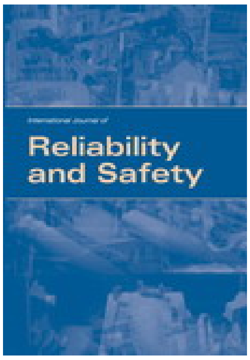

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Performance of batch service queue model with second optional service, repairable breakdown and standby server

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Abstract: In this paper, we investigate the performance of batch service queue model with second optional service, repairable breakdown and warm standby server. Both primary operating and warm standby servers provide First Essential Service (FES) and Second Optional Service (SOS) to customers, wherein FES is all arriving customers and only some of them may further request the SOS. The service times, failure times and repair times of both primary operating and warm standby server are assumed to follow exponential distributions. We use Runge-Kutta method to obtain the transient state probabilities and matrix-decomposition method to obtain the steady-state probabilities of the model. Also, a cost model is presented to determine the optimal service rates so that the expected cost is minimised. Finally, the effect of the model parameters on the system behaviour is demonstrated through numerical results and discussions.

Keywords: batch service queue model; first essential service; second optional service; repairable breakdown; standby server.

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1 Introduction

Availability of the system is an essential issue to measure the performance of a repairable system such as industrial system, manufacturing, power plants, telecommunication system, etc. A system may fail as time goes by, resulting in a loss of production, opportunities, goodwill, revenue, etc. The system can maintain a required or a high level of availability by keeping a standby server. If the primary operating server fails, the standby server is immediately replaced and continues to perform the assigned task. The provision of a standby server ensures the smooth functioning and availability of the server. The standby server can be classified as cold, warm, or hot standby. Suppose a standby server's failure rate is zero. In that case, it is termed a cold standby, a warm standby when the failure rate is non-zero but less than that of the primary operating server, and a hot standby when the failure rate is same as that of the primary operating server.

Machines (servers) are necessary for every industry, but they are also subjected to breakdown, resulting in interference. The repair facility should be available to fix the failed server otherwise the failed server has a negative effect on the system by lowering efficiency and hence increasing the total cost. In this scenario, several researchers have advocated the provision of standby servers to ensure the repairable system's efficiency and availability. Jain et al. (2014) investigated the performance of queuing system having a warm standby server and two heterogeneous servers. They computed the steady-state queuing size distribution using the matrix method for the Markovian server repair problem and switching failure. El-Said and El-Sherbeny (2005) studied the cold standby system with preventive maintenance and arbitrary change in units. They obtained the expected frequency of preventative maintenance, a busy period of the system, etc. Jain and Preeti (2014) presented the single server queue with repair and standbys, including the working vacation and server breakdown. They assumed that the server may have been breakdown whenever it has been in a busy state or a working vacation state. Also, they computed various performance measures using the recursive matrix method. Wang et al. (2006) evaluated distinct system configurations with warm standby server components and standby switching failures based on reliability and availability. Also, they construct explicit expressions for mean time to failure and the steady-state availability for four configurations. Ayyappan and Thilagavathy (2021) presented the queuing model with standby server, delayed repair, breakdown, single vacation, immediate feedback and impatient customers under the steady-state probability vector.

The authors use matrix analytic method to obtain the total number of customers present in the system. Readers may see other comprehensive research works of queuing models involving standby servers in Ke and Wu (2012); Kuo and Ke (2016); Hajee (2011); Jain and Rani (2013); Zhang and Zhang (2016); Jain and Gupta (2018); Radha et al. (2020) and Kotb and El-Ashkar (2022), etc.

The service process is one of the most significant aspects of a queuing system. Recently, there has been a lot of interest in queuing systems in which the service discipline involves more than one service, e.g., in bank operation, where bank clerks assist all arriving customers by creating bank accounts, making cash deposits and withdrawing money. Additionally, some customers may want additional services from bank clerks, such as online banking, after creating a bank account. This paper considers the server (primary or standby) providing two types of services, FES provided to all arriving customers. After completing FES, the customers may opt for SOS provided by the server (primary or standby) with probability r or leave the system with probability $(1-r)$. The previous studies dealing with the repairable server with SOS are found in Laxmi and Bhavani (2021) where they evaluated the availability of the SOS queuing model with combined warm standby and cold standby through indicator function. They obtain the steady state probabilities of the model using the matrix geometric method. The second optional repair service with warm standby is discussed by Gao (2021), who deals with the availability and reliability of a repairable fault-tolerant system. The reliability function and the meantime to the system's first failure are calculated using Laplace transform approach. Further, the steady-state probabilities of the system are obtained by solving the matrix equation. Laxmi et al. (2020) considered a queuing model with SOS, correlated reneging and working vacations. The authors use the matrix geometric method to derive the steady-state probabilities of the system.

Batch service refers to a queuing system in which customers are served in batches. It plays an essential role in many areas, such as manufacturing systems, telecommunication networks, transportation systems, etc. Numerous researchers have broadly studied the batch service queue, such as Ayyappan and Karpagam (2018) discussed a non-Markovian batch arrival general bulk service single server queuing system including server breakdown and repair, a standby server, several vacations, and re-service. They used a standby server when the primary server is under repair. Further, they computed the queue size at a random time using the probability generating function. Devipriya et al. (2014) investigated a fixed batch single server queuing system under multiple vacations with SOS. The probability of the number of customers in the queue during the server busy and on vacation is derived. Also, the average number of customers and variance of the system are computed using the probability generating function and Rouche's theorem.

In queuing theory, almost every process or system has a steady-state and transient state. Tarabia (2011) contributed to studying transient and steady-state behaviour. Tarabia considered the single server queue with balking, catastrophes, server failures and repairs, wherein transient and steady-state probabilities are obtained using probability generating function. Later, Kumar et al. (2019) studied queuing systems having customers' impatience with threshold under transient and steady-state. Using the probability generating function, they obtained average and variance of the number of customers in the system. For more work on queuing systems with transient state behaviour, one may refer Kumar and Sharma (2018, 2019); Legros (2019), Kumar and Sharma (2021), etc.

In real-life situations, queues with limited waiting space are more realistic. This situation may be seen in routers servicing arrival packets with varying speeds in a network. When the server is busy, arrival items wait in the queue, but when the waiting area is full, the arriving items are considered to be lost. A detailed survey on queues with limited waiting space can be found in Sikdar and Gupta (2008); Ammar et al. (2011); Gupta et al. (2019); Kuaban et al. (2020); Soodan and Kumar (2020); Sampath et al. (2020); Kumar et al. (2021), etc.

In existing literature reviews, no work has been reported for varying batch service queue model with SOS, repairable breakdown and standby server under transient and steady-state behaviours. This spurs us to investigate a batch service finite buffer queue model with SOS, repairable breakdown and warm standby server. The model becomes more adaptable by consideration of SOS along with varying batch size service. The key goals of this paper are as follows:

- To present the transient state probabilities using the Runge-Kutta method.
- To derive the steady-state probabilities recursively by using the matrix decomposition method.
- To optimise the service rates for both FES and SOS so that the expected cost is minimised.
- To examine the impact of various parameters on the system's performance measures.

The remaining part of the paper is structured as follows: Mathematical model description is presented in Section 2. Section 3 presents the transient state equations in matrix form. The steady-state is discussed in Section 4. The performance measures are presented in Section 5. Cost analysis is presented in Section 6. In Section 7, we present and discuss the numerical analysis followed by conclusions in Section 8.

2 Mathematical description of the model

This paper examines a batch queuing model with SOS in which a Poisson process determines the arrival with parameter λ. The system consists of two servers, one of which is the primary operating server and the other acts as a warm standby server. Initially, a primary server is operative and the second server is kept as a warm standby. When the primary server fails, the warm standby server replaces it. Both the primary operating server and the warm standby server are repairable. The primary operating server may fail at any time during FES (SOS) with exponential rate $\alpha_1\left(\alpha_2\right)$, and in such circumstances, the primary server immediately goes for a repair, which follows an exponential distribution with repair rates $\eta_{1F}\left(\eta_{1S}\right)$. Also, the warm standby server may fail during FES (SOS) with exponential distribution of parameter $\xi_1\left(\xi_2\right)$. Further, the switching standby server to replace the failed primary server may or may not be successful. We assume that switching the server to replace the failed server has a failure probability $\rho_1\left(\rho_2\right)$ for FES (SOS). Furthermore, if both the primary and standby servers fail, it is automatically routed to a repair facility, with repair times that follow an exponential distribution with parameter $\eta_{2F}\left(\eta_{2S}\right)$ for FES (SOS).

Both the primary and standby servers provide the FES to all incoming customers. After FES completion, the customers may opt for SOS provided by the server (primary or standby) with probability r or may exit the system with probability $\left(1-r\right)$. Service times in FES and SOS are exponential distributions with service rates μ_1 and μ_2, respectively, and services are offered in batches of no more than b customers. Therefore, if the server finds fewer or equal to b customers in the queue, he serves them all at once. If the server observes more than b customers in the queue, he accepts a batch of b on a first-come, first-served (FCFS) basis, while others continue waiting in the queue. The system has a limited capacity of size N, beyond which incoming customers are blocked from entering the system.

