Measuring the severity of a disturbance/disruption in a supply chain: a new quantitative definition, measure, and illustrations

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Abstract: An important factor affecting the performance of a supply chain (SC) is the severity of the disturbance to the chain. The performance measures of the chain could be expected to be adversely affected by an increase in the *severity* of the disturbance to the system, while a system with a less severe disturbance could be expected to show better performance under the same set of circumstances. And thereby the severity of the disturbance becomes a quantity of fundamental interest in measuring SC performance. To this end, this paper proposes a substantially new quantitative definition and measure of the severity of a disruption or disturbance to a SC, through the dynamic modelling framework. The measure is then illustrated for various types of disturbances. The measure proposed herein could be expected to be of a fundamental nature, and of significance in studying the responsiveness, resilience, and operational performance analysis of SCs.

Keywords: disruption; disturbance to a supply chain; severity of a disturbance; quantitative measure; quantitative definition.

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1 Introduction

In the recent past, there has been a large volume of work on measuring the performance of a supply chain (SC) and on developing SC performance metrics. These measures have used operational as well as cost parameters, and followed various approaches ranging from conceptual definitions to mathematical programming and simulation methods. These notwithstanding, an important feature of a SC is also its resilience to external disturbances and disruptions and is an area of growing interest and research (Melnyk et al., 2014; Fiksel et al., 2015; Tukamuhabwa et al., 2015). And consequently, in the recent past there has been a substantial volume of work on SC performance measurement as well as SC resilience and responsiveness. A fundamental performance requirement of a SC is that it should be able to respond quickly to unanticipated changes in demand, which under normal circumstances can be achieved through the maintenance of *adequate pipeline inventories* throughout the chain, to ensure an adequate fill rate. However, the presence of a replenishment lag in the system can magnify the effect of the demand disturbance, and further disrupt the chain and result in stock-outs with the system going into a severe back-order position. And since promptness of delivery is of paramount concern in SC performance, a large back-order position could destroy its performance characteristics as well as its responsiveness and resilience (as defined in Christopher and Peck, 2004). And one of the underlying factors impacting this phenomenon in a major way is the *severity* of the disturbance to the system. The performance of a SC as well as its resilience and responsiveness behaviour could be expected to be adversely affected by an increase in the severity of the disturbance to the system. While a less severe disturbance could be expected to result in better performance as well as better resilience and responsiveness under the same set of circumstances and all other conditions being identical and equal. And hence to measure SC performance as well as its responsiveness and resilience *more objectively*, one would need to make allowance for or factor out, the effect of the severity of the disturbance. Hence two SC systems can be compared on responsiveness, resilience, and operational performance only under identical disturbances or under disturbances of equal severity. And hence the severity of the disturbance becomes a quantity of fundamental interest in measuring SC performance as well as its resilience, responsiveness, and other such performance indicators.

Concomitantly, the setting of 'adequate' pipeline inventory levels and the 'design' inventory levels to achieve good performance is also governed by the *severity* of the disturbances that the system needs to be *designed* to handle. And hence, *SC design, SC performance, SC responsiveness, resilience, and other operational characteristics are impacted in a major way by the severity of the disturbance* to the chain. And hence studying the severity of a disturbance to a SC and finding ways to measure it precisely would be quintessential to SC design, SC performance analysis, SC responsiveness, and SC resilience.

Towards this end, this paper proposes a substantially new definition and quantitative measure of the severity of a disturbance to a SC. The measure is then illustrated for various types of disturbances.

Consequently, the important research question that this paper seeks to answer is the following:

- Can we develop a precise definition and measure of the *severity of a disturbance* to a SC system?

  If yes, this measure could then prove to be of significant value in measuring SC resilience, responsiveness, and other operational characteristics *more objectively*.

In this connection, we have included the recent work on SC performance measurement, SC resilience, and SC responsiveness in the list of references at the end of the paper, as also recent work on dynamic modelling for reference purposes.

