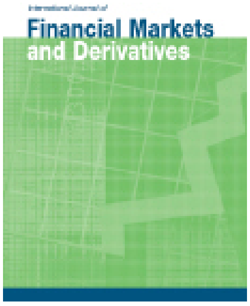

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Finite difference solutions of the CEV PDE

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Abstract: This work studies the valuation of European options under the constant elasticity of variance model. The model generalises the Black-Scholes framework for option pricing by incorporating a local instantaneous volatility term which is a function of the stock price. The model has the ability to fit certain implied volatility structures exhibited by market option prices, but the computation of the closed-form European option formula is not always stable and can be largely inaccurate for some parameter ranges because of the difficulties associated with the computation of the non-central chi-square distribution in the valuation formula. As an alternative to one line of research which aims at accelerating and stabilising the analytical price computation, we study finite difference techniques to obtain European option prices and associated hedging parameters. It is numerically demonstrated that a direct discretisation of the pricing equation in combination with an exponential integrator in time performs better than other schemes based on Crank-Nicolson discretisations of two transformed problems, one posed on an infinite domain and the other on a finite domain.

Keywords: option pricing; constant elasticity of variance; CEV; European options; finite difference scheme; exponential time differencing.

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1 Introduction

After its discovery, the constant volatility lognormal diffusion model of Black and Scholes (1973) for asset prices rapidly became the industry standard as the availability of closed-form solutions for European options made possible the dynamic hedging of financial risk. The Black-Scholes formula is a monotone function of the unobservable volatility parameter and for a quoted market option call or put price, the formula can be inverted to yield the market implied volatility. However over the years, limitations of Black-Scholes became apparent due to the existence of implied volatility surfaces known as smiles (U-shaped structure) and skews (decreasing with the option's strike for a given maturity). These structures are inconsistent with the constant volatility of asset returns assumed in Black-Scholes framework.

To provide a better fit to market option prices, several enhancements were proposed such as stochastic volatility models (Heston, 1993; Hagan et al., 2002) which lead to two dimensional pricing equations and jump diffusion models (Merton, 1976; Kou, 2002) which contain non-local integral terms. However, jump-diffusion and stochastic volatility models contain an extra factor which cannot be hedged by trading in the underlying asset only. Therefore, one-factor models are preferable for perfect replication. Mitra (2010) mentioned that stochastic volatility models suffer from analytical and calibration intractability and proposed using Fouque's option pricing method for pricing options under regime switching stochastic volatility models over short time scales.

Theoretic arguments and empirical evidence supported the hypothesis that there is an association between stock price and volatility and to deal with this observation, local volatility models were introduced which stay within the one-dimensional framework of the Black-Scholes model and thus lead to complete market models which mean that dynamic hedging portfolios can be constructed. The constant elasticity of variance (CEV) model (Cox, 1975) is a prototype of local volatility models in which the volatility is a deterministic function of the asset price. The CEV process is consistent with two empirical market observations. The model incorporates the negative relationship between the asset price and its return volatility and also, it captures the volatility skews observed in market option prices.

Similar to the case of the Black-Scholes model, European options admit analytical solutions under the CEV process. However, these formulas are in terms of the non-central chi-square distribution whose computations become slow when the maturity is low and the elasticity of the local volatility function is close to one (Schroder, 1989). To deal with some of the limitations of the approach based on the non-central chi-square distribution, Araneda and Villena (2021) developed a numerical method for computing European option prices under the CEV model by using the Feynman's path integral approximation technique and it is shown that the relative error is below 20% for most cases while for short maturity and low volatility, the errors decrease to less than 10%.

Recombining lattice methods for pricing options under CEV have been proposed by Costabile and Massabó (2010). For pricing American options under the Black-Scholes model which is a special case of the CEV model, Goldenberg (2010) showed that the binomial method of Cox and Rubinstein (1985) results in mispricing of the early exercise boundary in addition to the distribution and nonlinearity errors reported in the literature. Liu and Pang (2016) developed a time adjusted grid lattice method along with backward induction for the pricing of European and American options. For the CEV model, Javaid et al. (2022) recently considered pricing American put options with the use of Lie symmetry technique under three special cases of the CEV model which are the Black-Scholes model. the Cox-Ingersoll-Ross model and the Vasicek model. Other recent methods for option pricing under the CEV model can be found in Lee (2021) and Zhang et al. (2019).

A most successful approach to the solution of option pricing problems is based on solving the no-arbitrage partial differential equation (PDE) satisfied by the option price. This technique is particularly used for option pricing problems where no analytical solutions can be obtained. The CEV model has the advantage that it leads to a Black-Scholes-type one-dimensional PDE. A numerical example given by Wong and Zhao (2008) showed that a finite difference method may be much faster for some parameters in the CEV model. Since the CEV PDE cannot be transformed to as simple a form as the heat equation, we consider in this work three different approaches for the computation of the European option price. A first method considered has been derived by Wong and Zhao (2008) which develops a numerical method for a transformed problem posed on an infinite domain. We then propose a method which is based on a Crank-Nicolson discretisation of a transformed equation posed on a finite domain. These two procedures are compared against a finite difference scheme based on direct discretisation of the space derivatives in the CEV equation and which employs an exponential integrator for the resulting system of odes.

In previous works (Thakoor et al., 2013, 2015, 2018) we developed fourth-order numerical schemes for solving the CEV PDE and a sixth-order scheme in Thakoor (2022), but this higher-order accuracy is achieved at extra complications for implementation of the high-order schemes since grid refinement techniques or non-uniform grids need to be employed. Fixed income contracts and their derivatives under the CEV interest rate model were considered in Thakoor et al. (2012, 2017). In this work, because of the popularity of standard second-order discretisation schemes for the solution of finance PDEs, we develop a numerical scheme that is also fast and accurate. In addition, it is simple to implement and it necessitates only a one-step integration in time.