3 Transient state analysis of the model

3.1 Formulation of mathematical model

Let $X(t)$ be the number of customers in the system at time t, $S(t)$ be the server states at time t, which is given as

$$S(t)=\begin{cases} i, & \text{represents the state of the primary operating server}, i=0,1 \\ j, & \text{represents the state of the warm standby server}, j=0,1 \end{cases}$$

where 0 represents unavailability of the server and 1 represents availability of the server to service customers. Also, let $H(t)$ be the state of the service at time t, which is given as

$$H(t)=\begin{cases} 1, & \text{server is in FES} \\ 2, & \text{server is in SOS} \end{cases}$$

The process $\left\{\left(X(t),S(t),H(t)\right);t\geq 0\right\}$ defines a continuous time Markov process with state space

$$\Gamma=\left\{\left(n,s,h\right);0\leq n\leq N;s=\{0,1\}\cup\{0,1\};h=\{1,2\}\right\}$$

For the mathematical description of the birth-death process in a continuous time finite state space model at time t, the following notations are used:

- $P_{n,(1,1),1}(t)=$ transient state probability that there are n number of customer in the system, both primary operating server and warm standby server are available for the service and the server is providing FES, $0\leq n\leq N$.
- $P_{n,(1,1),2}(t)=$ transient state probability that there are n number of customer in the system, both primary operating server and warm standby server are available for the service and the server is providing SOS, $1\leq n\leq N$.

- $P_{n,(0,1),1}(t) =$ transient state probability that there are n number of customer in the system, primary operating server is in breakdown state and warm standby server is available for the service and the server is providing FES, $0 \le n \le N$.
- $P_{n,(0,1),2}(t) =$ transient state probability that there are n number of customer in the system, primary operating server is in breakdown state and warm standby server is available for the service and the server is providing SOS, $1 \le n \le N$.
- $P_{n,(1,0),1}(t) =$ transient state probability that there are n number of customer in the system, primary operating server is available for service and warm standby server is in breakdown and the server is providing FES, $0 \le n \le N$.
- $P_{n,(1,0),2}(t) =$ transient state probability that there are n number of customer in the system, primary operating server is available for service and warm standby server is in breakdown and the server is providing SOS, $1 \le n \le N$.
- $P_{n,(0,0),1}(t)\left(P_{n,(0,0),2}(t)\right) =$ transient state probability that there are n number of customer in the system, both primary operating server and warm standby server are in breakdown *while* is providing FES(SOS), $1 \le n \le N$.

The following set of equations are obtained using the probabilistic arguments in four cases:

Case I: $i = 1$ and $j = 1$, when both the primary and warm standby servers are available for the service.

$$P'_{0,(1,1),1}(t) = -\lambda P_{0,(1,1),1}(t) + (1-r)\mu_1 \sum_{k=1}^{b} P_{k,(1,1),1}(t) + \mu_2 \sum_{k=1}^{b} P_{k,(1,1),2}(t) \tag{1}$$

$$\begin{aligned} P'_{n,(1,1),1}(t) = &-(\lambda + \alpha_1 + \xi_1 + \mu_1) P_{n,(1,1),1}(t) + \lambda P_{n-1,(1,1),1}(t) \\ &+ (1-r)\mu_1 P_{n+b,(1,1),1}(t) + \mu_2 P_{n+b,(1,1),2}(t) + \eta_{1F} P_{n,(0,1),1}(t) \\ &+ \eta_{1F} P_{n,(1,0),1}(t) + \eta_{2F} P_{n,(0,0),1}(t), 1 \le n \le N-b \end{aligned} \tag{2}$$

$$\begin{aligned} P'_{n,(1,1),1}(t) = &-(\lambda + \alpha_1 + \xi_1 + \mu_1) P_{n,(1,1),1}(t) + \lambda P_{n-1,(1,1),1}(t) \\ &+ \eta_{1F} P_{n,(0,1),1}(t) + \eta_{1F} P_{n,(1,0),1}(t) + \eta_{2F} P_{n,(0,0),1}(t) \end{aligned}$$

$$N - b + 1 \le n \le N - 1 \tag{3}$$

$$\begin{aligned} P'_{N,(1,1),1}(t) = &-(\alpha_1 + \xi_1 + \mu_1) P_{N,(1,1),1}(t) + \lambda P_{N-1,(1,1),1}(t) \\ &+ \eta_{1F} P_{N,(0,1),1}(t) + \eta_{1F} P_{N,(1,0),1}(t) + \eta_{2F} P_{N,(0,0),1}(t) \end{aligned} \tag{4}$$

$$\begin{aligned} P'_{1,(1,1),2}(t) = &-(\lambda + \alpha_2 + \xi_2 + \mu_2) P_{1,(1,1),2}(t) + r\mu_1 P_{1,(1,1),1}(t) \\ &+ \eta_{1S} P_{1,(0,1),2}(t) + \eta_{1s} P_{1,(1,0),2}(t) + \eta_{2s} P_{1,(0,0),2}(t) \end{aligned} \tag{5}$$

$$\begin{aligned} P'_{n,(1,1),2}(t) = &-(\lambda + \alpha_2 + \xi_2 + \mu_2) P_{n,(1,1),2}(t) + \lambda P_{n-1,(1,1),2}(t) \\ &+ r\mu_1 P_{n,(1,1),1}(t) + \eta_{1S} P_{n,(0,1),2}(t) + \eta_{1s} P_{n,(1,0),2}(t) \\ &+ r\mu_1 P_{n,(1,1),1}(t) + \eta_{1S} P_{n,(0,1),2}(t) + \eta_{1s} P_{n,(1,0),2}(t) \end{aligned} \tag{6}$$

$$P'_{N,(1,1),2}(t) = -\left(\alpha_2 + \xi_2 + \mu_2\right) P_{N,(1,1),2}(t) + \lambda P_{N-1,(1,1),2}(t) + r\mu_1 P_{N,(1,1),1}(t) \\ + \eta_{1S} P_{N,(0,1),2}(t) + \eta_{1s} P_{N,(1,0),2}(t) + \eta_{2s} P_{N,(0,0),2}(t) \tag{7}$$

Case II: $i = 0$ and $j = 1$, when the primary server is in breakdown and warm standby server is available for service.

$$P'_{0,(0,1),1}(t) = -\lambda P_{0,(0,1),1}(t) + (1-r)\mu_1 \sum_{k=1}^{b} P_{k,(0,1),1}(t) + \mu_2 \sum_{k=1}^{b} P_{k,(0,1),2}(t) \tag{8}$$

$$P'_{n,(0,1),1}(t) = -\left(\lambda + \xi_1 + \eta_{1F} + \mu_1\right) P_{n,(0,1),1}(t) + \lambda P_{n-1,(0,1),1}(t) \\ + (1-r)\mu_1 P_{n+b,(0,1),1}(t) + \mu_2 P_{n+b,(0,1),2}(t) \\ + \alpha_1\left(1-\rho_1\right) P_{n,(1,1),1}(t), 1 \le n \le N-b \tag{9}$$

$$P'_{n,(0,1),1}(t) = -\left(\lambda + \xi_1 + \eta_{1F} + \mu_1\right) P_{n,(0,1),1}(t) + \lambda P_{n-1,(0,1),1}(t) \\ + \alpha_1\left(1-\rho_1\right) P_{n,(1,1),1}(t), N-b+1 \le n \le N-1 \tag{10}$$

$$P'_{N,(0,1),1}(t) = -\left(\xi_1 + \eta_{1F} + \mu_1\right) P_{N,(0,1),1}(t) + \lambda P_{N-1,(0,1),1}(t) + \alpha_1\left(1-\rho_1\right) P_{N,(1,1),1}(t) \tag{11}$$

$$P'_{1,(0,1),2}(t) = -\left(\lambda + \xi_2 + \eta_{1S} + \mu_2\right) P_{1,(0,1),2}(t) + r\mu_1 P_{1,(0,1),1}(t) + \alpha_2\left(1-\rho_2\right) P_{n,(1,1),2}(t) \tag{12}$$

$$P'_{n,(0,1),2}(t) = -\left(\lambda + \xi_2 + \eta_{1S} + \mu_2\right) P_{n,(0,1),2}(t) + \lambda P_{n-1,(0,1),2}(t) \\ + r\mu_1 P_{n,(0,1),1}(t) + \alpha_2\left(1-\rho_2\right) P_{n,(1,1),2}(t), 2 \le n \le N-1 \tag{13}$$

$$P'_{N,(0,1),2}(t) = -\left(\xi_2 + \eta_{1S} + \mu_2\right) P_{N,(0,1),2}(t) + \lambda P_{N-1,(0,1),2}(t) \\ + r\mu_1 P_{N,(0,1),1}(t) + \alpha_2\left(1-\rho_2\right) P_{N,(1,1),2}(t) \tag{14}$$

Case III: $i = 1$ and $j = 0$, when the primary server is available for service and warm standby server is in breakdown state.