We cite a few of them herein which were seminal in defining SC resilience and responsiveness.

In the current literature, SC resilience has been defined as the capability of SCs to operate in the face of disturbances and disruptions with or without a limited decrease in their performance (Christopher and Peck, 2004). Hence, a resilient SC would be enabled to effectively deal with disruptions, and hence could be expected to be at the heart of contemporary SC management thinking (Melnyk et al., 2014). More recently, SC resilience has been defined as the ability to prepare for and/or respond to disruptions, to make a timely and cost-effective recovery, and therefore progress to a post-disruption state of operations – ideally, a better state than prior to the disruption (Tukamuhabwa et al. 2015).

The above definitions, in essence, call for the enablement of a SC to meet and deal with disruptions, and hence building in a good degree of resilience could be expected to be one of the most important objectives in the design of a SC. Moreover, the above definitions given in terms of *responding to external stimuli* and *making an effective recovery, and restore operations*, lead quite naturally to the framework of *dynamic modelling and analysis,* which hence validates and reinforces the approach used in our paper. In this context, Spiegler et al. (2012) has also made use of these principles, and after a comprehensive set of simulations and analyses, conclude that optimum solutions for resilience do not automatically build-in robustness to lead-time variations.

Recently, a number of papers (Bhamra et al., 2011; Cardoso et al., 2015; Munoz and Dunbar, 2015; Datta et al., 2007; Falasca et al., 2008; Ratick et al., 2008; Colicchia et al., 2010; Carvalho et al., 2012; Tomlin, 2006; Tang and Tomlin, 2008; Mitra et al., 2008) have presented different definitions of resilience, a comprehensive taxonomy of which, and along with a thorough and complete comparison between them is presented in Spiegler et al. (2012).

Concomitantly, several performance metrics have been suggested in the above literature for measuring resilience of a SC ranging from cost/profit-based metrics, and extent of order fulfilment to recovery time, while Spiegler et al. (2012) uses control-theoretic concepts and proposes the integral of time absolute 'error' (ITAE (inventory deviation)] as a measure of SC resilience.

Several strategies for enhancing resilience have also been suggested in the above literature ranging from information sharing and collaboration, to capacity redundancy, production flexibility, inventory management, and system redesign (Adobor, 2019; Alderson et al., 2018; Chen and Miller-Hooks, 2012; Chopra and Meindl, 2015; Dolgui and Proth, 2010; Gustav et al., 2020; Hu and Kostamis, 2015; Hu et al., 2013; Ivanov, 2019; Ivanov and Sokolov, 2010; Ivanov et al., 2019; Li et al., 2017; Lin et al., 2004; MacKenzie et al., 2014; Marquez, 2010; Meisel et al., 2016; Mensah and Merkuryev, 2014; Moudgalya, 2007; Muharremoglu and Yang, 2010; Ponomarov and Holcomb, 2009; Sarimvies et al., 2008; Sheffi 2005; Simchi-Levi et al., 2004a, 2004b; Tang, 2006; Tang et al., 2014).

Some of the other recent works are those of Pant et al. (2014) which has proposed a modelling paradigm for quantifying resilience using the vulnerability (initial adverse impact of the disruption) and recoverability (speed of system recovery), while Dixit et al. (2016) have proposed the expected value of the fraction of demand satisfied post disruption as a measure of resilience.

In this context, Ribiero and Povoa (2018) provide a comprehensive literature review of quantitative modelling approaches to SC resilience and its definition and conclude that no consensus exists in defining SC resilience and call for further research into its quantitative definitions.

Concomitantly, responsiveness has also come to be seen as another important characteristic of a SC of late; and some definitions of responsiveness in the literature are cited below.