The rest of this paper proceeds as follows. The CEV diffusion for the stock price dynamics and the no-arbitrage PDE satisfied by the option price are described in Section 2. The numerical schemes derived in Section 3 and performance comparisons are done in Section 4. Finally, the study's conclusions are presented in Section 5.

2 European options under CEV

The CEV model assumes that under the risk-neutral measure $\mathbb{Q}$, the stock price dynamics S_t at time t is the solution of the stochastic differential equation given by

$$dS_t = (r-q)S_t dt + \delta S_t^{\alpha}\, dW_t, \tag{1}$$

where $\sigma\left(S_t\right) = \delta S_t^{\alpha-1}$ is the local volatility function, r is the risk-free rate, q is the dividend yield, δ is a scale parameter, W_t is standard Brownian motion under the probability measure $\mathbb{Q}$ and α is a parameter that relates the instantaneous variance of the percentage price change to the stock price.

Let $F_t = e^{r(T-t)}S_t$ be the forward price and $\nu^{-1} = 2-2\alpha$. Then

$$dF_t = \delta F_t^{\alpha} dW_t. \tag{2}$$

The forward price process (2) is related to the squared Bessel process X_t of dimension γ satisfying the stochastic differential equation

$$dX_t^{(\gamma)} = \gamma\, dt + 2\sqrt{X_t^{(\gamma)}} dW_t,$$

where $\gamma = 2-2\nu$ and $X_0 = 4\nu^2 F_0^{\frac{1}{\nu}}/\sigma^2$.

When the elasticity factor $\alpha < 1$, the dimension γ of the squared Bessel process is positive and in this case, the process X_t has a non-central chi-square distribution. The case when $\alpha > 1$, the forward process F_t is not a martingale since $\mathbb{E}^{\mathbb{Q}}[F_T|F_0] \neq F_0$ and this case has been recently considered by Lindsay and Brecher (2012).

In the index options market, it has been empirically observed that the elasticity factor can be strongly negative (Jackwerth and Rubinstein, 2001) and in this work we consider the case when $\alpha < 1$.

To allow comparisons between analytical and numerical approaches, we recall closed form expressions for the European put price (Schroder, 1989) and the two hedging sensitivities delta and gamma recently derived in Larguinho et al. (2013). Using the notations of Wong and Zhao (2008), let $Y_t = S_t^{\frac{1}{\nu}}$. Then using Itô's lemma, the dynamics of Y_t is given by

$$dY_t = (bY_t + \hbar)\, dt + \frac{\delta}{\nu}\sqrt{Y_t}\, dW_t,$$

where $b = \nu^{-1}(r-q)$, $a = \delta^2/(2\nu^2)$, $\hbar = (1-\nu)\, a$. The transition density $g\,(Y_T|Y_t)$ of the process Y_T for $T > t$ conditional to Y_t is given by (Hsu et al., 2008)

$$g(Y_T|Y_t) = kx^{\frac{\nu}{2}} z^{-\frac{\nu}{2}} e^{-x-z} I_\nu\left(2\sqrt{xz}\right),$$

where $k = (2\nu(r-q))/(\delta^2\left(e^{(r-q)(T-t)/\nu} - 1\right))$, $x = kS_t^{\frac{1}{\nu}} e^{(r-q)(T-t)/\nu}$, $z = kS_T^{\frac{1}{\nu}}$ and $I_w(y)$ is the modified Bessel function of the first kind of order w given by

$$I_w(y) = \sum_{r=0}^{\infty} \frac{(y/2)^{2r+w}}{r!\Gamma(r+1+w)}.$$

It then follows that the density S_T conditional on S_t is given by

$$f(S_T|S_t) = \frac{1}{\nu} S_T^{\frac{1-\nu}{\nu}} g\,(Y_T|Y_t) = \frac{1}{\nu} k^\nu x^{\frac{\nu}{2}} z^{1-\frac{3}{2}\nu} e^{-x-z} I_\nu\left(2\sqrt{xz}\right). \tag{3}$$

A European put option with maturity T and strike E has price $V(S,\, t)$ given by

$$V(S,\, t) = e^{-r(T-t)}\mathbb{E}\left[(E - S_T)^+ | S_t = S\right],$$

where $(E - S_T)^+ = \max(E - S_T,\, 0)$. This conditional expectation can be explicitly calculated in terms of the non-central chi-square distribution $\chi^2_\vartheta(\lambda)$ with ϑ degrees of freedom and non-centrality parameter λ whose density function is given by

$$\mathfrak{p}\,(u,\, \vartheta,\, \lambda) = \frac{1}{2} e^{-(\lambda+u)/2} \left(\frac{u}{\lambda}\right)^{(\vartheta-2)/4} I_{(\vartheta-2)/2}\left(\sqrt{\lambda u}\right).$$

The complementary distribution function $Q\,(x,\, \vartheta,\, \lambda)$ given by

$$Q\,(x,\, \vartheta,\, \lambda) = \int_x^\infty \frac{1}{2} e^{-(\lambda+u)/2} \left(\frac{u}{\lambda}\right)^{(\vartheta-2)/4} I_{(\vartheta-2)/2}\left(\sqrt{\lambda u}\right) du,$$

has the property (Hsu et al., 2008)

$$Q\,(x,\, 2-2\vartheta,\, \lambda) = 1 - Q\,(\lambda,\, 2\vartheta,\, x)\,. \tag{4}$$

Using the density function (3), the time zero European put price $V(S,\, 0)$ is given by

$$V(S,\,0) = Ee^{-rT}\int_0^E f(S_T|S_0 = S)dS_T - Se^{-rT}\int_0^E S_t f(S_T|S_0 = S)dS_T.$$