$$P'_{0,(1,0),1}(t) = -\lambda P_{0,(1,0),1}(t) + (1-r)\mu_1 \sum_{k=1}^{b} P_{k,(1,0),1}(t) + \mu_2 \sum_{k=1}^{b} P_{k,(1,0),2}(t) \tag{15}$$

$$P'_{n,(1,0),1}(t) = -\left(\lambda + \alpha_1 + \eta_{1F} + \mu_1\right) P_{n,(1,0),1}(t) + \lambda P_{n-1,(1,0),1}(t) \\ + (1-r)\mu_1 P_{n+b,(1,0),1}(t) + \mu_2 P_{n+b,(1,0),2}(t) + \xi_1 P_{n,(1,1),1}(t) \\ 1 \le n \le N-b \tag{16}$$

$$P'_{n,(1,0),1}(t) = -\left(\lambda + \alpha_1 + \eta_{1F} + \mu_1\right) P_{n,(1,0),1}(t) + \lambda P_{n-1,(1,0),1}(t) \\ + \xi_1 P_{n,(1,1),1}(t), N-b+1 \le n \le N-1 \tag{17}$$

$$P'_{N,(1,0),1}(t) = -\left(\alpha_1 + \eta_{1F} + \mu_1\right) P_{N,(1,0),1}(t) + \lambda P_{N-1,(1,0),1}(t) + \xi_1 P_{N,(1,1),1}(t) \tag{18}$$

$$P'_{1,(1,0),2}(t) = -\left(\lambda + \alpha_2 + \eta_{1S} + \mu_2\right) P_{1,(1,0),2}(t) + r\mu_1 P_{1,(1,0),1}(t) + \xi_2 P_{n,(1,1),2}(t) \tag{19}$$

$$P'_{n,(1,0),2}(t) = -\left(\lambda + \alpha_2 + \eta_{1S} + \mu_2\right) P_{n,(1,0),2}(t) + \lambda P_{n-1,(1,0),2}(t) \\ + r\mu_1 P_{n,(1,0),1}(t) + \xi_2 P_{n,(1,1),2}(t), 2 \le n \le N-1 \tag{20}$$

$$P'_{N,(1,0),2}(t) = -\left(\alpha_2 + \eta_{1S} + \mu_2\right)P_{N,(1,0),2}(t) + \lambda P_{N-1,(1,0),2}(t)$$
$$+r\mu_1 P_{N,(1,0),1}(t) + \xi_2 P_{N,(1,1),2}(t) \tag{21}$$

Case IV: $i = 0$ and $j = 0$, when both of primary server and warm standby server are in breakdown state.

$$P'_{1,(0,0),1}(t) = -\left(\lambda + \eta_{2F}\right)P_{1,(0,0),1}(t) + \alpha_1\rho_1 P_{1,(1,1),1}(t) + \alpha_1 P_{1,(1,0),1}(t) \tag{22}$$

$$P'_{n,(0,0),1}(t) = -\left(\lambda + \eta_{2F}\right)P_{n,(0,0),1}(t) + \lambda P_{n-1,(0,0),1}(t) + \alpha_1\rho_1 P_{n,(1,1),1}(t)$$
$$+\alpha_1 P_{n,(1,0),1}(t) + \xi_1 P_{n,(0,1),1}(t), 2 \le n \le N-1 \tag{23}$$

$$P'_{N,(0,0),1}(t) = -\eta_{2F} P_{N,(0,0),1}(t) + \lambda P_{N-1,(0,0),1}(t) + \alpha_1\rho_1 P_{N,(1,1),1}(t)$$
$$+\alpha_1 P_{N,(1,0),1}(t) + \xi_1 P_{N,(0,1),1}(t) \tag{24}$$

$$P'_{1,(0,0),2}(t) = -\left(\lambda + \eta_{2S}\right)P_{1,(0,0),2}(t) + \alpha_2\rho_2 P_{1,(1,1),2}(t) + \alpha_2 P_{1,(1,0),2}(t) + \xi_2 P_{1,(0,1),2}(t) \tag{25}$$

$$P'_{n,(0,0),2}(t) = -\left(\lambda + \eta_{2S}\right)P_{n,(0,0),2}(t) + \lambda P_{n-1,(0,0),2}(t) + \alpha_2\rho_2 P_{n,(1,1),2}(t)$$
$$+\alpha_2 P_{n,(1,0),2}(t) + \xi_2 P_{n,(0,1),2}(t), 2 \le n \le N-1 \tag{26}$$

$$P'_{N,(0,0),2}(t) = -\eta_{2S} P_{N,(0,0),2}(t) + \lambda P_{N-1,(0,0),2}(t) + \alpha_2\rho_2 P_{N,(1,1),2}(t)$$
$$+\alpha_2 P_{N,(1,0),2}(t) + \xi_2 P_{N,(0,1),2}(t) \tag{27}$$

Let $P(t) = \left[P_{0,(1,1),1}(t), P_1(t), P_2(t), P_{0,(0,1),1}(t), P_3(t), P_4(t), P_{0,(1,0),1}, P_5(t), P_6(t), ..., P_7(t), P_8(t)\right]^T$, $P(0) = [1, 0, 0, ..., 0]^T$ be the vectors of transient state probabilities, where

$$P_1(t) = \left(P_{1,(1,1),1}(t), ..., P_{N,(1,1),1}(t)\right),\ P_2(t) = \left(P_{1,(1,1),2}(t), ..., P_{N,(1,1),2}(t)\right)$$

$$P_3(t) = \left(P_{1,(0,1),1}(t), .., P_{N,(0,1),1}(t)\right),\ P_4(t) = \left(P_{1,(0,1),2}(t), ..., P_{N,(0,1),2}(t)\right)$$

$$P_5(t) = \left(P_{1,(1,0),1}(t), ..., P_{N,(1,0),1}(t)\right),\ P_6(t) = \left(P_{1,(1,0),2}(t), ..., P_{N,(1,0),2}(t)\right)$$

$$P_7(t) = \left(P_{1,(0,0),1}(t), ..., P_{N,(0,0),1}(t)\right),\ P_8(t) = \left(P_{1,(0,0),2}(t), ..., P_{N,(0,0),2}(t)\right)$$

we obtain the system of equations as

$$\frac{d}{dt}P(t) = QP(t) \tag{28}$$

Equation (28) is derived using Chapman-Kolmogorov equations. The reader may refer Theorem 2.15 in Shortle et al. (2018) for more details.

The co-efficient matrix of the above system of equations is denoted by Q and it is presented as follows:

$$Q = \begin{pmatrix} -\lambda & A_{1,2} & A_{1,3} & 0 & A_{1,5} & A_{1,6} & 0 & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & A_{2,9} & A_{2,10} & A_{2,11} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & A_{3,9} & A_{3,10} & A_{3,11} \\ 0 & A_{4,2} & A_{4,3} & -\lambda & A_{4,5} & A_{4,6} & 0 & A_{4,8} & A_{4,9} & A_{4,10} & A_{4,11} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} & A_{5,8} & A_{5,9} & A_{5,10} & A_{5,11} \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} & A_{6,9} & A_{6,10} & A_{6,11} \\ 0 & A_{7,2} & A_{7,3} & 0 & A_{7,5} & A_{7,6} & -\lambda & A_{7,8} & A_{7,9} & A_{7,10} & A_{7,11} \\ A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8} & A_{8,9} & A_{8,10} & A_{8,11} \\ A_{9,1} & A_{9,2} & A_{9,3} & A_{9,4} & A_{9,5} & A_{9,6} & A_{9,7} & A_{9,8} & A_{9,9} & A_{9,10} & A_{9,11} \\ A_{10,1} & A_{10,2} & A_{10,3} & A_{10,4} & A_{10,5} & A_{10,6} & A_{10,7} & A_{10,8} & A_{10,9} & A_{10,10} & A_{10,11} \\ A_{11,1} & A_{11,2} & A_{11,3} & A_{11,4} & A_{11,5} & A_{11,6} & A_{11,7} & A_{11,8} & A_{11,9} & A_{11,10} & A_{11,11} \end{pmatrix}$$

is a $(8N+3)\times(8N+3)$ square matrix. The entries of the matrix Q are listed below:

$$A_{1,2} = \begin{pmatrix} \lambda & 0 & \cdots & 0 \end{pmatrix}_{1\times N},\ A_{2,8} = \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_1 \end{pmatrix}_{N\times N},\ A_{4,5} = \begin{pmatrix} \lambda & 0 & \cdots & 0 \end{pmatrix}_{1\times N}$$

$$A_{2,1} = \begin{pmatrix} a_{i,1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{N\times 1},\ A_{2,2} = \begin{pmatrix} \beta_1-\lambda & \lambda & 0 & \cdots & 0 & 0 \\ M_{1,0} & \beta_1-\lambda & \lambda & \cdots & 0 & 0 \\ M_{2,0} & M_{2,1} & \beta_1-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & M_{N-2,2} & \cdots & \beta_1-\lambda & \lambda \\ M_{N-1,0} & M_{N-1,1} & M_{N-1,2} & \cdots & M_{N-1,N-2} & \beta_1 \end{pmatrix}_{N\times N}$$

where

$$\beta_1 = -(\alpha_1 + \xi_1 + \mu_1), a_{i,1} = (1-r)\mu_1, \text{for } i = 1, 2, ..., b \text{ and } b \le N$$

$$M_{i,j} = \begin{cases} (1-r)\mu_1, & \text{if } \dfrac{i-j}{b} = 1 \text{ for } 1 \le i \le N-1, 0 \le j \le N-2, b \le N \\ 0, & \text{otherwise} \end{cases}$$

$$A_{2,10} = \begin{pmatrix} \alpha_1\rho_1 & 0 & \cdots & 0 \\ 0 & \alpha_1\rho_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1\rho_1 \end{pmatrix}_{N\times N}, A_{2,5} = \begin{pmatrix} \alpha_1(1-\rho_1) & 0 & \cdots & 0 \\ 0 & \alpha_1(1-\rho_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1(1-\rho_1) \end{pmatrix}_{N\times N}$$