In Hun and Partar (2014), responsiveness is defined as the maximum probability that an order placed is serviced within a time period of t time units, (with completion times being Erlang and phase-type distributions) subject to a budget constraint, and as a function of t. Whereas, in Nilakantan (2014), responsiveness is indirectly defined as being inversely related to the *total cumulative back-orders till restoration to normalcy*, defined by: $-\sum_{k=0}^{\infty} \min\{0, x_i(k)\}$ in stage i of the chain, and whose magnitude has an inverse relationship with responsiveness and fill-rate. Thus, the lower the cumulative back-order position, the higher would be taken to be the responsiveness, and the vice versa.

More importantly, in a thorough examination of the earlier literature on these topics, we have not come across any *precise study of the severity of a disturbance to a supply chain* in the literature hitherto. And hence there appears to be a discernible gap in the literature on this topic.

This paper attempts to bridge this apparent gap and proposes a substantially new quantitative definition and measure of the severity of a disruption or disturbance to a SC through the dynamic modelling framework.

And since the severity of a disturbance to a SC is expected to have a major impact on its resilience, responsiveness, and operational performance, it is felt that a study of the severity of a disturbance to SC and ways to measure it precisely and quantitatively will be quintessential to the study of SC responsiveness, resilience and operational performance.

The measure proposed herein could hence be expected to be of fundamental relevance in studies of responsiveness, resilience, and other operational characteristics of SCs.

To this end, our paper firstly having taken note above that a precise definition and quantitative measure of the severity of a disturbance or disruption to a SC was not come across in the existing literature, proceeds next to the presentation of the basic dynamic modelling framework and the system equations in Section 2 below. These are used subsequently in the definition of severity. Thereafter, a precise definition of the *severity of a disturbance* is presented in Section 3 based on the dynamic modelling approach and is also illustrated therein. Section 4 then presents numerical illustrations of severity computations, and also illustrates the direct effect of the severity of disturbance on system performance. Section 5 then presents the managerial implications, while Section 6 concludes the paper mentioning the main contributions and their research significance, along with scope for further work.

2 The dynamic modelling framework and notation

We take a simple serial system with three stations in the SC as explained below.

The notation, and model equations used in our paper with *deviation* variables, is standard in the literature on both control theory and SC dynamics (e.g., Axsater, 1985; Ortega and Lin, 2004; Popplewell and Bonney 1987; Weindahl and Breithaupt, 2000; Nilakantan, 2019):

$x_i(k)$ is the inventory deviation at time k, in stage i of the chain

$q_i(k)$ is the material flow deviation in period $(k-1, k]$, into stage i of the chain

$r_3(k)$ is the deviation in demand observed at the warehouse in $(k-1, k]$.

with

$i = 1$ representing the upstream end of the production facility (raw material)

$i = 2$ representing the downstream end of the production facility (finished goods)

$i = 3$ representing the finished goods warehouse.

The deviation variable is defined as the difference between the actual value and its planned/design value. Thus,

$$x_i(k) \equiv y_i(k) - y_i^o(k),\ q_3(k) = \overline{q}_3(k) - q_3^0(k), r_3(k) = \overline{r}_3(k) - r_3^0(k) \tag{2.1}$$

where

$y_i(k)$ is the actual (observed) value of inventory at time k

$y_i^o(k)$ is the planned or design (Nominal) value of the inventory at time k

$\overline{q}_3(k)$ is the actual (observed) material flow in (k–1, k]

$q_3^0(k)$ is the design (nominal) material flow in (k–1, k]

$\overline{r}_3(k)$ is the actual (observed) demand at the warehouse in (k–1, k]

$r_3^0(k)$ is the design (nominal) demand at the warehouse in (k–1, k].

The dynamic equations of the system are then given by:

$$x_i(k+1) = x_i(k) + q_i(k+1) - q_{i+1}(k+1) \tag{2.2}$$

where for $i = 3$, $q_4(k) = r_3(k)$ is the demand outflow from the warehouse.

