Cayley bipolar fuzzy graphs associated with bipolar fuzzy groups

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Abstract: Recently, bipolar fuzzy graph is a growing research topic as it is the generalisation of fuzzy graphs. Let G be a non-trivial group and S be a non-empty subset of G such that not containing the identity element of G and $S = S^{-1} = \{s^{-1} | s \in S\}$. The Cayley graph $\Gamma = Cay(G, S)$ is the graph whose vertex set $V(\Gamma)$ is G and edge set $E(\Gamma)$ is $\{\{g, gs\} | g \in G, s \in S\}$. A non-empty subset S of G such that not containing the identity and $S = S^{-1}$ is referred to as a Cayley subset of G, and the Cayley graph $\Gamma = Cay(G, S)$ is referred to as a Cayley graph of G relative to S. In this paper, we introduce the concept of Cayley bipolar fuzzy graphs on the bipolar fuzzy groups. Also some properties of Cayley bipolar fuzzy graphs as connectivity and transitivity are provided.

Keywords: fuzzy graph; Cayley fuzzy graphs; bipolar fuzzy group; Cayley bipolar fuzzy graph; automorphism; isomorphism; level set; (s, t)-level set; level graph.

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1 Introduction

The notion of fuzzy sets was introduced by Zadeh (1965) as a method of representing uncertainty and vagueness. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines. Kaufmanns initial definition of a fuzzy graph

(Akram et al., 2013) was based on Zadeh's (1965) fuzzy relations. Later Rosenfeld (1975) introduced the fuzzy analogue of several basic graph theoretic concepts. Mordeson and Nair (2001) defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Sahoo et al. (2015a, 2015b, 2016a, 2016b, 2017) gave some properties on the intuitionistic fuzzy graphs.

Akram and Wieslaw (2011, 2012) and Akram et al. (2013) defined interval-valued fuzzy graphs, regular bipolar fuzzy graphs, certain types of interval-valued fuzzy graphs, intuitionistic fuzzy hyper graphs with applications and regularity in vague intersection graphs and vague line graphs. In 1878, the Cayley graph was considered for finite groups by Cayley. Max Dehn introduced Cayley graphs under the name Gruppenbid (group diagram) from 1909 to 1910. His most important application was the solution of the word problem for the fundamental group of surfaces with genus, which is equivalent to the topological problem of deciding which closed curves on the surface contract to a point. Rosenfeld (1911, 1975) used the concept of fuzzy sets to develop the theory of fuzzy group and considered the fuzzy relation on fuzzy sets and developed the theory of fuzzy graphs in 1972. Alshehri and Akram (2013) defined Cayley bipolar fuzzy graphs on a group. Akram et al. (2014) introduced the notation of Cayley intuitionistic fuzzy graphs on groups and investigate some of their properties. Borzooei and Rashmanlou (2016) defined Cayley interval-valued fuzzy graghs. Dudek and Talebi (2016) considered operations on level graphs of bipolar fuzzy graphs. Talebi (2018) introduced the concept of Cayley fuzzy graphs of a fuzzy group and investigate some properties of Cayley fuzzy graph and investigate some well-known operations of fuzzy graphs on the Cayley fuzzy graphs. It is evident that Cayley graphs are very useful tool in theoretical computer science. Cayley graphs are excellent models for interconnection networks (Cooperman and Finkelstein, 1992; Heydemann, 1997). It has relations with many practical problems in graph and group theory, computer science, biology and coding theory. In this paper, we develop the concept of Cayley bipolar fuzzy graphs on the bipolar fuzzy groups. Also some properties of Cayley bipolar fuzzy graphs are provided.

2 Preliminaries

In this section, we give some necessary concepts of bipolar fuzzy graphs and bipolar fuzzy subgroups.

Definition 2.1 (Zhang, 1994): Let X be a non-empty set. A bipolar fuzzy set B in X is an object having the form

$$B = \left\{ \left\langle x, \mu_B^P(x), \mu_B^N(x) \right\rangle \mid x \in X \right\}$$

where $\mu_B^P : X \to [0, 1]$ and $\mu_B^N : X \to [-1, 0]$ are mappings.

We use the positive membership degree μ_B^P to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B and the negative membership degree μ_B^N to denote the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar fuzzy set B.