$$A_{2,3} = \begin{pmatrix} r\mu_1 & 0 & \cdots & 0 \\ 0 & r\mu_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r\mu_1 \end{pmatrix}_{N\times N}, A_{3,6} = \begin{pmatrix} \alpha_2(1-\rho_2) & 0 & \cdots & 0 \\ 0 & \alpha_2(1-\rho_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_2(1-\rho_2) \end{pmatrix}_{N\times N}$$

$$A_{3,1} = \begin{pmatrix} s_{i,1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{N\times 1}, A_{3,2} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ B_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ B_{2,0} & B_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{N-2,0} & B_{N-2,1} & B_{N-2,2} & \cdots & 0 & 0 \\ B_{N-1,0} & B_{N-1,1} & B_{N-1,2} & \cdots & B_{N-1,N-2} & 0 \end{pmatrix}_{N\times N}$$

where

$$s_{i,1} = \mu_2, \text{for } i = 1, 2, ..., b \text{ and } b \le N$$

$$B_{i,j} = \begin{cases} \mu 2, & if \dfrac{i-j}{b} = 1 \text{ for } 1 \le i \le N-1, 0 \le j \le N-2, b \le N, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{3,3} = \begin{pmatrix} \beta_2 - \lambda & 0 & \cdots & 0 \\ 0 & \beta_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_2 \end{pmatrix}_{N\times N}, A_{3,9} = \begin{pmatrix} \xi_2 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_2 \end{pmatrix}_{N\times N}$$

where

$$\beta_2 = -(\alpha_2 + \xi_2 + \mu_2)$$

$$A_{3,11} = \begin{pmatrix} \alpha_2\rho_2 & 0 & \cdots & 0 \\ 0 & \alpha_2\rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_2\rho_2 \end{pmatrix}_{N\times N}, A_{5,2} = \begin{pmatrix} \eta_{1F} & 0 & \cdots & 0 \\ 0 & \eta_{1F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{1F} \end{pmatrix}_{N\times N}$$

$$A_{5,4}=\begin{pmatrix} a_{i,1}\\ \vdots\\ 0\\ 0\end{pmatrix}_{N\times 1},\quad A_{5,5}=\begin{pmatrix} \beta_3-\lambda & \lambda & 0 & \cdots & 0 & 0\\ M_{1,0} & \beta_3-\lambda & \lambda & \cdots & 0 & 0\\ M_{2,0} & M_{2,1} & \beta_3-\lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ M_{N-2,0} & M_{N-2,1} & M_{N-2,2} & \cdots & \beta_3-\lambda & \lambda\\ M_{N-1,0} & M_{N-1,1} & M_{N-1,2} & \cdots & M_{N-1,N-2} & \beta_3\end{pmatrix}_{N\times N}$$

where

$$\beta_3=-\left(\xi_1+\eta_{1F}+\mu_1\right),\ a_{i,1}=(1-r)\mu_1,\ \text{for } i=1,2,\ldots,b\ \text{ and } b\le N$$

$$M_{i,j}=\begin{cases}(1-r)\mu_1, & \text{if } \dfrac{i-j}{b}=1 \text{ for } 1\le i\le N-1,\ 0\le j\le N-2,\ b\le N\\ 0, & \text{otherwise}\end{cases}$$

$$A_{5,10}=\begin{pmatrix} \xi_1 & 0 & \cdots & 0\\ 0 & \xi_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \xi_1\end{pmatrix}_{N\times N},\quad A_{6,3}=\begin{pmatrix} \eta_{1S} & 0 & \cdots & 0\\ 0 & \eta_{1S} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \eta_{1S}\end{pmatrix}_{N\times N}$$

$$A_{6,4}=\begin{pmatrix} s_{i,1}\\ \vdots\\ 0\\ 0\end{pmatrix}_{N\times 1},\quad A_{6,5}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ B_{1,0} & 0 & 0 & \cdots & 0 & 0\\ B_{2,0} & B_{2,1} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ B_{N-2,0} & B_{N-2,1} & B_{N-2,2} & \cdots & 0 & 0\\ B_{N-1,0} & B_{N-1,1} & B_{N-1,2} & \cdots & B_{N-1,N-2} & 0\end{pmatrix}_{N\times N}$$

where

$$s_{i,1}=\mu_2,\ \text{for } i=1,2,\ldots,b\ \text{ and } b\le N$$

$$B_{i,j}=\begin{cases}\mu_2, & \text{if } \dfrac{i-j}{b}=1 \text{ for } 1\le i\le N-1,\ 0\le j\le N-2,\ b\le N\\ 0, & \text{otherwise}\end{cases}$$

$$A_{6,6}=\begin{pmatrix} \beta_4-\lambda & \lambda & 0 & \cdots & 0 & 0\\ 0 & \beta_4-\lambda & \lambda & \cdots & 0 & 0\\ 0 & 0 & \beta_4-\lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_4-\lambda & \lambda\\ 0 & 0 & 0 & \cdots & 0 & \beta_4\end{pmatrix}_{N\times N},\quad A_{7,8}=\begin{pmatrix}\lambda & 0 & \cdots & 0\end{pmatrix}_{1\times N}$$

where

$$\beta_4 = -\left(\xi_2 + \eta_{1S} + \mu_2\right)$$

$$A_{5,6} = \begin{pmatrix} r\mu_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r\mu_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r\mu_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r\mu_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & r\mu_1 \end{pmatrix}_{N\times N}, A_{8,2} = \begin{pmatrix} \eta_{1F} & 0 & \cdots & 0 \\ 0 & \eta_{1F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{1F} \end{pmatrix}_{N\times N}$$

$$A_{8,9} = \begin{pmatrix} r\mu_1 & 0 & \cdots & 0 \\ 0 & r\mu_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r\mu_1 \end{pmatrix}_{N\times N}, A_{8,10} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1 \end{pmatrix}_{N\times N}$$

$$A_{8,7} = \begin{pmatrix} a_{i,1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{N\times 1}, A_{8,8} = \begin{pmatrix} \beta_5-\lambda & \lambda & 0 & \cdots & 0 & 0 \\ M_{1,0} & \beta_5-\lambda & \lambda & \cdots & 0 & 0 \\ M_{2,0} & M_{2,1} & \beta_5-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & M_{N-2,2} & \cdots & \beta_5-\lambda & \lambda \\ M_{N-1,0} & M_{N-1,1} & M_{N-1,2} & \cdots & M_{N-1,N-2} & \beta_5 \end{pmatrix}_{N\times N}$$

where

$$\beta_5 = -\left(\alpha_1 + \eta_{1F} + \mu_1\right), a_{i,1} = (1-r)\mu_1, \text{for } i = 1, 2, ..., b \text{ and } b \le N$$

$$M_{i,j} = \begin{cases} (1-r)\mu_1, & \text{if } \dfrac{i-j}{b} = 1 \text{ for } 1 \le i \le N-1, 0 \le j \le N-2, b \le N \\ 0, & \text{otherwise} \end{cases}$$

$$A_{9,3} = \begin{pmatrix} \eta_{1S} & 0 & \cdots & 0 \\ 0 & \eta_{1S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{1S} \end{pmatrix}_{N\times N}, A_{10,2} = \begin{pmatrix} \eta_{2F} & 0 & \cdots & 0 \\ 0 & \eta_{2F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{2F} \end{pmatrix}_{N\times N}$$

$$A_{9,7} = \begin{pmatrix} s_{i,1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{N\times 1}, A_{9,8} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ B_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ B_{2,0} & B_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{N-2,0} & B_{N-2,1} & B_{N-2,2} & \cdots & 0 & 0 \\ B_{N-1,0} & B_{N-1,1} & B_{N-1,2} & \cdots & B_{N-1,N-2} & 0 \end{pmatrix}_{N\times N}$$

where

$$s_{i,1} = \mu_2, \text{ for } i = 1, 2, ..., b \text{ and } b \le N$$

$$B_{i,j} = \begin{cases} \mu_2, & \text{if } \dfrac{i-j}{b} = 1 \text{ for } 1 \le i \le N-1, 0 \le j \le N-2, b \le N \\ 0, & \text{otherwise} \end{cases}$$

$$A_{9,9} = \begin{pmatrix} \beta_6 - \lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & \beta_6 - \lambda & \lambda & \cdots & 0 & 0 \\ 0 & 0 & \beta_6 - \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_6 - \lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & \beta_6 \end{pmatrix}_{N \times N}, A_{9,11} = \begin{pmatrix} \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_2 \end{pmatrix}_{N \times N}$$

where

$$\beta_6 = -\left(\alpha_2 + \eta_{1S} + \mu_2\right)$$

$$A_{6,11} = \begin{pmatrix} \xi_2 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_2 \end{pmatrix}_{N \times N}, A_{10,10} = \begin{pmatrix} \beta_7 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & \beta_7 & \lambda & \cdots & 0 & 0 \\ 0 & 0 & \beta_7 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_7 & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\eta_{2F} \end{pmatrix}_{N \times N}$$

where

$$\beta_7 = -\left(\lambda + \eta_{2F}\right)$$

$$A_{11,3} = \begin{pmatrix} \eta_{2S} & 0 & \cdots & 0 \\ 0 & \eta_{2S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{2S} \end{pmatrix}_{N \times N}, A_{11,11} = \begin{pmatrix} \beta_8 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & \beta_8 & \lambda & \cdots & 0 & 0 \\ 0 & 0 & \beta_8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_8 & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\eta_{2S} \end{pmatrix}_{N \times N}$$

where

$$\beta_8 = -\left(\lambda + \eta_{2S}\right)$$

The remaining matrices are zero matrices with appropriate dimensions. To find the transient solution of the model, we employ a numerical approach using Runge-Kutta method of fourth order. Some of the numerical results are presented in Section 7.