The system is assumed to be at zero deviation at time $k = 0$, and the first deviation in demand and/or supply, is felt at the end of the first period, at $k = 1$, yielding the standard initial conditions (ICs) for the system: $x_i(k) = 0$ and $q_i(k) = 0$ for all $k \leq 0$; and $r_3(k) = 0$ for all $k \leq 0$.

The control variables are the replenishment flows, through which control is effected/achieved in the system. The replenishment control flows are given by functions of the latest available/fully observed inventory and demand deviations as under:

$$q_i(k+1) = f\left(x_3\left(k-1-l_i\right), r_3\left(k-1-l_3\right)\right) \text{ for i} = 1,2,3 \tag{2.3}$$

where the l_is are the lags in the various sections of the system. Under conditions of *zero lag*, i.e., with $l_i = 0$ in all stages, the ordered consignments would arrive in the next/following period. However, the presence of replenishment lags in the system will result in delays in the delivery of consignments, and consequently result in depletion of the inventory levels, thereby causing *disruptions in the system.*

Since the warehouse end of the chain is the interface between the supply system and the external market, and which is what would determine the performance of the chain and its responsiveness and resilience, we henceforth focus our attention on the warehouse end of the chain.

Since demand and supply flows would have a stochastic component in real settings, the *net random variation* in the flows is represented in these models by a white noise process with mean zero and variance σ^2, added to the RHS of the difference equation of the system, or equivalently, superimposed on the demand flows which are on the RHS of the system difference equation. This is illustrated for the case of a step demand disturbance as under:

$$r_3(k+1) \equiv -b_0 + \varepsilon(k+1), \varepsilon(k+1) \approx WN\left(0,\sigma^2\right) \tag{2.4}$$

The first term above represents a sudden increase in *mean* demand of magnitude b_0 units, and the second the stochastic component represented by a white noise process.

With this background, we next proceed to investigate the severity of a disturbance to a SC.

3 The severity of a disturbance and its quantitative measurement

In this section we attempt to measure and quantify the severity of a disturbance to a SC using the dynamic modelling approach delineated above. We first start with some basic definitions below.

3.1 Some basic definitions

There are two parameters which can be seen to be important and would play a role in defining the severity of the disturbance. These are:

1 the *magnitude* of the disturbance

2 the *time duration* of the disturbance/disruption, i.e., the time over which the disturbance acts on the system.

The magnitude of the disturbance is usually the magnitude of the flow disturbance (either the demand outflows, or the supply inflows, or both), and is measured by the *deviations* from the nominal/design values. The time duration of the disturbance is the time for which the disturbance acts on or is felt by the system.

The severity of the disturbance is dependent on both and is given by the *impulse* of the disturbance, (akin to the impulse of a force in physics) which is the product of these two parameters. This is defined more precisely below. It can be understood as the *potential of the disturbance to disrupt* the system, or its *disruption potential*.

Before we can define the impulse of a disturbance precisely, we need to define the following quantities:

a *Response initiation time*: This is the time taken to *initiate rectification action* in the system, and is usually instantaneous, and in our discrete time models we can take it as one period, since the corrective action will be *applicable only from the next succeeding* period.

b *Response time* T_{Resp}: This is the time taken for the corrective action (the corrective flows) to be felt by the system *following the disruption*, and is equal to the sum of the *response initiation time* and the *lags* present in the system Thus, we have:

$$Response\ Time = Response\ Initiation\ Time + system\ lag$$

And hence, we have:

$$Response\ Time, T_{\mathrm{Re}sp} = 0 + 1 + l_3 = 1 + l_3.$$

c *Recovery time*: This is the time taken for the system to be restored to its normal operating levels *measured from the time of disruption*.