For the sake of simplicity, we shall use the symbol $B = (\mu_B^P, \mu_B^N)$ for the bipolar fuzzy set

$$B = \left\{ \left\langle x, \mu_B^P(x), \mu_B^N(x) \right\rangle \mid x \in X \right\}.$$

The family of all bipolar fuzzy set on V is written as $BFS[V]$.

Definition 2.2: Let $L_* = \{(s, t): t \in [0, 1], s \in [-1, 0]\}$. For any $(s_1, t_1), (s_2, t_2) \in L_*$, the orders $\leq$ and $<$ on L_* are defined as

$$(s_1, t_1) \leq (s_2, t_2) \leftrightarrow s_1 \geq s_2 \text{ and } t_1 \leq t_2,$$

$$(s_1, t_1) < (s_2, t_2) \leftrightarrow (s_1, t_1) \leq (s_2, t_2) \text{ and } s_1 > s_2 \text{ or } t_1 < t_2.$$

By Definition 2.2, it is easy to see that, $(L_*, \leq)$ constitutes a complete lattice with minimum element $(0, 0)$ and maximum element $(-1, 1)$.

Definition 2.3: Let $B = (\mu_B^P, \mu_B^N)$ be a bipolar fuzzy set. For every $(s, t) \in L_*$, we define

$$B_{(s,t)} = \left\{ x \in V : \mu_B^P(x) \geq t, \mu_B^N \leq s \right\}.$$

Then, $B_{(s,t)}$ is called (s, t) level set. The set $\{x \mid x \in V : \mu_B^P(x) \neq 0 \text{ or } \mu_B^N \neq 0\}$ led the support A and is denoted by A^*.

Definition 2.4: Let G be a classical group. Then $B = (\mu_B^P(x), \mu_B^N(x)) \in BFS[G]$ is called an bipolar fuzzy subgroup on G if the following conditions are satisfied for all $x, y \in G$

1 $\mu_B^P(xy) \geq \min\left\{\mu_B^P(x), \mu_B^P(y)\right\}, \mu_B^N(xy) \leq \max\left\{\mu_B^N(x), \mu_B^N(y)\right\}$

2 $\mu_B^P(x) = \mu_B^P(x^{-1}), \mu_B^N(x) = \mu_B^N(x^{-1}).$

The set of all bipolar fuzzy subgroups on G is denoted by $BF[G]$.

For the remainder of this paper, G denotes an arbitrary classical group with identity e.

Proposition 2.5 (Samanta and Pal, 2014): Let $A, B \in BFS[V]$. For every $(s, t) \in L_*$, if $A \subseteq B$, then $A_{(s,t)} \subseteq B_{(s,t)}$.

Definition 2.6: Let G and H be two groups and let $A \in BF[G]$ and $B \in BF[H]$. An isomorphism from G to H is called an isomorphism from A to B if $A = B \circ \phi$, i.e., for every $x \in G$,

$$\mu_A^P(x) = \mu_B^P(\phi(x)) \quad \text{and} \quad \mu_A^N(x) = \mu_B^N(\phi(x)).$$

Let V be a finite non-empty set. Denote by $\tilde{V}^2$ the set of all two-element subsets of V. A graph on V is a pair (V, E) where $E \subseteq \tilde{V}^2$, V and E are called vertex set and edge set, respectively. For simplicity, the subset of the form $\{x, y\}$ will be denoted by (x, y).

Definition 2.7: Let V be a finite non-empty set, $A \in BFS[V]$ and $B \in BFS[\tilde{V}^2]$. The triple $X = (V, A, B)$ is called a bipolar fuzzy graph on V, if for every $(x, y) \in \tilde{V}^2$

$$\mu_B^P(x, y) \leq \mu_A^P(x) \wedge \mu_A^P(y) \text{ and } \mu_B^N(x, y) \geq \mu_A^N(x) \vee \mu_A^N(y).$$

The set of all bipolar fuzzy graphs on V is denoted by $BFG[V]$. For given $X = (V, A, B) \in BFG[V]$, in this study, suppose that $A^* = V$.