4 Steady-state analysis of the model

The steady-state equation for (28) can be expressed in matrix-form as

$$0 = QP \tag{29}$$

From equation (29), the following steady-state equations can be written.

$$0 = -\lambda P_{0,(1,1),1} + P_1 A_{2,1} + P_2 A_{3,1} \tag{30}$$

$$0 = P_{0,(1,1),1} A_{1,2} + P_1 A_{2,2} + P_2 A_{3,2} + P_3 A_{5,2} + P_5 A_{8,2} + P_7 A_{10,2} \tag{31}$$

$$0 = P_1 A_{2,3} + P_2 A_{3,3} + P_4 A_{6,3} + P_6 A_{9,3} + P_8 A_{11,3} \tag{32}$$

$$0 = -\lambda P_{0,(0,1),1} + P_3 A_{5,4} + P_4 A_{6,4} \tag{33}$$

$$0 = P_1 A_{2,5} + P_{0,(0,1),1} A_{4,5} + P_3 A_{5,5} + P_4 A_{6,5} \tag{34}$$

$$0 = P_2 A_{3,6} + P_3 A_{5,6} + P_4 A_{6,6} \tag{35}$$

$$0 = -\lambda P_{0,(1,0),1} + P_5 A_{8,7} + P_6 A_{9,7} \tag{36}$$

$$0 = P_1 A_{2,8} - P_{0,(1,0),1} A_{7,8} + P_5 A_{8,8} + P_6 A_{9,8} \tag{37}$$

$$0 = P_2 A_{3,9} + P_5 A_{8,9} + P_6 A_{9,9} \tag{38}$$

$$0 = P_1 A_{2,10} + P_3 A_{5,10} + P_5 A_{8,10} + P_7 A_{10,10} \tag{39}$$

$$0 = P_2 A_{3,11} + P_4 A_{6,11} + P_6 A_{9,11} + P_8 A_{11,11} \tag{40}$$

Equation (37) yields

$$P_5 = P_1 \Psi_1 + P_{0,(1,0),1} \Psi_2 + P_6 \Psi_3 \tag{41}$$

where $\Psi_1 = -A_{2,8} A_{8,8}^{-1}, \Psi_2 = -A_{7,8} A_{8,8}^{-1}$ and $\Psi_3 = -A_{9,8} A_{8,8}^{-1}$

Using equation (41) into equation (38), we obtain

$$P_6 = P_1 \Psi_4 + P_{0,(1,0),1} \Psi_5 + P_2 \Psi_6 \tag{42}$$

where $\Psi_4 = -\Psi_1 A_{8,9} \left(\Psi_3 A_{8,9} + A_{9,9} \right)^{-1}, \Psi_5 = -\Psi_2 A_{8,9} \left(\Psi_3 A_{8,9} + A_{9,9} \right)^{-1}$ and $\Psi_6 = -A_{3,9} \left(\Psi_3 A_{8,9} + A_{9,9} \right)^{-1}$

Using (42) into (40), we obtain

$$P_8 = P_1 \Psi_7 + P_{0,(1,0),1} \Psi_8 + P_2 \Psi_9 \tag{43}$$

where $\Psi_7 = -\Psi_4 A_{9,11} A_{11,11}^{-1}, \Psi_8 = -\Psi_5 A_{9,11} A_{11,11}^{-1}$ and $\Psi_9 = -\left(A_{3,11} + \Psi_6 A_{9,11} \right) A_{11,11}^{-1}$.

Using equation (43) into equation (32), we get

$$P_4 = P_1 \Psi_{10} + P_{0,(1,0),1} \Psi_{11} + P_2 \Psi_{12} \tag{44}$$

where $\Psi_{10} = -\left(A_{2,3} + \Psi_4 A_{9,3} + \Psi_7 A_{11,3}\right)A_{6,3}^{-1}$, $\Psi_{11} = -\left(\Psi_5 A_{9,3} + \Psi_8 A_{11,3}\right)A_{6,3}^{-1}$ and $\Psi_{12} = -\left(A_{3,3} + \Psi_6 A_{9,3} + \Psi_9 A_{11,3}\right)A_{6,3}^{-1}$.

Using equation (44) into equation (35), we obtain

$$P_3 = P_1\Psi_{13} + P_{0,(1,0),1}\Psi_{14} + P_2\Psi_{15} \tag{45}$$

where $\Psi_{13} = -\Psi_{10}A_{6,6}A_{5,6}^{-1}$, $\Psi_{14} = -\Psi_{11}A_{6,6}A_{5,6}^{-1}$ and $\Psi_{15} = -(A_{3,6} + \Psi_{12}A_{6,6})A_{5,6}^{-1}$.

Using equations (45) into equation (33), we have

$$P_{0,(0,1),1} = P_1\Psi_{16} + P_{0,(1,0),1}\Psi_{17} + P_2\Psi_{18} \tag{46}$$

where $\Psi_{16} = \frac{1}{\lambda}\left(\Psi_{13}A_{5,4} + \Psi_{10}A_{6,4}\right)$, $\Psi_{17} = \frac{1}{\lambda}\left(\Psi_{14}A_{5,4} + \Psi_{11}A_{6,4}\right)$

and $\Psi_{18} = \frac{1}{\lambda}\left(\Psi_{15}A_{5,4} + \Psi_{12}A_{6,4}\right)$

Substituting equations (44), (45) and (46) into equation (34), we have

$$P_2 = P_1\Psi_{19} + P_{0,(1,0),1}\Psi_{20} \tag{47}$$

where $\Psi_{19} = -\left(A_{2,5} + \Psi_{16}A_{4,5} + \Psi_{13}A_{5,5} + \Psi_{10}A_{6,5}\right)\left(\Psi_{18}A_{4,5} + \Psi_{15}A_{5,5} + \Psi_{12}A_{6,5}\right)^{-1}$, and $\Psi_{20} = -\left(\Psi_{17}A_{4,5} + \Psi_{14}A_{5,5} + \Psi_{11}A_{6,5}\right)\left(\Psi_{18}A_{4,5} + \Psi_{15}A_{5,5} + \Psi_{12}A_{6,5}\right)^{-1}$.

Substituting equation (47) into equations (41) to (46), respectively, we obtain

$$P_5 = P_1\Psi_{21} + P_{0,(1,0),1}\Psi_{22} \tag{48}$$

where $\Psi_{21} = \Psi_1 + \Psi_4\Psi_3 + \Psi_{19}\Psi_6$, and $\Psi_{22} = \Psi_2 + \Psi_5\Psi_3 + \Psi_{20}\Psi_6$.

$$P_6 = P_1\Psi_{23} + P_{0,(1,0),1}\Psi_{24} \tag{49}$$

where $\Psi_{23} = \Psi_4 + \Psi_{19}\Psi_6$, and $\Psi_{24} = \Psi_5 + \Psi_{20}\Psi_6$.

$$P_8 = P_1\Psi_{25} + P_{0,(1,0),1}\Psi_{26} \tag{50}$$

where $\Psi_{25} = \Psi_7 + \Psi_{19}\Psi_9$, and $\Psi_{26} = \Psi_8 + \Psi_{20}\Psi_9$.

$$P_4 = P_1\Psi_{27} + P_{0,(1,0),1}\Psi_{28} \tag{51}$$

where $\Psi_{27} = \Psi_{10} + \Psi_{19}\Psi_{12}$, and $\Psi_{28} = \Psi_{11} + \Psi_{20}\Psi_{12}$.

$$P_3 = P_1\Psi_{29} + P_{0,(1,0),1}\Psi_{30} \tag{52}$$

where $\Psi_{29} = \Psi_{13} + \Psi_{19}\Psi_{15}$, and $\Psi_{30} = \Psi_{14} + \Psi_{20}\Psi_{15}$.

$$P_{0,(0,1),1} = P_1\Psi_{31} + P_{0,(1,0),1}\Psi_{32} \tag{53}$$

where $\Psi_{31} = \Psi_{16} + \Psi_{19}\Psi_{18}$, and $\Psi_{32} = \Psi_{17} + \Psi_{20}\Psi_{18}$.

Using equations (48) and (52) into (39), we get

$$P_7 = P_1\Psi_{33} + P_{0,(1,0),1}\Psi_{34} \tag{54}$$

where $\Psi_{33} = -\left(A_{2,10} + \Psi_{29}A_{5,10} + \Psi_{21}A_{8,10}\right)A_{10,10}^{-1}$, and

$\Psi_{34} = -(\Psi_{30}A_{5,10} + \Psi_{22}A_{8,10})A_{10,10}^{-1}$.