Note: In Spiegler et al. (2012) recovery time is taken as the time to restore the system less the Response Time. However, in our definition, we would prefer to include the response time within the recovery time, and measure it from time of disruption, since this could be expected to be of greater significance in a practical setting. And hence, we have

$$\text{Recovery time (our paper)} = \text{Recovery time (Spiegler et al., 2012)} + \text{Response time}$$

d *Time duration of a disturbance* T_D: This is the time for which the disturbance is felt by the system. For example, for a step disturbance it would be infinity.

e *The evaluation time* T_E: This is the *maximum time duration* following the disruption/disturbance *within which we would like to have the system restored to normal operation again*. For example, if the system time is measured in days, then we might specify that the system should be restored to its original state within seven days or ten days say, after which we might not be interested in evaluating its performance. Thus, the performance of the system would cease to be of interest to us beyond this point of time (if it cannot be restored to normal operation within this specified time of T_E days/periods).

This quantity proves particularly useful in cases where the time duration of the disturbance is infinite, e.g., in the case of a step disturbance. In such cases the time duration is restricted to T_E, and eliminates the possibility of the impulse having a value of infinity (This is discussed further subsequently).

f Now, the *impulse time* of the disturbance denoted by T_I can be defined as the minimum of these two times, i.e., $T_I = \min\{T_D, T_E\}$, and is the time for which the disturbance is taken to act on the system for the purpose of calculating its *impulse* value.

3.2 Severity of a disturbance in controlled systems

In controlled systems and in our subsequent analysis, the *replenishment inflows* into a stage are taken to be the *control variables, i.e.,* the variables that are under the control of the planner and *through which control is exercised* over *the system.* In a SC, at every stage, the outflows are the demands that have to be met at that stage, and the inflows are the replenishment control flows through which the system is controlled. And hence we do not include these control variables in defining the impulse of a disturbance in such controlled systems. And hence in controlled systems, we can define the *impulse of a disturbance* as under:

$$Impulse = \int_0^\infty \Big[\big\{\big((l_i(t)+1\big)\big(\bar{D}^{out}(t)+D^{out}(t)\big)\big\} - \big\{\big(l_i^0(t)+1\big)\bar{D}^{out}(t)\Big]dt$$

in continuous time, and (3.1)

$$= \sum_{k=0}^\infty \Big[\big\{\big(l_i(k)+1\big)\big(D^{out}(k)+\bar{D}^{out}(k)\big\} - \big\{\big(l_i^0(k)+1\big)\bar{D}^{out}(k)\big\}\Big]$$

in discrete time.

where

$l_i^0(k), l_i(k)$ are respectively, the normal (planned for) lag in the system, and the actual lag at time = k.

$\bar{D}^{out}(k), D^{out}(k)$ are respectively, the normal (planned for) demand, and demand deviation at time k.

The definition above allows and accounts for both types of disturbances in the system, i.e., demand as well as replenishment lag disturbances simultaneously present in system.

This can be explicitly seen by splitting the terms in the definition above as under:

$$Impulse = \int_0^\infty \begin{bmatrix} \big\{\big(l_i^o(t)+\big(l_i(t)-l_i^0(t)\big)+1\big)\big(\bar{D}^{out}(t)+D^{out}(t)\big)\big\} \\ -\big\{\big(l_i^0(t)+1\big)\bar{D}^{out}(t) \end{bmatrix} dt$$

in continuous time, (3.1a)

$$= \sum_{k=0}^\infty \begin{bmatrix} \big\{\big(l_i^0(k)+\big(l_i(k)-l_i^0(k)\big)+1\big)\big(D^{out}(k)+\bar{D}^{out}(k)\big\} \\ -\big\{\big(l_i^0(k)+1\big)\bar{D}^{out}(k)\big\} \end{bmatrix}$$

in discrete time.

where $l_i^0(k), (l_i(k)-l_i^0(k))$ are respectively, the normal (planned for) lag in the system, and the increase in lag at time = k.

This quantity is essentially the difference between actual and planned values of the product of the magnitude of the disturbance times the time duration of occurrence (and hence *incorporates both the magnitude and time duration of the disturbance*, since both contribute to its *severity*). This is captured by the integral/summation of the magnitude of the disturbance over time. Thus, the difference between the actual and planned values of this product is taken as the impulse.