Letting $d = kE^{\frac{1}{\nu}}$ and using the fact that for integer ν, $I_\nu(y) = I_{-\nu}(y)$, we can show that

$$\int_0^E f(S_T|S_0 = S)dS_T = \int_0^{2d} \mathfrak{p}\left(u,\,2-2\nu,\,2x\right)\,du = 1 - Q\left(2d,\,2-2\nu,\,2x\right),$$

and

$$\begin{aligned}\int_0^E S_T f(S_T|S_0 = S)dS_T &= Se^{(r-q)T}\int_0^{2d} \mathfrak{p}\left(u,\,2\nu+2,\,2x\right)\,du \\ &= Se^{(r-q)T}\left(1 - Q\left(2d,\,2+2\nu,\,2x\right)\right).\end{aligned}$$

Using equation (4), it then follows that

$$V(S,\,0) = Ee^{-rT}Q\left(2x,\,2\nu,\,2d\right) - Se^{-qT}\left(1 - Q\left(2d,\,2+2\nu,\,2x\right)\right). \tag{5}$$

Analytical delta (Δ) and gamma (Γ) for the European put have been recently derived by Larguinho et al. (2013). The closed-form expression for delta is given by

$$\begin{aligned}\Delta &= -e^{-qT}\left[1 - Q(2d,\,2(1+\nu),\,2x)\right] \\ &\quad + \frac{2x}{\nu S}\left[Se^{-qT}\mathfrak{p}(2d,\,4+2\nu,\,2x) - Ee^{-rT}\mathfrak{p}(2x,\,2\nu,\,2d)\right],\end{aligned}$$

and for gamma, we have

$$\begin{aligned}\Gamma &= \frac{2x}{\nu^2 S}e^{-qT}\left[(1+\nu-x)\mathfrak{p}(2d,\,4+2\nu,\,2x) + x\mathfrak{p}(2d,\,6+2\nu,\,2x)\right] \\ &\quad + \frac{2x}{\nu^2 S^2}Ee^{-rT}\left[x\mathfrak{p}(2x,\,2\nu,\,2d) - d\mathfrak{p}(2x,\,2(1+\nu),\,2d)\right].\end{aligned}$$

3 Finite difference solutions

The efficient computation of the complementarity non-central chi-square distribution function in equation (5) have been investigated in Benton and Krishnamoorthy (2003), Ding (1992), Dyrting (2004) and Schroder (1989) and a comparison of the different approaches is carried out in a recent work of Larguinho et al. (2013). The main findings of their work is that although some methods are fast, they differ significantly in accuracy depending on the parameter range chosen.

Our aim is to obtain an efficient technique which can be used over the whole set of parameters and the technique we propose is based on solving the PDE satisfied by the option price. Letting $\tau = T - t$ denote the time to maturity, the price $V(S,\,\tau)$ solves the problem

$$\begin{aligned}&\frac{\partial V}{\partial \tau} = \frac{1}{2}\delta^2 S^{2\alpha}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV,\;\; S \geq 0,\, 0 \leq \tau \leq T, \\ &V(S,\,0) = \max(E - S,\,0),\;\; S \geq 0, \\ &V(0,\,\tau) = Ee^{-r\tau},\;\; 0 \leq \tau \leq T, \\ &V(S,\,\tau) = 0 \text{ as } S \to +\infty,\;\; 0 \leq \tau \leq T.\end{aligned} \tag{6}$$

3.1 Standardised-form transformation

In, the following, we develop numerical methods for the computation of V. The first scheme described has been proposed in Wong and Zhao (2008) and is based on a transformation of equation (6) to a standardised form. The substitutions $V = e^{-r\tau}V_1$, $\tilde{S} = Se^{(r-q)\tau}$ and $V_1(S,\tau) = U(\tilde{S},\tau)$ transform the PDE (6) into

$$e^{-(r-q)\frac{\tau}{\nu}} \frac{\partial U}{\partial \tau} = \frac{1}{2}\delta^2 \tilde{S}^{2-\frac{1}{\nu}} \frac{\partial^2 U}{\partial \tilde{S}^2}. \tag{7}$$

Then using $X = \tilde{S}^{\frac{1}{\nu}}$, equation (7) becomes

$$2\nu^2 e^{-(r-q)\frac{\tau}{\nu}} \frac{\partial U}{\partial \tau} = \delta^2 X \frac{\partial^2 U}{\partial X^2} + \delta^2(1-\nu)\frac{\partial U}{\partial X}. \tag{8}$$

Letting $\tilde{\tau} = \left(e^{\frac{(r-q)\tau}{\nu}} - 1\right)/(2\nu(r-q))$, equation (8) simplifies further to

$$\frac{\partial U}{\partial \tilde{\tau}} = \delta^2 X \frac{\partial^2 U}{\partial X^2} + \delta^2(1-\nu)\frac{\partial U}{\partial X}.$$

A final substitution using $T_{\max} = (\exp\{(r-q)T/\nu\} - 1)/(2\nu(r-q))$ and $Y = \frac{2}{\delta}\sqrt{X}$, gives rise to the problem

$$\begin{aligned}
&\frac{\partial U}{\partial \tilde{\tau}} = \frac{\partial^2 U}{\partial Y^2} + \left(\frac{1-2\nu}{Y}\right)\frac{\partial U}{\partial Y},\ 0 \le Y \le \infty,\ 0 \le \tilde{\tau} \le T_{\max},\\
&U(Y,0) = \max\left(E - \left(\frac{\delta}{2}Y\right)^{2\nu}, 0\right),\ 0 \le Y \le \infty,\\
&U(0,\tilde{\tau}) = E,\ 0 \le \tilde{\tau} \le T_{\max},\\
&U(Y,\tilde{\tau}) = 0 \text{ as } Y \to -\infty,\ 0 \le \tilde{\tau} \le T_{\max}.
\end{aligned} \tag{9}$$

The method of Wong and Zhao localises the problem to the domain $\{Y : 0 \le Y \le a\}$ where

$$a = \max\left(2(S_{\max}e^{(r-q)T})^{\frac{1}{2\nu}}/\sigma,\ 2(S_{\max})^{\frac{1}{2\nu}}/\sigma\right),$$

with the asset price $S_{\max}$ chosen large enough for obtaining numerical solutions with sufficient accuracy and the right boundary condition becomes $U(Y_M,\tilde{\tau}) = 0$, $0 \le \tilde{\tau} \le T_{\max}$.