Definition 2.8: Let $X_1 = (V_1, A_1, B_1)$ and $X_2 = (V_2, A_2, B_2)$ be two bipolar fuzzy graphs. An isomorphism $\varphi: X_1 \to X_2$ is a bijective mapping $\varphi: V_1 \to V_2$ which satisfies the following conditions:

1 $\mu_{A_1}^P(x) = \mu_{A_2}^P(\varphi(x)),\ \mu_{A_1}^N(x) = \mu_{A_2}^N(\varphi(x))$, for $x \in V_1$

2 $\mu_{B_1}^P(x, y) = \mu_{B_2}^P(\varphi(x), \varphi(y)),\ \mu_{B_1}^N(x, y) = \mu_{B_2}^N(\varphi(x), \varphi(y))$, for $(x, y) \in \tilde{V}^2$.

An isomorphism from a bipolar fuzzy graph X to itself is called an automorphism of X. The set of all automorphisms of X forms a group, which is called the automorphism group of X and denoted by $Aut(X)$.

The image of an element $v \in V$ under a permutation $g \in Sym(V)$ is denoted by v^g.

Definition 2.9: Let $X = (V, A, B)$ be a bipolar fuzzy graph and G be a subgroup of $Aut(X)$.

1 we say that G acts transitively on V, if for any $x, y \in V$, there is a $g \in G$ such that $x^g = y$.

2 we say that G acts regularly on V, if G acts transitively on V and $G_x = \{x \in G \mid x^g = x\}$, the stabiliser x under G is identity, for all $x \in V$.

The bipolar fuzzy graph $X = (V, A, B)$ is called vertex-transitive, if $Aut(X)$ acts transitively on V.

A path P of length n in an bipolar fuzzy graph $X = (V, A, B)$ is a sequence of distinct vertices $u_0, u_1, \ldots, u_n$ such that

$$\left(\mu_B^P(x, y)(u_{i-1}, u_i),\ \mu_B^N(x, y)(u_{i-1}, u_i)\right) > (0, 0), 1 \le i \le n,$$

P is called a path between u_0 and u_n. If $u_0 = u_n$, $n \ge 3$, P is called a cycle. The bipolar fuzzy graph $X = (V, A, B)$ is called connected if for every two vertices $x, y \in V$, there is a path between x and y.

Suppose that $X = (V, A, B)$ and $Y = (V, A', B')$ are bipolar fuzzy graphs such that $W \subseteq V$, then Y is called a bipolar fuzzy subgraph of X, if

1 $\mu_{A'}^P(x) \le \mu_A^P(x),\ \mu_{A'}^N(x) \ge \mu_A^N(x)$, for all $x \in W$

2 $\mu_B^P(x, y) \le \mu_B^P(x, y),\ \mu_B^N(x, y) \ge \mu_B^N(x, y)$, for all $(x, y) \in \tilde{W}^2$.

The bipolar fuzzy subgraph $X[W] = (W, A_{|w}, B_{|w})$ of X = (V, A, B) is called the induced bipolar fuzzy subgraph in X by W and

$$\mu_{A_{|w}}^P(x) = \mu_A^P(x),\ \mu_{A_{|w}}^N(x) = \mu_A^N(x),\ x \in W, \text{ and}$$

$$\mu_{B_{|w}}^P(x, y) = \mu_B^P(x, y),\ \mu_{B_{|w}}^N(x, y) = \mu_B^N(x, y),\ (x, y) \in \tilde{W}^2.$$

Any maximal connected induced bipolar fuzzy subgraph in X is called a connected component of X.

Let G be a non-trivial group and S be a non-empty subset of G such that not containing the identity element of G and $S = S^{-1} = \{s^{-1} \mid s \in S\}$. The Cayley graph $\Gamma = Cay(G, S)$ is the graph whose the vertex set $V(\Gamma)$ is G and edge set $E(\Gamma)$ is $\{\{g, gs\} \mid g \in G, s \in S\}$. A non-empty subset S of G such that not containing the identity and $S = S^{-1}$ is referred to as a Cayley subset of G and the Cayley graph $\Gamma = Cay(G, S)$ is referred to as a Cayley graph of G relative to S.

Definition 2.10: Let G be a non-trivial group with identity e and let $A \in BF[G]$. Suppose that $C \subseteq A$ such that

1 the support of C, C^*, is a non-empty subset of G and $C(e) = \left(\mu_B^P(e), \mu_B^N(e)\right) = (0, 0)$

2 $C(x) = (\mu_C^P(x), \mu_C^N(x)) = (\mu_C^P(x^{-1}), \mu_C^N(x^{-1})) = C(x^{-1})$

3 $\mu_C^P(xy^{-1}) \le \mu_A^P(x) \wedge \mu_A^P(y), \mu_C^N(xy^{-1}) \ge \mu_A^N(x) \vee \mu_A^N(y)$, for all $x, y \in G$.

