.

Wu, W., Zhou, W. and Chen, T. (2008) 'Cluster synchronization of linearly coupled complex networks under pinning control', *IEEE Transactions on Circuits and Systems I: Regular Papers*, Vol. 56, No. 4, pp.829–839.

Wu, Z., Xu, X.J., Chen, G. and Fu, X. (2012) 'Adaptive synchronization and pinning control of colored networks', *American Institute of Physics, Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 22, No. 4, p.43137.

Xia, W. and Cao, J. (2009) 'Pinning synchronization of delayed dynamical networks via periodically intermittent control', *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol. 19, No. 1, p.13120.

Yeh, P.C. and Kokotovic, P.V. (1995) 'Adaptive control of a class of nonlinear discrete-time systems', *International Journal of Control*, Vol. 62, No. 2, pp.303–324.

Zhang, Y., Hu, G., Cerdeira, H.A., Chen, S., Braun, T. and Yao, Y. (2001) 'Partial synchronization and spontaneous spatial ordering in coupled chaotic systems', *Physical Review E*, Vol. 63, No. 2, p.26211.