Let $\Delta Y = a/M$ be the uniform spacing along the Y-direction, let $Y_m = m\Delta Y$ and let $U_m(\tau) = U(Y_m,\tau)$ for $0 \le m \le M$. Consider the finite differences Δ_Y and Δ_Y^2 given by

$$\begin{aligned}
&\Delta_Y U_m(\tau) = U_{m+1}(\tau) - U_{m-1}(\tau),\\
&\Delta_Y^2 U_m(\tau) = U_{m+1}(\tau) - 2U_m(\tau) + U_{m-1}(\tau).
\end{aligned}$$

Then, we have

$$\begin{aligned}
U_m'(\tau) = &\frac{1}{(\Delta Y)^2}\Delta_Y^2 U_m(\tau)\\
&+ \frac{1}{2\Delta Y}\left(\frac{1-2\nu}{Y_m}\right)\Delta_Y U_m(\tau),\ 1 \le m \le M-1.
\end{aligned} \tag{10}$$

Letting $\eta_m = (1-2\nu)/(2m)$ and $U(\tau) = [U_1(\tau), U_2(\tau), ..., U_{M-1}(\tau)]^T$, $U'(\tau) = \left[U_1'(\tau), U_2'(\tau), ..., U_{M-1}'(\tau)\right]^T$ and writing equation (10) in matrix form, we obtain the system of odes given by

$$U'(\tau) = \mathbf{A}_1 U(\tau) + \mathbf{b}_1, \tag{11}$$

where $\mathbf{A}_1 \in \mathbb{R}^{(M-1)\times(M-1)}$ is a tridiagonal matrix given by

$$\mathbf{A}_1 = \frac{1}{(\Delta Y)^2}\begin{pmatrix} -2 & 1+\eta_1 & & & \\ 1-\eta_2 & -2 & 1+\eta_2 & & \\ \ddots & \ddots & \ddots & & \\ & 1-\eta_{M-2} & -2 & 1+\eta_{M-2} \\ & & 1-\eta_{M-1} & -2 \end{pmatrix},$$

and $\mathbf{b}_1$ which incorporates the boundary condition is given by

$$\mathbf{b}_1 = [(1-\eta_1)U_0(\tau),\, 0,\, ...,\, 0,\, (1+\eta_{M-1})U_M(\tau)]^T\,.$$

Let $\Delta\tilde{\tau} = T_{\max}/N$ be the time step, then $\tilde{\tau}_n = n\Delta\tilde{\tau}$, $0 \le n \le N$ and let $U^n = \left[U_1^n, U_2^n, ..., U_{M-1}^n\right]$ denote the vector of unknowns at time level n. Applying the Crank-Nicolson time stepping to equation (11), we obtain the tridiagonal system

$$\left(\mathbf{I} - \frac{\Delta\tilde{\tau}}{2}\mathbf{A}_1\right)U^{n+1} = \left(\mathbf{I} + \frac{\Delta\tilde{\tau}}{2}\mathbf{A}_1\right)U^n + \tilde{\mathbf{b}}_1, \tag{12}$$

where $\mathbf{I} \in \mathbb{R}^{(M-1)\times(M-1)}$ is the identity matrix and $\tilde{\mathbf{b}}_1$ is now given by

$$\tilde{\mathbf{b}}_1 = \left[\frac{\Delta\tilde{\tau}}{2}(1-\eta_1)(U_0^{n+1}+U_0^n),\, 0,\, ...,\, 0,\, \frac{\Delta\tilde{\tau}}{2}(1+\eta_{M-1})(U_M^{n+1}+U_M^n)\right]^T.$$

3.2 Finite domain transformation

The second transformed problem is posed on the finite space domain (0, 1). We employ substitutions described in Zhu et al. (2004) to transform equation (6) to a problem posed on the finite domain $\Omega_x = (0, 1) \times [0, T]$. The substitution $S = xE/(1-x)$ transforms the local volatility function $\sigma(S)$ to

$$\sigma(x) = \delta\left(\frac{Ex}{1-x}\right)^{\alpha-1}.$$

Then the transformation

$$\bar{x} = x(1-x),\ r_q = (r-q),\ V(S,\tau) = EP(x,\tau)/(1-x),$$

gives rise to the following problem

$$\begin{aligned} &\frac{\partial P}{\partial \tau} = \frac{1}{2}\sigma^2(x)\bar{x}^2\frac{\partial^2 P}{\partial x^2} + r_q\bar{x}\frac{\partial P}{\partial x} + (r_q x - r)\,P,\ 0<x<1,\\ &P(x,0) = \max(1-2x,0),\ 0<x<1,\\ &P(0,\tau) = 0,\ 0\le\tau\le T,\\ &P(1,\tau) = e^{-r\tau},\ 0\le\tau\le T. \end{aligned} \tag{13}$$