From equation (30), we have

$$P_{0,(1,1),1} = P_1\Psi_{35} + P_{0,(1,0),1}\Psi_{36} \tag{55}$$

where $\Psi_{35} = \frac{1}{\lambda}\left(A_{2,1} + \Psi_{19}A_{3,1}\right)$ and $\Psi_{36} = \frac{1}{\lambda}\left(\Psi_{20}A_{3,1}\right)$.

From equation (31), we have

$$P_1 = P_{0,(1,0),1}\Psi_{37} \tag{56}$$

and Ψ_{37} is given by

$\Psi_{37} = -BC^{-1}$.

where $B = \Psi_{36}A_{1,2} + \Psi_{20}A_{3,2} + \Psi_{30}A_{5,2} + \Psi_{22}A_{8,2} + \Psi_{34}A_{10,2}$

$C = \Psi_{35}A_{1,2} + A_{2,2} + \Psi_{19}A_{3,2} + \Psi_{29}A_{5,2} + \Psi_{21}A_{8,2} + \Psi_{33}A_{10,2}$

Substituting equation (56) into equations (47) to (55), respectively, we get

$$P_2 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{19} + \Psi_{20}\right) \tag{57}$$

$$P_5 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{21} + \Psi_{22}\right) \tag{58}$$

$$P_6 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{23} + \Psi_{24}\right) \tag{59}$$

$$P_8 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{25} + \Psi_{26}\right) \tag{60}$$

$$P_4 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{27} + \Psi_{28}\right) \tag{61}$$

$$P_3 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{29} + \Psi_{30}\right) \tag{62}$$

$$P_{0,(0,1),1} = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{31} + \Psi_{32}\right) \tag{63}$$

$$P_7 = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{33} + \Psi_{34}\right) \tag{64}$$

$$P_{0,(1,1),1} = P_{0,(1,0),1}\left(\Psi_{37}\Psi_{35} + \Psi_{36}\right) \tag{65}$$

Now all probabilities have been expressed as a function of $P_{0,(1,0),1}$. The normalisation condition is

$$P_{0,(1,1),1} + P_{0,(1,0),1} + P_{0,(0,1),1} + P_1e + P_2e + P_3e + P_4e + P_5e + P_6e$$

$$+P_7e + P_8e = 1 \tag{66}$$

where e is a vector of dimensions $(N \times 1)$ with all of the entries equal to one.

Substituting equations (56) to (65) into equation (66), we get

$$P_{0,(1,0),1} = \frac{1}{\Psi_{38}}, \tag{67}$$

where

$$\Psi_{38} = \Psi_{37}\Psi_{35} + \Psi_{36} + 1 + \Psi_{37}\Psi_{31} + \Psi_{32} + \Psi_{37}e + \Psi_{37}\Psi_{19}e + \Psi_{20}e$$
$$+\Psi_{37}\Psi_{29}e + \Psi_{30}e + \Psi_{37}\Psi_{27}e + \Psi_{28}e + \Psi_{37}\Psi_{21}e + \Psi_{22}e + \Psi_{37}\Psi_{23}e$$
$$+\Psi_{24}e + \Psi_{37}\Psi_{33}e + \Psi_{34}e + \Psi_{37}\Psi_{25}e + \Psi_{26}e$$

The derivation is complete for all steady-state probabilities, which can be used to find the measures of performance of the model.

5 Measures of performance

The analysis of this study is based on the following system performance measures.

- Expected system length during FES is given by

$$L_s FES(t) = \sum_{n=1}^{N} nP_{n,(1,1),1}(t) + \sum_{n=1}^{N} nP_{n,(0,1),1}(t)$$
$$+\sum_{n=1}^{N} nP_{n,(1,0),1}(t) + \sum_{n=1}^{N} nP_{n,(0,0),1}(t)$$

- Expected system length during SOS is given by

$$L_s SOS(t) = \sum_{n=1}^{N} nP_{n,(1,1),2}(t) + \sum_{n=1}^{N} nP_{n,(0,1),2}(t)$$
$$+\sum_{n=1}^{N} nP_{n,(1,0),2}(t) + \sum_{n=1}^{N} nP_{n,(0,0),2}(t)$$

- Overall expected system length is given by

$$L_s(t) = \sum_{n=1}^{N}\sum_{i=1}^{2} nP_{n,(1,1),i}(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} nP_{n,(0,1),i}(t)$$
$$+\sum_{n=1}^{N}\sum_{i=1}^{2} nP_{n,(1,0),i}(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} nP_{n,(0,0),i}(t)$$

- Let $P_{block}FES(t)$ denote the blocking probability of the customers to enter the system when the server provides FES

$$P_{block}FES(t) = P_{N,(1,1),1}(t) + P_{N,(0,1),1}(t) + P_{N,(1,0),1}(t) + P_{N,(0,0),1}(t)$$

- Let $P_{block}SOS(t)$ denote the blocking probability of the customers to enter the system when the server provides SOS

$$P_{block}SOS(t) = P_{N,(1,1),2}(t) + P_{N,(0,1),2}(t) + P_{N,(1,0),2}(t) + P_{N,(0,0),2}(t)$$

- Probability of the system being under repair

$$P_r(t) = P_rFES(t) + P_rSOS(t)$$

$$= \sum_{n=1}^{N}\sum_{i=1}^{2} P_{n,(1,0),i}(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} P_{n,(0,1),i}(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} P_{n,(0,0),i}(t)$$

- The effective arrival rate is given by

$$\lambda_{eff}(t) = \lambda(1 - P_{block}) = \lambda\left(\sum_{n=1}^{N-1}\sum_{i=1}^{2} P_{n,(1,1),i}(t) + \sum_{n=1}^{N-1}\sum_{i=1}^{2} P_{n,(0,1),i}(t)\right.$$

$$\left. + \sum_{n=1}^{N-1}\sum_{i=1}^{2} P_{n,(1,0),i}(t) + \sum_{n=1}^{N-1}\sum_{i=1}^{2} P_{n,(0,0),i}(t)\right)$$

- Availability of the server is given by

$$A_s(t) = (1-r)\sum_{n=0}^{N}\left(P_{n,(1,1),1}(t) + P_{n,(1,0),1}(t) + P_{n,(0,1),1}(t)\right)$$

$$+ \sum_{n=1}^{N}\left(P_{n,(1,1),2}(t) + P_{n,(1,0),2}(t) + P_{n,(0,1),2}(t)\right)$$

- The expected waiting time in the system during FES using Little's law, is given by

$$W_sFES(t) = \frac{L_sFES(t)}{\lambda_{eff}}$$

- The expected waiting time in the system during SOS using Little's law, we get

$$W_sSOS(t) = \frac{L_sSOS(t)}{\lambda_{eff}}$$

- The overall expected waiting time in the system using Little's law, we get

$$W_s(t) = \frac{L_s(t)}{\lambda_{eff}}$$

- Probability that the server is idle is given by

$$P_0(t) = P_{0,(1,1),1}(t) + P_{0,(1,0),1}(t) + P_{0,(0,1),1}(t)$$

- Fixing failure of warm-standby

$$F_f(t) = \alpha_1\rho_1\sum_{n=1}^{N} P_{n,(1,1),1}(t) + \alpha_2\rho_2\sum_{n=1}^{N} P_{n,(1,1),2}(t)$$

Note that one can obtain the steady-state performance measures by replacing the transient state probabilities with their steady-state counterparts.

6 Cost analysis of the model

In this section, we develop the total expected cost per customer per unit time using the performance measures mentioned in the previous section. The primary objective is to minimise the cost as much as possible by setting the optimal service rates. We define the cost components of the model as follows:

- $c_0 =$ cost per unit time when the server is idle,
- $c_1 =$ cost per unit per customer present in the system,
- $c_2 =$ cost per unit time when a failed server is present in the system,
- $c_3 =$ cost per unit time of the server under repair,
- $c_4 =$ cost per unit time for each server available for the service,
- $c_f\left(c_s\right) =$ fixed cost per unit time of FES(SOS).

Using the above definitions of each cost component and its associated system performance measure, the total expected cost function may be mathematically represented as follows:

$$F\left(\mu_1, \mu_2\right) = c_0 P_0 + c_1 L_s + c_2 F_f + c_3 P_r + c_4 A_s + \left(1 - P_0\right)\left(c_f \mu_1 + c_s \mu_2\right)$$

Let us consider the following optimisation problem

$$F\left(\mu_1^*, \mu_2^*\right) = \underset{s.t\ \mu_1, \mu_2 > 0}{\text{Minimise}} F\left(\mu_1, \mu_2\right) \tag{68}$$

We employ the Quasi-Newton method to search $\left(\mu_1, \mu_2\right)$ until the minimum values of $F\left(\mu_1, \mu_2\right)$ is achieved, say $F\left(\mu_1^*, \mu_2^*\right)$. The steps of Quasi-Newton method you may refer Laxmi and George (2020).

7 Numerical investigation

In this section, we illustrate the applicability of the Runge-Kutta approach in transient state and the matrix-decomposition method in steady-state. We compute the model numerically by taking the arbitrary model parameters that have close incidence with the practical situations as

$N = 7, b = 3, \lambda = 4.5, \mu_1 = 5.5, \mu_2 = 4.5, r = 0.2, \rho_1 = 0.65, \rho_2 = 0.55, \alpha_1 = 1.5, \alpha_2 = 1.1,$ $\eta_{1F} = 0.8, \eta_{1S} = 1.0, \eta_{2F} = 0.6, \eta_{2S} = 0.5, \xi_1 = 0.35, \xi_2 = 0.5, c_0 = 3, c_1 = 12, c_2 = 6, c_3 = 4, c_f = 6, c_s = 8$, with the assumption that $b \le N$. Tables and graphs are used to present the numerical computations.