Note 1: The upper limit of the integral/summation is taken as infinity in the most general case. In most cases of practical interest this would be finite and is discussed subsequently.

The logic for the factor ($l_3(t)$ + 1) in the integrand above is that, in the *presence of lag* in the system a sudden increase in demand or decrease in supply or both, would *cause cumulating inventory deviations* for as many periods as equal to the lag present in the system. This is because any corrective replenishment flow would take at least that many periods ((l_i + 1) periods) to be felt in the system. And hence a flow disturbance is *magnified by the presence of lag*, and it is this magnification of the disturbance that the impulse definition above attempts to capture and quantify. We thus attempt to quantify the disruption *potential* of the disturbance.

The value of the impulse could well be infinity in some cases, e.g., in the case of a step demand. In such cases, we can truncate the integral at $t = T_I$, which can be taken as the time duration of the disturbance when calculating the impulse for practical purposes. And hence we could have the equivalent definitions as:

$$Impulse = \int_0^{T_I} \begin{bmatrix} \left\{ \left(l_i^o(t) + \left(l_i(t) - l_i^0(t) \right) + 1 \right) \left(\bar{D}^{out}(t) + D^{out}(t) \right) \right\} \\ -\left\{ \left(l_i^0(t) + 1 \right) \bar{D}^{out}(t) \right. \end{bmatrix} dt$$

in continuous time,

$$= \sum_{k=0}^{T_I} \begin{bmatrix} \left\{ \left(l_i^0(k) + \left(l_i(k) - l_i^0(k) \right) + 1 \right) \left(D^{out}(k) + \bar{D}^{out}(k) \right\} \right. \\ -\left\{ \left(l_i^0(k) + 1 \right) \bar{D}^{out}(k) \right\} \end{bmatrix} \tag{3.1b}$$

in discrete time.

Note 2: The actual outflow from the system may not be equal to the demand when the inventory deviation is negative (shortfall). Nevertheless, when calculating the impulse (severity) of a disturbance, we are essentially interested in the *potential deviation that the disturbance could cause*, and hence we use the demand deviation instead of the actual outflow.

Note 3: We can note that performance shortfall in a SC is measured by *unfulfilled cumulative demand* which in turn, is equivalently indicated and measured by inventory level depletion [using equation (2.2)]. And hence a decrease in demand (negative demand deviation) or a decrease in replenishment lag *would not lead to cumulative performance shortfall* since it would result in full demand fulfilment and increase in inventory levels. Hence a disturbance with a negative impulse value over the impulse time would not lead to performance shortfall. On the other hand, an increase in demand (a positive demand deviation) or an increase in replenishment lag would lead to a positive value of impulse and could lead to performance shortfall; And thus, when the impulse value is (–)ve, it would not cause sustained underperformance of the system. But when the impulse value is positive, it would cause disruptions in the system which could lead to performance shortfall.

Similarly, when the impulse value is zero, it would mean that the net impact of the disturbance over the impulse time is zero. However, and nevertheless, a zero-valued impulse can still cause disruptions in the system and lead to performance shortfall, as is illustrated subsequently.

Note 4: the impulse function: Though the impulse as defined above specifies the evaluation horizon either as infinity or the impulse time, T_I, from the definition of the impulse we can see that the impulse value will vary with time, k. And hence, it would also be meaningful to talk of an impulse function, which is the value of the impulse at a point of time *k*. And hence, we can also define the impulse function at time *t* or *k* as:

$$Impulse\ Function(t/k) = \int_0^t \begin{bmatrix} \left\{ \left(l_i^o(t) + \left(l_i(t) - l_i^0(t) \right) + 1 \right) \left(\bar{D}^{out}(t) + D^{out}(t) \right) \right\} \\ -\left\{ \left(l_i^0(t) + 1 \right) \bar{D}^{out}(t) \right. \end{bmatrix} dt$$

in continuous time,

$$= \sum_{k=0}^{k} \begin{bmatrix} \left\{ \left(l_i^0(k) + \left(l_i(k) - l_i^0(k) \right) + 1 \right) \left(D^{out}(k) + \bar{D}^{out}(k) \right\} \right. \\ -\left\{ \left(l_i^0(k) + 1 \right) \bar{D}^{out}(k) \right\} \end{bmatrix} \tag{3.2}$$

in discrete time.

thereby showing the *variation* of the impulse with time. This is illustrated in the examples below.