To solve problem (13), we choose spacings $\Delta x = 1/M$ and $k = T/N$ and we consider a uniform grid $\Omega_x^{\Delta x,k} = \{(x_m, \tau_n), x_m = m\Delta x, 0 \le m \le M, \tau_n = nk, 0 \le n \le N\}$ on Ω_x. Letting $\lambda = k/(\Delta x)^2$ and $\mu = k/\Delta x$, a second-order discretisation with a Crank-Nicolson time stepping for equation (13) is given by

$$\mathbf{A}_2 P^{n+1} = \mathbf{B}_2 P^n + \mathbf{b}_2, \tag{14}$$

where $\mathbf{A}_2, \mathbf{B}_2 \in \mathbb{R}^{(M-1)\times(M-1)}$ are tridiagonal matrices given by

$$\mathbf{A}_2 = \begin{pmatrix} \xi_1^+ & -\varrho_1^+ & & & \\ -\varrho_2^- & \xi_2^+ & -\varrho_2^+ & & \\ \ddots & \ddots & \ddots & & \\ & -\varrho_{M-2}^- & \xi_{M-2}^+ & -\varrho_{M-2}^+ & \\ & & -\varrho_{M-1}^- & \xi_{M-1}^+ & \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} \xi_1^- & \varrho_1^+ & & & \\ \varrho_2^- & \xi_2^- & \varrho_2^+ & & \\ \ddots & \ddots & \ddots & & \\ & \varrho_{M-2}^- & \xi_{M-2}^- & \varrho_{M-2}^+ & \\ & & \varrho_{M-1}^- & \xi_{M-1}^- & \end{pmatrix},$$

and $\mathbf{b}_2$ is given by

$$\mathbf{b}_2 = \left[\varrho_1^-(P_0^{n+1} + P_0^n), 0, ..., 0, \varrho_{M-1}^+(P_M^{n+1} + P_M^n)\right]^T,$$

where for $1 \le m \le M-1$, $\varrho_m^{\pm} = (\varepsilon_m \pm \zeta_m)$, $\xi_m^{\pm} = 1 \pm 2\varepsilon_m \pm \epsilon_m$, $\varepsilon_m = \lambda\sigma^2(x_m)$ $\bar{x}_m^2/4$, $\zeta_m = \mu r_q \bar{x}_m/4$ and $\epsilon_m = k\left(r_q x_m - r\right)/2$.

3.3 Exponential time differencing scheme

We consider a direct discretisation of equation (6) by localising the problem to the domain $\Omega_S = [0, S_{\max}] \times [0, T]$. Considering a set of grid points (S_m, τ_n) where $S_m = mh$ with $h = S_{\max}/M$ and $\tau_n = nk$ with $k = T/N$, let $V_m^n = V(S_m, \tau_n)$ and consider central-difference approximations to the first and second-order derivatives given by $\Delta_S V_m = (V_{m+1} - V_{m-1})/(2h)$ and $\Delta_S^2 V_m = (V_{m+1} - 2V_m + V_{m-1})/h^2$. Then letting $f_m = \delta^2 S_m^{2\alpha}/2$ and $g_m = (r-q)S_m$, central difference approximations at the interior grid points of equation (6) gives

$$V_m'(\tau) = f_m \Delta_S^2 V_m + g_m \Delta_S V_m - rV_m, \ 1 \le m \le M-1. \tag{15}$$

At the end points, we can either employ the financial boundary conditions given by $V_0(\tau) = Ee^{-r\tau}$ and $V_M(\tau) = 0$ or use one-sided boundary conditions. Our scheme uses one-sided approximations which is described next.

Let $\Delta_+ V_m = V_{m+1} - V_m$, $\Delta_- V_m = V_m - V_{m-1}$ and consider the operators $\delta_- = \left(\Delta_- + \frac{1}{2}\Delta_-^2\right)/(2h)$, $\delta_+ = \left(\Delta_+ + \frac{1}{2}\Delta_+^2\right)/(2h)$ and $\delta_+^2 = \left(\Delta_+^2 - \Delta_+^3\right)/h^2$. Then at the left boundary point where $m = 0$, Δ_S and Δ_S^2 are replaced by δ_+ and δ_+^2 as given by

$$V_0'(\tau) = f_0 \delta_+^2 V_0 + g_0 \delta_+ V_0 - rV_0,$$

and at the right boundary where $m = M$, Δ_S is replaced by δ_- and since the option has a payoff which is at most linear in S, we use the linear asymptotic boundary condition $V_{SS}(S, \tau) = 0$ as $S \to \infty$, implemented as $V_{SS}(S_M, \tau) = 0$ to obtain the approximation

$$V_M'(\tau) = g_M \delta_- V_M - rV_M.$$

Then letting $\mathbf{V} = [V_0(\tau), V_1(\tau), ..., V_M(\tau)]^T$, we obtain the semi-discrete system

$$\mathbf{V}'(\tau) = \mathbf{L}\mathbf{V}(\tau), \tag{16}$$

where $\mathbf{L} \in \mathbb{R}^{(M+1)\times(M+1)}$ is given by

$$\mathbf{L} = \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{b}_0 & \mathfrak{c}_0 & \mathfrak{d}_0 & 0 & 0 \\ \mathfrak{a}_1 & \mathfrak{b}_1 & \mathfrak{c}_1 & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & \mathfrak{a}_{M-1} & \mathfrak{b}_{M-1} & \mathfrak{c}_{M-1} \\ 0 & \cdots & 0 & \mathfrak{a}_M & \mathfrak{b}_M & \mathfrak{c}_M \end{pmatrix}, \tag{17}$$

with

$$\mathfrak{a}_m = \frac{f_m}{h^2} - \frac{g_m}{2h},\ \mathfrak{b}_m = -\frac{2f_m}{h^2} - r,\ \mathfrak{c}_m = \frac{f_m}{h^2} + \frac{g_m}{2h},\ 1 \le m \le M-1,$$

$$\mathfrak{a}_0 = \frac{2f_0}{h^2} - \frac{3g_0}{2h} - r,\ \mathfrak{b}_0 = -\frac{5f_0}{h^2} + \frac{2g_0}{h},\ \mathfrak{c}_0 = \frac{4f_0}{h^2} - \frac{g_0}{2h},\ \mathfrak{d}_0 = -\frac{f_0}{h^2},$$

and

$$\mathfrak{a}_M = \frac{g_M}{2h},\ \mathfrak{b}_M = -\frac{2g_M}{h},\ \mathfrak{c}_M = \frac{3g_M}{2h} - r.$$

An exponential time integration of equation (16) gives

$$\mathbf{V}(T) = e^{\mathbf{L}T}\mathbf{V}(0). \tag{18}$$

To obtain an efficient scheme, we require a fast computation of the matrix exponential in equation (18). This can be achieved by using the Carathéodory-Fejér (CF) procedure in Trefethen and Gutknecht (1983) to compute best rational approximations of $e^{\mathbf{L}T}$ (Cody et al., 1969). To describe this procedure, consider the computation of the integral

$$I = \frac{1}{2\pi i}\int_{\Upsilon} e^z f(z)\, dz, \tag{19}$$

where the function f is analytic in the neighbourhood of the negative real axis and Υ is a Hankel contour [see Figure 5.1 in Schmelzer and Trefethen (2007)].