Table 1 Waiting time in the system for different μ_1, α_1 and ξ_1 at steady-state

(μ_1,α_1)	W_sFES	W_sSOS	W_sFES	W_sSOS
	$\xi_1 = 0.2$		$\xi_1 = 0.3$	
(4.5, 0.9)	0.823173	0.239404	0.859868	0.245128
(4.5, 1.2)	0.958233	0.230523	1.00485	0.236029
(4.5, 1.5)	1.076750	0.222975	1.13263	0.228284
	$\xi_1 = 0.2$		$\xi_1 = 0.3$	
(5.5, 0.9)	0.681825	0.246254	0.712115	0.251926
(5.5, 1.2)	0.797443	0.237071	0.835707	0.242508
(5.5, 1.5)	0.899323	0.229248	0.945034	0.234473
	$\xi_1 = 0.2$		$\xi_1 = 0.3$	
(6.5, 0.9)	0.585429	0.252596	0.611258	0.258212
(6.5, 1.2)	0.687302	0.243212	0.719773	0.248581
(6.5, 1.5)	0.777417	0.235188	0.816095	0.240338

Table 1 shows the effect of waiting time for both FES and SOS in the system for different μ_1, α_1 and ξ_1 at steady-state. In this Table, we observe that:

- For a fixed failure rates $\left(\alpha_1\left(\xi_1\right)\right)$, waiting time in FES decreases while in SOS increases as μ_1 increases. Increasing μ_1, the customers are served faster so that the waiting time decrease during FES, and the customer opting for SOS spend much time in SOS because the service rate $\left(\mu_2\right)$ is kept constant.
- The waiting time during FES increases as the failure rate $\left(\xi_1\right)$ increases when $\left(\mu_1\left(\alpha_1\right)\right)$ is kept constant. The reverse trend is observed during SOS.
- For a fixed $\mu_1\left(\xi_1\right)$, as failure rate $\left(\alpha_1\right)$ increases, the waiting time during FES increases. However, the opposite trend is observed during SOS. This indicates that as the failure rate increases during FES, fewer customers opt for SOS, resulting in less waiting time during SOS and more waiting time during FES.

From Table 2, an increase in failure rate α_1 and the probability of switching failure ρ_1 lead to a decrease in the server's availability. This agrees with our intuition.

Table 3 shows optimal service rates and cost on the different probability of switching failure $\left(\rho_1\right)$ and probability of opting for SOS $\left(r\right)$. In this Table, when ρ_1 is kept constant, we observe that as r increases, the optimal service rate $\left(\mu_2\right)$ and cost increases while the optimal service rate $\left(\mu_1\right)$ decreases. The reason is that, as more customers opt for SOS, the server increases μ_2 while μ_1 decreases to balance the system profitably. Further, for fixed r, as probability of switching failure increases, the optimal cost increases while optimal service rates for both FES and SOS decreases. This is because as probability of switching failure increases, the server tends to be inactive, resulting in slower service.

Table 2 Impact of α_1 on A_s with different probability of switching failure

	A_s		
α_1	$\rho_1 = 0.55$	$\rho_1 = 0.60$	$\rho_1 = 0.65$
0.5	0.590178	0.584093	0.577964
0.6	0.574038	0.566938	0.559778
0.7	0.559410	0.551356	0.543226
0.8	0.546099	0.537148	0.528105
0.9	0.533941	0.524147	0.514245
1.0	0.522797	0.512211	0.501500
1.1	0.512550	0.501218	0.489746
1.2	0.503100	0.491066	0.478876
1.3	0.494359	0.481664	0.468797
1.4	0.486254	0.472935	0.459430
1.5	0.478720	0.464812	0.450703

Table 3 The optimal service rates (μ_1, μ_2) and cost function $f(\mu_1, \mu_2)$ obtained in variation of r and ρ_1 at steady-state

		μ_1^*	μ_2^*	$f(\mu_1^*, \mu_2^*)$
$r = 0.3$	$\rho_1 = 0.2$	7.57193	5.88470	75.8115
	$\rho_1 = 0.4$	7.14676	5.22095	80.5383
	$\rho_1 = 0.6$	6.59207	4.50464	85.3466
$r = 0.4$	$\rho_1 = 0.2$	6.86070	6.11661	79.3859
	$\rho_1 = 0.4$	6.46623	5.41546	83.9408
	$\rho_1 = 0.6$	5.94114	4.65734	88.4880
$r = 0.5$	$\rho_1 = 0.2$	6.34801	6.29105	82.3816
	$\rho_1 = 0.4$	5.96509	5.55460	86.7738
	$\rho_1 = 0.6$	5.43876	4.74946	91.0851

In Table 4, we present the impact of the primary server's failure rate (α_1) and repair rate (η_{1F}) on the optimal cost and service rates. We can see that, as α_1 kept constant, the optimal service rates and cost increases as η_{1F} increases, which is true. Furthermore, for a fixed η_{1F}, as α_1 increases, the optimal cost increases while the service rate decreases.

Table 4 The optimal service rates (μ_1, μ_2) and cost function $f(\mu_1, \mu_2)$ obtained in variation of α_1 and η_{1F} at steady-state

		μ_1^*	μ_2^*	$f(\mu_1^*, \mu_2^*)$
$\alpha_1 = 0.8$	$\eta_{1F} = 0.5$	7.96017	4.75736	77.2618
	$\eta_{1F} = 0.7$	7.96606	4.77579	77.3422
	$\eta_{1F} = 0.9$	7.97532	4.79770	77.4020
$\alpha_1 = 1.0$	$\eta_{1F} = 0.5$	7.66956	4.39425	79.9129
	$\eta_{1F} = 0.7$	7.67429	4.39840	80.1180
	$\eta_{1F} = 0.9$	7.68166	4.40935	80.2801
$\alpha_1 = 1.2$	$\eta_{1F} = 0.5$	7.36684	4.08863	82.1546
	$\eta_{1F} = 0.7$	7.36663	4.07874	82.4759
	$\eta_{1F} = 0.9$	7.36613	4.07674	82.7342

Figure 1 plots the transient state probabilities when the server is idle versus time. The graph show the sharp decrease in $P_{0,(1,1),1}$ (t) and increases in $P_{0,(0,1),1}(t)$ and $P_{0,(1,0),1}(t)$ with time until it attains the steady-state.

Figure 1 Transient state probabilities when the server is idle versus time

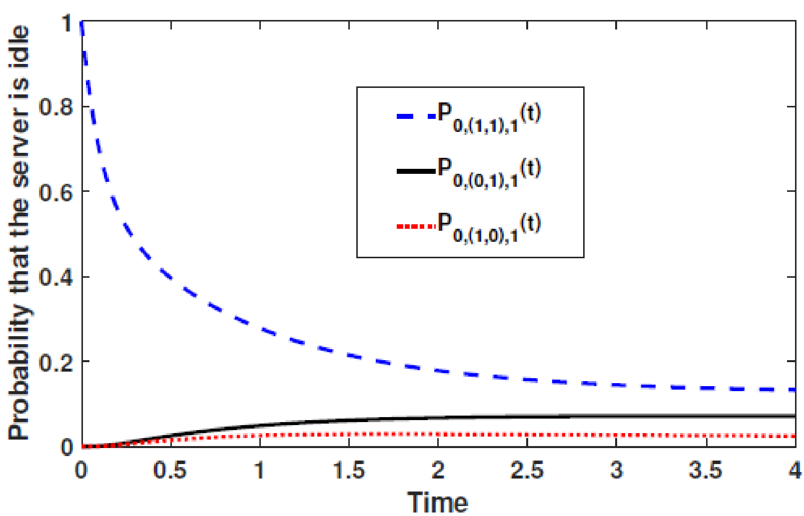
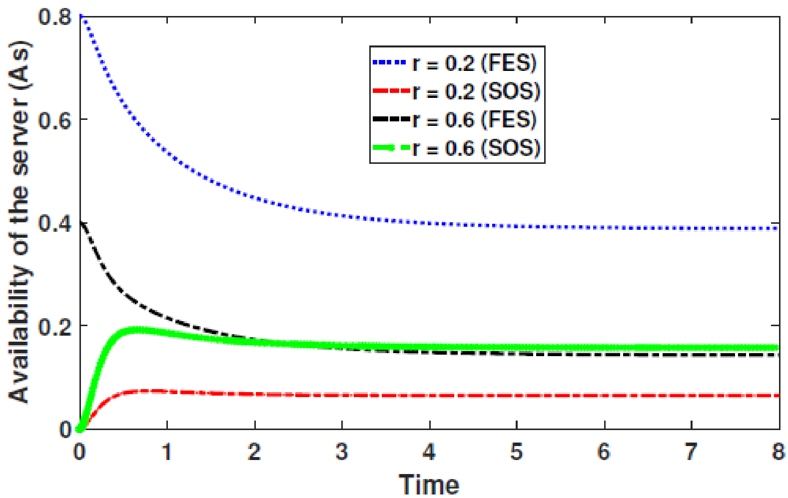