Note 5: We need to note the difference between the *impulse function* defined in equation (3.2) above and the *impulse value* as defined in equation (3.1). The impulse value (which is the integral of the impulse function) is what will be used to measure the severity of a disturbance, while the impulse function merely gives the variation of the impulse value with time. However, and as will be discussed subsequently, in cases wherein the impulse value becomes zero, the impulse function proves useful and provides a better understanding of the impulse. This is discussed in detail subsequently.

3.2.1 The impulse value for a SC

Now since the driving force in a SC is the demand at the warehouse at the downstream end, we define the impulse of the disturbance to the SC as that at the warehouse end, and hence as

$$Impulse\ of\ the\ Disturbance = \int_0^{\infty/T_I} \left\{ \left(l_3(t) + 1 \right) r_3(t) \right\} dt,$$

in continuous time, or equivalently, (3.3)

$$Impulse\ of\ the\ Disturbance = \sum_{k=0}^{\infty/T_I} \left\{ \left(l_3(k) + 1 \right) r_3(k) \right\}, \text{ in discrete time.}$$

The units of impulse would be 'unit-periods' or unit-days' for discrete items, and in general 'quantity-periods' for the general case (e.g., 'kilogram-days').

Hence in our further discussion, we take the impulse of a disturbance at the warehouse as defined above to be the *quantitative measure of the severity* of a disturbance to a SC.

For completeness, we also give below the impulse definitions for uncontrolled systems.

3.3 Severity of a disturbance in uncontrolled systems

In uncontrolled systems, the replenishment inflows, along with the demands at each stage are *both environmental variables* and form part of the disturbance to the system, and hence are included in the definition of the impulse. And hence, we have, in uncontrolled systems:

$$Impulse = -\int_0^\infty \begin{Bmatrix} \{(l_i(t)+1)[F^{in}(t) - D^{out}(t)]\} \\ -\{l_i^0(t)+1)(\bar{F}^{in}(t) - \bar{D}^{out}(t))\} \end{Bmatrix} dt$$

in continuous time, and

$$= -\sum_{k=1}^{\infty} \begin{bmatrix} [(l_i(k)+1)[F^{in}(k) - D^{out}(k)]\} \\ -\{(l_i^0(t)+1)(\bar{F}^{in}(k) - \bar{D}^{out}(k))\} \end{bmatrix} \tag{3.4}$$

in discrete-time.

where $F^{in}(t), F^{in}(k) / \bar{F}^{in}(t), \bar{F}^{in}(k)$ are the supply inflow deviations/planned inflows, respectively.

And hence as in the case of controlled systems above, an increase in supply at a station (positive supply flow deviation), or a decrease in demand (negative demand deviation), or both, *would not lead to cumulative performance shortfall* since both would result in full demand fulfilment and increase in inventory levels. Whereas an increase in demand (positive demand deviation) or a decrease in supply inflow (negative supply deviation) would lead to performance shortfall.

Also, from equation (3.2), we can note that *under conditions of constant lag*, i.e., $l(k) \equiv l_3$, $\forall k$, the value of the impulse for an uncontrolled system is equal to $-(l_3 + 1)(x_i(T_l) - x_i(0))$, and for the warehouse, $x_3(T_l) - x_3(0) = x_3(T_l)$, since the system is at rest at time zero with $x_3(0) = 0$. And hence the impulse of a disturbance under constant lag can be very simply measured in uncontrolled systems by $-(l_3 + 1)x_3(T_l)$.