Then the bipolar fuzzy graph $X = (G, A, B)$ such that B is defined by:

$$\mu_B^P(x, y) = \mu_C^P\left(xy^{-1}\right), \mu_B^N(x, y) = \mu_C^N\left(xy^{-1}\right), \text{for all } (x, y) \in \tilde{G}^2$$

is called the Cayley bipolar fuzzy graph (CBFG) of A relative to C and is denoted by $CayBF(G, A, C)$. The bipolar fuzzy subset C with the properties introduced in the above definition is referred to as a Cayley bipolar fuzzy subset (CBFS) of A in G.

Lemma 2.11: Let $A \in BF[G]$ and $C \subseteq A$ such that $C^* \neq \emptyset$, $C(e) = (\mu_C^P(e), \mu_C^N(e)) = (0, 0)$ and $\mu_C^P(xy^{-1}) \le \mu_A^P(x) \wedge \mu_A^P(y), \mu_C^N(xy^{-1}) \ge \mu_A^N(x) \vee \mu_A^N(y)$, for all $x, y \in G$. Then C is a *CBFS* of A if and only if $C_{(s,t)}$ is a Cayley subset of $A_{(s,t)}$, for all $(s, t) \in L_* - \{(0, 0)\}$, $C_{(s,t)} \neq \emptyset$.

Proof: Let $C \in BFS[G]$ be a *CBFS* of $A \in BF[G]$. If $C_{(s,t)} \neq \emptyset$, for $(s, t) \in L_*-\{(0,0)\}$, then $A_{(s,t)}$ is a non-trivial subgroup of G. Because $C \subseteq A$ and from Proposition 2.5, $C_{(s,t)} \subseteq A_{(s,t)}$, clearly, $e \notin C_{(s,t)}$. Now for any $e \neq x \in G$

$$\begin{aligned} x \in C_{(s,t)} &\leftrightarrow \mu_C^P(x) \ge t, \mu_C^N(x) \le s \\ &\leftrightarrow \mu_C^P(x^{-1}) \ge t, \mu_C^N(x^{-1}) \le s \\ &\leftrightarrow x^{-1} \in C_{(s,t)}. \end{aligned}$$

Therefore, $C_{(s,t)}$ is a Cayley subset of $A_{(s,t)}$.

Conversely, it is enough to show that for any $e \neq x \in G$,

$$C(x) = \left(\mu_C^P(x), \mu_C^N(x)\right) = \left(\mu_C^P\left(x^{-1}\right), \mu_C^N\left(x^{-1}\right)\right) = C\left(x^{-1}\right).$$

Without loss generality, suppose that $\mu_C^P(x) = t, \mu_C^p(x^{-1}) = t', \mu_C^N(x) = s$ and $\mu_C^N(x^{-1}) = s'$, such that $(s, t) \neq (0, 0)$. Then $x \in C_{(s,t)} \neq \emptyset$ and so $x^{-1} \in C_{(s,t)} \neq \emptyset$, because by hypothesis $C_{(s,t)}$ is a Cayley subset of $A_{(s,t)}$. Hence $t' = \mu_C^P(x^{-1}) \ge t, s' = \mu_C^N(x^{-1}) \le s$ and $(s', t') \neq (0, 0)$.

Similarly, we can conclude that $t' \le t$ and $s \le s'$. Therefore, $t' = t$ and $s' = s$, the proof is complete.

Proposition 2.12 (Dudek and Talebi, 2016): Let V be a finite non-empty set, $A \in BFS[V]$ and $B \in BFS[\tilde{V}^2]$. Then $X = (V, A, B) \in BFG[V]$ if and only if $X_{(s,t)} = (A_{(s,t)}, B_{(s,t)})$ is a graph called (s, t)-level graph of X, for all $(s, t) \in L_*$.

Theorem 2.13: Let $X = CayBF(G, A, C)$. If $C_{(s,t)} \neq \emptyset$, $(s, t) \in L_* - \{(0, 0)\}$, then $X_{(s,t)} = Cay(A_{(s,t)}, C_{(s,t)})$.