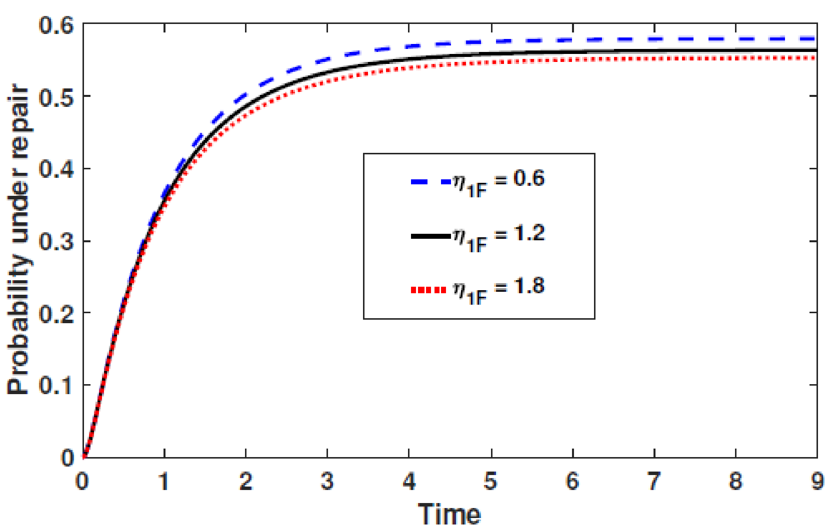
Figure 2 shows the effect of time on the availability of the server (A_s) with different values of probability of opting for SOS (r). We observe that as time progresses, A_s in FES shows the decreasing trend until it attains a steady-state. However, a reverse trend is observed for A_s in SOS. Furthermore, when $r = 0.6$, we observe the intersection points of the curves of A_s for both FES and SOS at $t = 2.5$, for $t < 2.5$ A_s in FES is maximum and for $t > 2.5$ A_s in SOS is maximum.

Figure 2 Effect of time on an availability of the server with different values of r



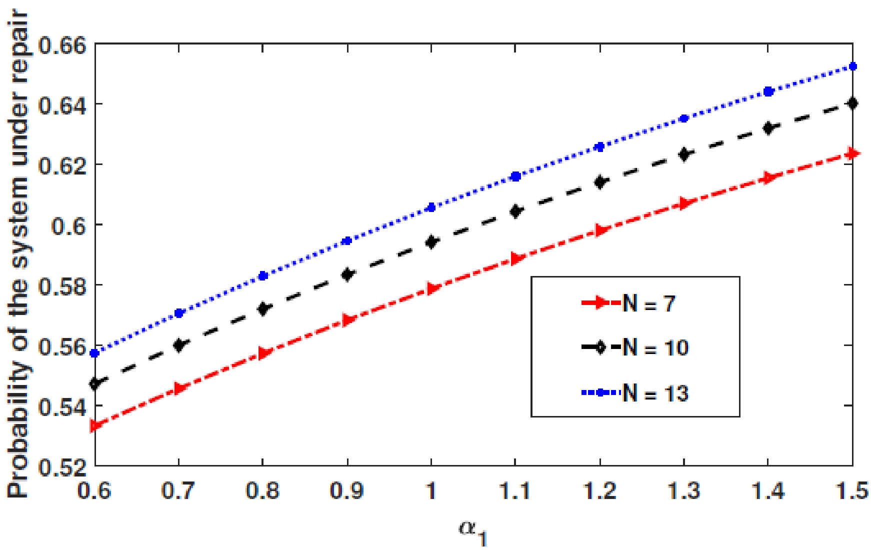
The impact of time on the probability of the system being under repair (P_r) with different values of repair rates (η_{1F}) is demonstrated in Figure 3. The Figure shows that as time progresses, P_r increases until it reaches a steady-state. Also, as η_{1F} increases, P_r decreases because the server repaired faster which results in reducing the probability of the system being under repair.

Figure 3 Impact of time on the probability of the system being under repair with different values of repair rates



The probability of the system being under repair (P_r) and the failure rate of the primary operating server (α_1) with different buffer size (N) is shown in Figure 4. As failure rate in FES increases, we observer the increasing trend of P_r for different values of N. Similarly, for a fixed α_1, P_r is high for larger values of N, which is true.

Figure 4 Impact of failure rate in FES on the probability of the system being under repair



In Figure 5, we show the effect of probability of opting for SOS (r) on the blocking probability (P_{block}). We observe that as r raises, P_{block} shows decreasing trend in FES and opposite trend observed in SOS. Moreover, in case of $N = 7$, we observe an interesting behaviour, for $r < 0.8$ P_{block} in FES is larger, for $r > 0.8$ P_{block} in SOS is larger and at $r = 0.8$, they coincide. Similarly, the same trend observed in case of $N = 13$, for $r < 0.76, r > 0.76$ and $r = 0.76$. This reveals the fact that as more customers opting for SOS as more blocking probability observed when the server is in SOS.

In Figure 6, we plot the expected system length (L_s) versus arrival rate (λ). It is obvious that as λ increases, the inflow of customers to the system increases, which tend to a longer queue. Moreover, L_s shows an opposite trend with the increase of batch size taken for the service, as we expect.

Figure 5 Effect of probability of opting for SOS on the blocking probability

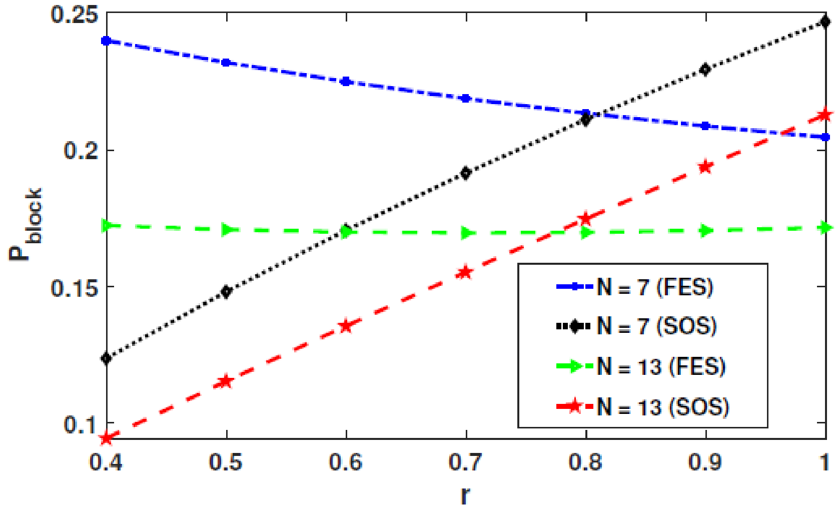
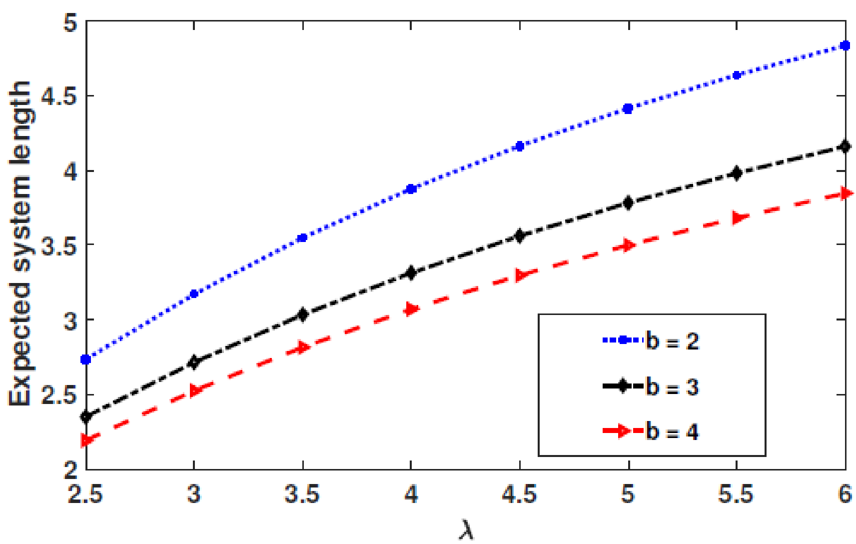


Figure 6 Impact of arrival rate on the expected system length



8 Conclusions

This paper presents the performance of the batch service queue model with SOS, repairable breakdown, and warm standby server. We compute the model in the transient state using the Runge-Kutta method and the steady-state using the matrix decomposition method. Also, we present various performance measures of the system, such as the mean number of the customers in the system, the blocking probability, availability of the server in the system, probability of the system being under repair, etc. Further, the cost model analysis has been presented to determine the minimum cost and optimal service rates during FES and SOS. Numerically we find the following:

- An increase in failure rate leads to a decrease in the server's availability, as observed in Table 2, which is intuitively true.
- The blocking probability is reduced when buffer size increases, as depicted in Figure 5.
- As shown in Figure 4, the probability of the system being under repair obviously increases as the failure rate increase.
- Increase in batch size for the service decreases the expected system length, as shown in Figure 6.
- From Table 4, an increase in the failure rate of the primary operating server decreases the optimum service rates and increases the optimum expected cost.

In future work, one can incorporate the concepts of working vacations and vacation interruption batch arrivals, Markovian arrival process, etc.

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