Let $\mathfrak{R}_\eta(z) = \mathcal{P}(z)/\mathcal{Q}(z)$ be a rational function of two polynomials $\mathcal{P}$ of degree $\eta - 1$ and $\mathcal{Q}$ of degree η that is a good approximation to e^z on $(-\infty, 0)$. Suppose $\mathfrak{R}_\eta$ has poles at z_1, z_2, ..., z_η and let c_1, c_2, ..., c_η be the corresponding residues of $\mathfrak{R}_\eta$. Then, if Υ' is a contour lying between $(-\infty, 0)$ and the points $(z_j)_{j=1}^{\eta}$, the integral

$$I_\eta = \frac{1}{2\pi i}\int_{\Upsilon'} \mathfrak{R}_\eta(z) f(z)\, dz,$$

is a good approximation to (19). Using the partial fraction expansion of $\mathfrak{R}_\eta(z)$ in the form

$$\mathfrak{R}_\eta(z) = \sum_{j=1}^{\eta} \frac{c_j}{z - z_j}, \tag{20}$$

we see that

$$I_\eta = \sum_{j=1}^{\eta} c_j f(z_j). \qquad (21)$$

This shows that equation (21) is a quadrature formula for approximating the integral I in equation (19).

Algorithm 1 ETD-CF algorithm

begin

1. Initialisation:

(1a). Input elasticity factor α, r, q, initial stock price S, E, T, σ and compute $\delta = \sigma S^{1-\alpha}$.

(1b). Choose space steps M and set $\eta = 12$.

(1c). Set $S_{\max} = 2E$; compute $h = S_{\max}/M$.

(1d). Construct the S-grid: $\mathbf{S} = [S_{\min}, S_1, ..., S_{M-1}, S_M = S_{\max}]$.

(1e). Set payoff vector $\mathbf{V}(0) = \max(E - \mathbf{S}, 0)$.

(1f). Initialise $\mathbf{V}(T) \in \mathbb{R}^{M+1}$ as a vector of zeros.

2. Build ODE system:

(2a). Construct matrix $\mathbf{L}$ in equation (17).

3. Time integration:

(3a). Compute the poles $\mathbf{z} = [z_1, z_2, ..., z_\eta]$ and residues $\mathbf{c} = [c_1, c_2, ..., c_\eta]$ using the CF procedure.

for $j = 1 : 2 : \eta$ **do**

(3b). Find d_j from $(\mathbf{L}T - \mathbf{z}_j\mathbf{I})d_j = \mathbf{V}(0)$.

(3c). $\mathbf{V}(T) = \mathbf{V}(T) + \mathbf{c}_j d_j$.

end

(3d). $\mathbf{V}(T) = 2\mathbf{V}(T)$.

4. Output option price:

(4a). Interpolate $\mathbf{V}(T)$ to obtain the option price at the initial stock price and output price.

end

Now if we let Υ to be a contour that encloses the spectrum of $\mathbf{L}T$ and $\mathbf{I} \in \mathbb{R}^{(M+1)\times(M+1)}$ be the identity matrix, then we have

$$e^{\mathbf{L}T}\mathbf{V}(0) = \frac{1}{2\pi i}\int_{\Upsilon} e^{sT}\left(s\mathbf{I} - \mathbf{L}T\right)^{-1}\mathbf{V}(0)\, ds.$$

Generalising equation (20) to the matrix $e^{\mathbf{L}T}$, we find that a rational approximation to $e^{\mathbf{L}T}\mathbf{V}(0)$ is given by the partial fraction expansion

$$e^{\mathbf{L}T}\mathbf{V}(0) \approx \sum_{j=1}^{\eta} c_j(\mathbf{L}T - z_j\mathbf{I})^{-1}\mathbf{V}(0). \qquad (22)$$

The discretisation matrix $\mathbf{L}$ is a tridiagonal real matrix, which means that the poles and residues appear in complex conjugate pairs. Therefore the algorithm requires only $\eta/2$ tridiagonal solves for computing the price and this results in a linear computational complexity of $\mathcal{O}(M)$.

3.3.1 Pseudocode

A pseudocode demonstrating how to implement the exponential time differencing scheme with the CF procedure (ETD-CF) algorithm is given in Algorithm 1.

4 Numerical results

The numerical procedures are assessed on various examples. For the schemes based on transformed PDEs, the scheme in equation (12) is denoted by FDM-WZ and the scheme in equation (14) by ZW. The exponential time differencing scheme (18) with the CF procedure (22) is denoted ETD-CF. Comparison is also done against a Crank-Nicolson discretisation of equation (6) which is denoted by CN.

For a set of options, comparisons between the different methods are carried on the basis of their computational time required to price all the options in the set and on the accuracy measure, the root mean square error (RMSE). We choose $S_{\max} = 2E$ and this choice of $S_{\max}$ is motivated by the fact that in the literature, it is common taken to be twice the strike price. For the FDM-WZ, ZW and CN schemes we use the same number of spatial and temporal steps while for ETD-CF we use M spatial steps and a single time step is required with η = 12. All numerical tests have been performed using Mathematica 9 on a Core i7 laptop with 8GB RAM and speed 3.20 GHz.