Proof: From Lemma 2.11, $C_{(s,t)}$ is a Cayley subset of $A_{(s,t)}$. Suppose that $X = (G, A, B)$, i.e., $\mu_B^P(x, y) = \mu_C^P(xy^{-1}), \mu_B^N(x, y) = \mu_C^N(xy^{-1}), (x, y) \in \tilde{G}^2$. Then, by Proposition 2.12, $X_{(s,t)} = (A_{(s,t)}, B_{(s,t)})$. Now for any two elements $x, y \in A_{(s,t)}$, we have

$$\begin{aligned}(x, y) \in B_{(s,t)} &\leftrightarrow \mu_B^P(x, y) \geq t, \mu_B^N(x, y) \leq s\\ &\leftrightarrow \mu_C^P(xy^{-1}) \geq t, \mu_C^N(xy^{-1}) \leq s\\ &\leftrightarrow xy^{-1} \in C_{(s,t)}.\end{aligned}$$

Therefore, $X_{(s,t)} = Cay(A_{(s,t)}, C_{(s,t)})$.

Remark 2.14: Theorem 2.13 says every level graph of a Cayley bipolar fuzzy graph is a Cayley graph.

If $A \in BF[G]$ and C is a CBFS of A, then we can conclude that C^* is a Cayley subset of $A^* = G$. Thus, similar to Theorem 2.13, we can have the following proposition.

Proposition 2.15: If $X = CayBF(G, A, C)$, then $X^* = Cay(G, C^*)$.

Theorem 2.16: Let $X = CayBF(G, A, C)$, then X is connected if and only if $< C^* > = G$.

Proof: We know that X is connected if and only if X^* is connected, so it follows from Proposition 2.15 and the fact that $Cay(G, C^*)$ is connected if and only if $< C^* > = G$.

Theorem 2.17: Let $X = CayBF(G, A, C)$ such that A is a constant function and $H = < C^* > \neq G$. Then, the number of connected component of X is $[G: H]$, the number of the cosets of H in G and each connected component of X is isomorphic to $CayBF(H, A_{|H}, C_{|H})$.

Proof: Let $H_{x_0}, H_{x_1}, \ldots, H_{x_{n-1}}$, with $x_0 = e$ be a transversal of the cosets of H in G. From Theorem 2.16, $CayBF(H, A_{|H}, C_{|H})$ is connected. We prove that $X[Hx_i] \simeq CayBF(H, A_{|H}, C_{|H})$, $0 \leq i \leq n - 1$. Define $\varphi: H \to Hx_i$ by $\varphi(h) = hx_i$, $h \in H$. Then φ is an injection and

$$\mu_{A_{|H}}^P(h) = \mu_A^P(h) = \mu_A^P(hx_i) = \mu_A^P(\varphi(h)),$$

$$\mu_{A_{|H}}^N(h) = \mu_A^N(h) = \mu_A^N(hx_i) = \mu_A^N(\varphi(h)), \text{ for all } h \in H.$$

Also, for any $(h_1, h_2) \in \tilde{H}^2$, we have

$$\begin{aligned}\mu_{C_{|H}}^P(h_1h_2^{-1}) &= \mu_C^P(h_1h_2^{-1})\\ &= \mu_C^P\left(h_1x_i(h_2x_i)^{-1}\right)\\ &= \mu_{C_{|H}}^P\left(\varphi(h_1)\varphi(h_2)^{-1}\right),\end{aligned}$$

and

$$\begin{aligned}\mu^N_{C_{|H}}\left(h_1h_2^{-1}\right) &= \mu^N_C\left(h_1h_2^{-1}\right)\\ &= \mu^N_C\left(h_1x_i\left(h_2x_i\right)^{-1}\right)\\ &= \mu^N_{C_{|H}}\left(\varphi(h_1)\varphi(h_2)^{-1}\right).\end{aligned}$$

Therefore, φ is an isomorphism from $CayBF(H, A_{|H}, C_{|H})$ to $X[Hx_i]$. Now suppose that $x \in Hx_i$, $y \in Hx_j$, $0 \le i, j \le n-1$ such that $C(xy^{-1}) = \left(\mu^P_C(xy^{-1}), \mu^N_C(xy^{-1})\right) \ne (0, 0)$. Then there are $h_1, h_2 \in H$ such that $x = h_1x_i$, $y = h_2x_j$ and

$$\left(\mu^P_C\left(h_1x_i\left(h_2x_j\right)^{-1}\right), \mu^N_C\left(h_1x_i\left(h_2x_j\right)^{-1}\right)\right) \ne (0, 0).$$

So $h_1x_ix_j^{-1}h_2^{-1} \in C^* \subseteq H$, implying $Hx_i = Hx_j$. Therefore $i = j$, the proof is complete.