Example 1

A set of nine European put options, each having a maturity of four months are priced under a CEV process with parameters α = 0.875, S = 40, q = 0 and r = 0.05. Table 1 shows computed prices for different values of σ and E. We observe that although ETD-CF(512) and FDM-WZ(512) have similar accuracies, the exponential time integration scheme is faster with timings approximately half of those required by the other methods.

Table 1 European put options for α = 0.875

σ	E	*PDE-methods*					*Closed form*
		FDM-WZ (512)	*ZW (512)*	*CN (512)*	*ETD-CF (512)*	*ETD-CF (1,024)*	
0.2	35	0.20032	0.20042	0.20100	0.20083	0.20078	0.20071
	40	1.51205	1.51375	1.51472	1.51491	1.51490	1.51481
	45	4.75303	4.75567	4.75611	4.75637	4.75618	4.75610
0.3	35	0.69720	0.69805	0.69860	0.69852	0.69492	0.69843
	40	2.41796	2.41911	2.41976	2.41988	2.41987	2.41980
	45	5.49468	5.49723	5.49753	5.49790	5.49778	5.49774
0.4	35	1.34750	1.32848	1.34895	1.34891	1.34891	1.34891
	40	3.32432	3.32519	3.32569	3.32577	3.32574	3.32571
	45	6.33607	6.33823	6.33846	6.33880	6.33872	6.33868
RMSE		1.3e-3	4.6e-4	2.3e-4	1.1e-4	5.9e-5	
cpu(s)		0.620	1.894	0.784	0.252	0.351	1.243

Refining the grids with M = 1,024 leads to ETD-CF numerical solutions in very close agreements with analytical solutions.

Example 2

Our second numerical example considers the computation of a set of 20 European put options, each having a maturity of six months. The initial stock price is S = 100 and α = –1. Table 2 shows computed put prices for different values of the triplet (r, q, σ) evaluated at different strike prices using M = 512. We observe that although CN achieves a similar RMSE as ETD-CF, the exponential time differencing scheme is more than three times faster. RMSE values and computational times for the numerical schemes using discretisation matrices with different number of spatial nodes M = 64, 128, 256, 512 and 1,024 were also computed and the log-log plot illustrated in Figure 1 clearly demonstrates the superiority of ETD-CF.

Table 2 European put options for α = –1, T = 0.5, S = 100, M = 512

(r, q, σ)	*E*	*FDM-WZ*	*ZW*	*CN*	*ETDCF*	*Closed form*
(0.07, 0.03, 0.2)	80	0.58509	0.58429	0.58464	0.58464	0.58466
	90	1.75897	1.75735	1.75854	1.75854	1.75857
	100	4.59650	5.59457	4.59613	4.59613	4.59666
	110	9.98186	9.98009	9.98176	9.98164	9.98164
	120	17.84267	17.84230	17.84251	17.84259	17.84235
(0.07, 0.03, 0.4)	80	4.67905	4.59191	4.59239	4.59239	4.59247
	90	6.93147	6.84297	6.89373	6.89373	6.89376
	100	10.27142	10.18254	10.18328	10.18328	10.18355
	110	14.96738	14.87815	14.87923	14.87923	14.87921
	120	21.14677	21.05775	21.05859	21.05859	21.05859
(0.07, 0, 0.3)	80	2.05757	2.04021	2.04073	2.04073	2.04081
	90	3.78722	3.76947	3.77043	3.77043	3.77049
	100	6.77743	6.75933	6.76037	6.76037	6.76073
	110	11.47547	11.45738	11.45877	11.45870	11.45874
	120	18.05168	18.03430	18.03507	18.03504	18.03502
(0.03, 0.07, 0.3)	80	2.99945	2.97520	2.97571	2.97571	2.97576
	90	5.39389	5.36903	5.37003	5.37003	5.37003
	100	9.33954	9.31464	9.31565	9.31565	9.31601
	110	15.18431	15.15924	15.16059	15.16054	15.16055
	120	22.85442	22.82972	22.83048	22.83046	22.83047
RMSE		1.6e-2	4.6e-4	7.8e-5	7.5e-5	
cpu(s)		1.280	2.721	2.283	0.608	

We also report the L_2 and L_∞ error norms calculated using the formulas

$$\|\text{error}\|_{L_2} = \left(\frac{1}{M} \sum_{i=0}^{M-1} |V_i - \text{exact}_i|^2 \right)^{\frac{1}{2}}, \quad \|\text{error}\|_{L_\infty} = \max_{0 \le i \le M-1} |V_i - \text{exact}_i|,$$

where M is the number of spatial grid nodes over the domain $(0, 2E]$ for the set of parameters α = –1, S = 100, E = 100, r = 0.07, q = 0.03, σ = 0.2 and T = 0.5 in Table 3. These results indicate a smooth second-order convergence rate in both L_2 and L_∞ error norms.

Figure 1 Plot of log RMSE against log cpu (see online version for colours)

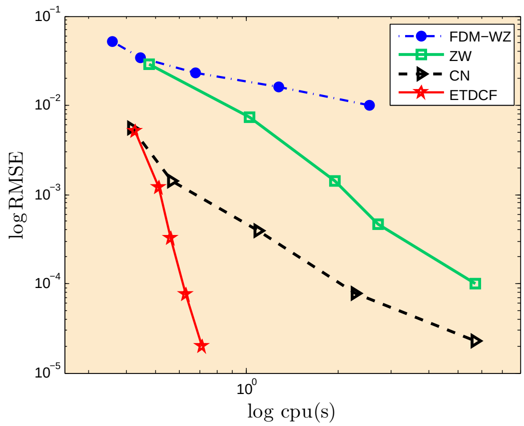

Table 3 Error norms and convergence rates

M	*Price*	*Error*	*Order*	L_2	*Order*	L_∞	*Order*
32	4.45477	1.4e-1	-	2.6e-2	-	1.4e-1	-
54	4.56230	3.4e-2	2.046	6.4e-3	2.021	3.4e-2	2.046
128	4.58813	8.5e-3	2.010	1.6e-3	1.984	8.5e-3	2.010
256	4.59453	2.1e-3	2.003	4.1e-4	1.987	2.1e-3	2.003
512	4.59613	5.4e-4	2.004	9.6e-5	2.091	5.3e-4	2.003
1,024	4.59653	1.3e-4	2.011	2.4e-5	1.935	1.3e-4	2.011
Exact				4.59666			