Theorem 2.18: Let $X = CayBF(G, A, C)$ such that μ^P_A and μ^N_A are constant functions. Then $Aut(X)$ contains a subgroup is isomorphic to G which acts regularly on the vertex set G.

Proof: For each $g \in G$ the mapping ρ_g: $x \to xg$ is a permutation of the elements of G. From elementary group theory, we know that the permutations ρ_g form a group is isomorphic to G. Set $\rho_G = \{\rho_g: g \in G\}$. For each $g \in G$, $x, y \in G$, we have

$$\mu^P_C\left(\rho_g(x)\rho_g(y)^{-1}\right) = \mu^P_C\left(xgg^{-1}y^{-1}\right) = \mu^P_C\left(xy^{-1}\right),$$

$$\mu^N_C\left(\rho_g(x)\rho_g(y)^{-1}\right) = \mu^N_C\left(xgg^{-1}y^{-1}\right) = \mu^N_C\left(xy^{-1}\right),$$

also, since μ^P_A and μ^N_A are constant, it is trivial that

$$\mu^P_A\left(\rho_g(x)\right) = \mu^P_A(x), \mu^N_A\left(\rho_g(x)\right) = \mu^N_A(x).$$

Therefore, $\rho_g \in Aut(X)$ and so ρ_G is a subgroup of $Aut(x)$. If $x, y \in G$, there exists $g = xy^{-1} \in G$ such that $\rho_g(x) = x(xy^{-1}) = y$ and also, if $\rho_b(x) = y$, $b \in G$, then $xb = y$, hence $b = x^{-1}y = g$. Hence, $\rho_{x^{-1}y}$ is a unique element of ρ_G such that $\rho_{x^{-1}y}(x) = y$. Therefore, ρ_G acts regularly on G.

Theorem 2.19: Let $X = (V, A, B) \in BFG[V]$, for a non-empty set V, such that has not isolated vertex. If a subgroup G of $Aut(X)$ acts regularly on V, then X is isomorphic to a Cayley bipolar fuzzy graph.

Proof. Assume that a subgroup G of $Aut(X)$ acts regularly on V. Then X is vertex-transitive, hence μ^P_A and μ^N_A are constant functions. Let $\mu^P_A(x) = t$, $\mu^N_A(x) = s, x \in V$, for a constant $(s, t) \in L_*$. Choose a fixed vertex $x_0 \in V$. For each $y \in V$, there is a unique element $g \in G$, such that $y = x_0^g$, because G acts regularly on V. Therefore, $V = \{x_0^g \mid g \in G\}$. Define $A \in BFS[G]$ such that $\mu^P_A(x) = t, \mu^N_A(x) = s$, for $x \in G$ and $C \in BFS[G]$ by $\mu^P_C(g) = \mu^P_B\left(x_0, x_0^g\right), \mu^N_C(g) = \mu^N_B\left(x_0, x_0^g\right), g \ne e$ and $\mu^P_C(e) = 0, \mu^N_C(e) = 0$. Clearly, $A \in BF[G]$. Prove that C is a CBFS of A. There is $g \in G$

such that $C(g)=(\mu_C^P(g), \mu_C^N(g))=(\mu_B^P(x_0, x_0^g), \mu_B^N(x_0, x_0^g))\neq(0,0)$, because x_0 is not isolated, hence $C^*\neq 0$. For each $g\in G$, we have

$$\begin{aligned}C(g^{-1})&=\left(\mu_C^P(g^{-1}), \mu_C^P(g^{-1})\right)=\left(\mu_B^P\left(x_0, x_0^{g^{-1}}\right), \mu_B^N\left(x_0, x_0^{g^{-1}}\right)\right)\\&=\left(\mu_B^P\left(x_0^g, x_0\right), \mu_B^N\left(x_0^g, x_0\right)\right)\\&=\left(\mu_C^P(g), \mu_C^P(g)\right)\\&=C(g).\end{aligned}$$

Also, if $a, b\in G$, $a\neq b$, then

$$\begin{aligned}\mu_C^P(ab^{-1})&=\mu_B^P\left(x_0, x_0^{ab^{-1}}\right)\\&\leq\mu_A^P(x_0)\wedge\mu_A^P\left(x_0^{ab^{-1}}\right)\\&=t\\&=\mu_A^P(a)\wedge\mu_A^P(b),\end{aligned}$$

and

$$\begin{aligned}\mu_C^N(ab^{-1})&=\mu_B^N\left(x_0, x_0^{ab^{-1}}\right)\\&\geq\mu_A^N(x_0)\vee\mu_A^N\left(x_0^{ab^{-1}}\right)\\&=s\\&=\mu_A^N(a)\vee\mu_A^N(b).\end{aligned}$$

Now claim that X and $CayBF(G, A, C)$ are isomorphic. Define φ: $G\rightarrow V$ by $\varphi(g)=x_0^g$, $g\in G$. Clearly, φ is a bijection and

$$\mu_A^P(g)=t=\mu_A^P(\varphi(g)),\ \mu_A^N(g)=s=\mu_A^N(\varphi(g)),\ \text{for all } g\in G,$$

$$\mu_C^P(ab^{-1})=\mu_B^P\left(x_0, x_0^{ab^{-1}}\right)=\mu_B^P\left(x_0^b, x_0^a\right)=\mu_B^P(\varphi(a), \varphi(b)),$$

$$\mu_C^N(ab^{-1})=\mu_B^N\left(x_0, x_0^{ab^{-1}}\right)=\mu_B^N\left(x_0^b, x_0^a\right)=\mu_B^N(\varphi(a), \varphi(b)),$$

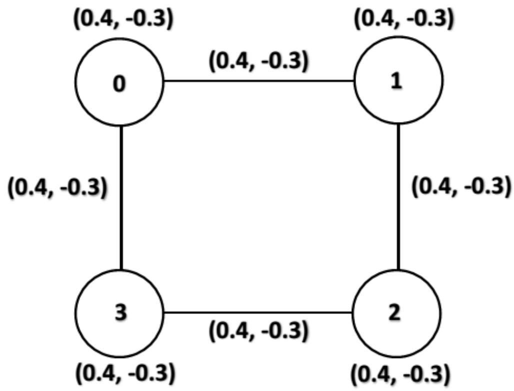
for all $a, b\in G$. Therefore, φ is an isomorphism from $CayBF(G, A, C)$ to X.

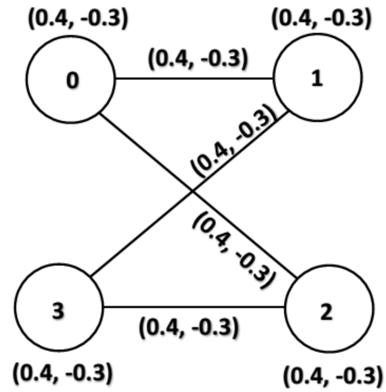
Example 2.20: Figure 1, the bipolar fuzzy graph $X=(Z_4, A, B)$ on the cycle group $Z_4=\{0, 1, 2, 3\}$, such that $\mu_A^P(x)=0.4, \mu_A^N(x)=-0.3, x\in Z_4$ is isomorphic to CayBF (Z_4, A, C) where $\mu_C^P(1)=\mu_C^P(3)=0.4, \mu_C^N(1)=\mu_C^N(3)=-0.3$ and $\mu_C^P(0)=\mu_C^P(2)=0$, $\mu_C^N(0)=\mu_C^N(2)=0$.