Example 3

In Table 4, we show the ability of the ETD-CF procedure to accurately price European put options when the elasticity factor is strongly negative. The data chosen for this example are: an initial stock price of $S = 100$, a maturity of $T = 0.5$ years, $E = 110$, $r = 0.05$, $q = 0$ and $\sigma = 0.2$. The results indicate that ETD-CF yields a smooth second-order convergence rate and the computed prices have sufficient accuracy.

Table 4 European put options for strongly negative values of the elasticity factor

M	$\alpha = -3$				$\alpha = -4$			
	Price	*Error*	*Order*	*cpu(s)*	*Price*	*Error*	*Order*	*cpu(s)*
64	9.32215	2.6e-2	-	0.022	9.14804	2.5e-2	-	0.019
128	9.34183	6.7e-3	1.971	0.035	9.16724	6.6e-3	1.967	0.030
256	9.34685	1.7e-3	1.966	0.043	9.17214	1.6e-3	1.962	0.046
512	9.34813	4.4e-4	1.973	0.059	9.17340	4.3e-4	1.971	0.052
1,024	9.34846	1.1e-4	2.013	0.067	9.17373	1.1e-4	2.015	0.063
Exact	9.34857				9.17383			

4.1 Hedging sensitivities

Example 4

We show that ETD-CF accurately computes the hedging sensitivities delta and gamma. We consider a set of nine European put options for α = –4, different strike prices with low and high volatilities and computed results are shown in Table 5. We observe that the computed greeks are the same as the analytical solutions (Larguinho et al., 2013).

Table 5 European put prices and greeks for $\alpha = -4$, $T = 0.5$, $S = 100$, $r = 0.05$, $q = 0$ and different at the money volatilities and strikes using M = 1,024 spatial steps

		European put					
σ	E	ETD-CF			Closed form		
		Price	Delta	Gamma	Price	Delta	Gamma
0.2	90	2.56671	–0.38182	0.03367	2.56677	–0.38182	0.03367
	100	4.56570	–0.55535	0.03367	4.56583	–0.55536	0.03366
	110	9.17386	–0.80798	0.02234	9.17383	–0.80799	0.02234
0.3	90	5.75334	–0.54567	0.02429	5.75338	–0.54568	0.02429
	100	7.63678	–0.64904	0.02335	7.63687	–0.64904	0.02335
	110	11.35665	–0.79116	0.01766	11.35664	–0.79117	0.01766
0.4	90	8.83063	–0.62045	0.01636	8.83065	–0.62045	0.01636
	100	10.64983	–0.70307	0.01633	10.64988	–0.70308	0.01633
	110	13.77012	–0.80092	0.01378	13.77012	–0.80093	0.01378
RMSE		2.8e-5	5.5e-6	1.7e-6			
cpu(s)			0.549				

Example 5

In this last example, we compute call option prices and corresponding greeks using ETD-CF. Table 6 shows the result for a set of 15 European call options for varying values of the elasticity parameter evaluated at different strike prices. At-the-money, in-the-money and out-of-money options are all accurately priced. Moreover, an average of 64 milliseconds is sufficient for computing each option price and the two greeks.

5 Conclusions

Among the different option pricing models proposed to capture the volatility skew implied by market option prices, we observed that the CEV process has some important properties. However, the valuation of European options and hedging sensitivities is not as simple as in the case of the Black-Scholes model as the functions involved can give rise to unstable and slow computations for some specific cases. As an alternative to the analytical price computation, we proposed an easy-to-implement finite difference scheme using an exponential time discretisation for the price computation. We gave a wide range of examples illustrating fast and accurate computations of option prices. The exponential time integration scheme leads to a very fast technique for European option pricing since a single step in the time stepping is required.

Table 6 European call option prices and greeks for S = 100, σ = 0.25, T = 0.5, r = 0.1, q = 0, M = 1,024 for different strikes and values of the elasticity factor

		European call					
		ETD-CF			*Closed form*		
E	α	*Price*	*Delta*	*Gamma*	*Price*	*Delta*	*Gamma*
95	0.5	12.66291	0.72925	0.01893	12.66292	0.72925	0.01893
	0	12.74260	0.71178	0.01979	12.74262	0.71179	0.01979
	–1	12.91973	0.67348	0.02180	12.91975	0.67348	0.02180
	–2	13.12136	0.62855	0.02442	13.12138	0.62855	0.02442
	–3	13.39483	0.57427	0.02775	13.39481	0.57429	0.02774
100	0.5	9.58443	0.62820	0.02162	9.58454	0.62820	0.02162
	0	9.59141	0.61134	0.02223	9.59152	0.61134	0.02223
	–1	9.62050	0.57625	0.02365	9.62061	0.57625	0.02365
	–2	9.67460	0.53802	0.02548	9.67471	0.53802	0.02547
	–3	9.76370	0.49462	0.02784	9.76379	0.49462	0.02784
105	0.5	7.01699	0.52019	0.02278	7.01700	0.52020	0.02278
	0	6.94029	0.50279	0.02313	6.94030	0.50280	0.02313
	–1	6.80351	0.46860	0.02399	6.80352	0.46860	0.02399
	–2	6.68902	0.43437	0.02509	6.68903	0.43437	0.02509
	–3	6.59985	0.39880	0.02654	6.59985	0.39880	0.02654
RMSE		2.6e-5	6.8e-6	4.7e-6			
cpu(s)			0.960				

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