For given finite non-empty set V, if $X=(V, A, B)$ is isomorphic to a Cayley bipolar fuzzy graph, then V forms a non-trivial group, $A\in BF[V]$ and there is a CBFS C of A such that

$$\begin{aligned}&\mu_B^P(x, y)=\mu_C^P\left(xy^{-1}\right),\\&\mu_B^N(x, y)=\mu_C^N\left(xy^{-1}\right),\ \text{for all } (x, y)\in\tilde{V}^2.\end{aligned}$$

Figure 1 Cayley bipolar fuzzy graph X (a) CayBF(Z_4, A, C) (b) X = (Z_4, A, B)





Theorem 2.21: Let $A \in BF[G]$ and $X = (G, A, B) \in BFG[G]$. Then X is a Cayley bipolar fuzzy graph relative to a CBFS of A if and only if

$$\mu_B^P(x, y) = \mu_B^P\left(xy^{-1}, e\right), \mu_B^N(x, y) = \mu_B^N\left(xy^{-1}, e\right), \text{ for all } (x, y) \in \tilde{G}^2.$$

Proof: Suppose that $X = CayBF(G, A, C)$ for a CBFS C of A. Then

$$\mu_B^P(x, y) = \mu_C^P\left(xy^{-1}\right) = \mu_B^P\left(xy^{-1}, e\right),$$
$$\mu_B^N(x, y) = \mu_C^N\left(xy^{-1}\right) = \mu_B^N\left(xy^{-1}, e\right), \text{ for all } (x, y) \in \tilde{G}^2.$$

Conversely, let $\mu_B^P(x, y) = \mu_B^P\left(xy^{-1}, e\right), \mu_B^N(x, y) = \mu_B^N\left(xy^{-1}, e\right),$ for all $(x, y) \in \tilde{G}^2$.

Define $C \in BFS[G]$ by $\mu_C^P(e) = 0, \mu_C^N(e) = 0$ and

$$\mu_C^P(x) = \mu_B^P(x, e), \mu_C^N(x) = \mu_B^N(x, e), \quad e \neq x \in G.$$

Then, for all $e \neq x \in G$ we have

$$\begin{aligned}\mu_C^P\left(x^{-1}\right) &= \mu_B^P\left(x^{-1}, e\right) = \mu_B^P\left(e, x^{-1}\right) \\ &= \mu_B^P(ex, e) = \mu_B^P(x, e) = \mu_C^P(x),\end{aligned} \tag{1}$$

and

$$\begin{aligned}\mu_C^N\left(x^{-1}\right) &= \mu_B^N\left(x^{-1}, e\right) = \mu_B^N\left(e, x^{-1}\right) \\ &= \mu_B^N(ex, e) = \mu_B^N(x, e) = \mu_C^N(x).\end{aligned}$$

$$\begin{aligned}&\mu_C^P\left(xy^{-1}\right) = \mu_B^P\left(xy^{-1}, e\right) = \mu_B^P(x, y) \le \mu_A^P(x) \wedge \mu_A^P(y), \\ &\mu_C^N\left(xy^{-1}\right) = \mu_B^N\left(xy^{-1}, e\right) = \mu_B^N(x, y) \ge \mu_A^N(x) \vee \mu_A^N(y), (x, u) \in \tilde{G}^2.\end{aligned} \tag{2}$$

Therefore, C is a CBFS of A and the relations

$$\mu_B^P(x, y) = \mu_B^P(xy^{-1}, e) = \mu_C^P\left(xy^{-1}\right),$$

$$\mu_B^N(x, y) = \mu_B^N(xy^{-1}, e) = \mu_C^N\left(xy^{-1}\right).$$

Complete the proof that $X = CayBF(G, A, C)$.

The following example shows that it is possible for a $A \in BF[G]$ that a $X = (G, A, B) \in BFG[G]$ is isomorphic to a Cayley bipolar fuzzy graph, but $X \neq CayBF(G, A, C)$, for each CBFS of A.

Example 2.22: In Example 2.20, the bipolar fuzzy graph $X = (Z_4, A, B)$ is isomorphic to $CayBF(Z_4, A, C)$ such that

$$\mu_C^P(1) = \mu_C^P(3) = 0.4,\ \mu_C^N(1) = \mu_C^N(3) = -0.3,$$

$$\mu_C^P(0) = \mu_C^P(2) = 0,\ \mu_C^N(0) = \mu_C^N(2) = 0,$$

but by Theorem 2.21, $X \neq CayBF(Z_4, A, C)$, for any CIFS C of A.

3 Conclusions

Cayley graphs are excellent models for interconnection networks and very useful tools theoretical computer science. In this paper, we introduced the notation of Cayley bipolar fuzzy graphs of a bipolar fuzzy group and discussed some properties of these fuzzy graphs. For future works, plan to investigate the products of Cayley bipolar fuzzy graphs and develop rough Cayley bipolar fuzzy graphs.

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