3.4 Equivalence between the two types of systems

We can note that supply induced disturbances in uncontrolled systems, can be equivalently treated as a combination of an increase in lag along with an increase in demand in controlled systems, as under:

a Firstly, a complete supply disruption for 'm' periods starting at time = k, can be equivalently viewed as a sudden increase in replenishment lag from $l_i^0(k)$ (existing lag) to

$$l_i(k) = l_i^0(k) + m \tag{3.5}$$

Or alternatively as an increase in demand (equal to the supply shortfall) during the period of disruption, i.e., from $D_i^{out.eqv}(k)$ to $D_i^{out}(k) - F_i^{in}(k)$ during the period of complete disruption, where $F_i^{in}(k)$ is the deviation of supply from its scheduled values.

b A partial supply shortfall without additional delay can be treated as an equivalent increase in demand as under:

$$D_i^{out,eqv}(k) \equiv \left(D_i^{out}(k) - F_i^{in}(k)\right) \tag{3.6}$$

c And finally, a complete supply disruption for 'm' periods with partial shortfall of supply thereafter under increased lag can be treated as a sudden increase in lag at time = k as in equation (3.5) above, coupled with an equivalent demand increase as in equation (3.6) above. And hence the impulse component at time = k as defined in equation (4.3) above for demand-induced disturbances in controlled systems can be written as:

$$\left(l_i^0(k) + m + 1\right) D_i^{out,eqv}(k) \equiv -\left(l_i(k) + 1\right)\left(F_i^{in}(k) - D_i^{out}(k)\right) \tag{3.7}$$

which is precisely as in the definition of impulse in uncontrolled systems.

And hence and hereafter, without loss of generality, we can take up controlled systems in our study, using the *equivalence and correspondence* between the two to simplify our analysis.

The advantage of this approach is that *all environmental variables can be clubbed with the demand as an 'equivalent demand'* and can be *separated from the control flows* which are independently controlled for restoration of the system.

3.5 Illustrations of impulse calculations (for controlled systems)

In this section, we present examples of various types of demand disturbances and compute their impulse values. We take up controlled systems only since these are the ones with greater practical significance.

We take the replenishment lag to be time-invariant in the illustrations below for ease of computation:

1 *Step disturbance* of magnitude b_0 units in the first period: $r_3(k) \equiv b_0$, $k \geq 1$, Lag = $l_3(k) \equiv l_3 \forall k$ (i.e., sudden and sustained demand *increase* under constant time-invariant lag of l_3 periods)

$$Impulse = \sum_{k=0}^{\infty} \left(l_3(k) + 1\right) r_3(k) \equiv \sum_{k=0}^{\infty} (l_3 + 1) r_3(k) = \lim_{k \to \infty} (l_3 + 1) b_0 k \tag{3.8}$$
$$\to \infty \; as \; k \to \infty$$

The values are (+)ve and increase without bound as impulse time increases.

2 *Sinusoidal disturbance* (seasonal variation) about the mean forecasted demand with amplitude of A:

$$r_3(k) \equiv ACoswk,\; k \geq 0,\; Lag = l_3(k) \equiv l_3 \forall k$$

$$\begin{aligned} Impulse &= \sum_{k=0}^{\infty} \left(l_3(k)+1\right) r_3(k) \equiv \sum_{k=0}^{\infty} (l_3+1) r_3(k) \\ &= \lim_{k\to\infty} \sum_{l=0}^{k} (l_3(l)+1) ACoswl \\ &= \lim_{k\to\infty} (l_3+1) ASin\left(w(k-1/2)\right) \big/ 2Sin(w/2),\ w \neq 2\pi, \end{aligned} \tag{3.9}$$

is oscillatory with amplitude of $(l_3 + 1)A / 2Sin(w/2)$ about a centre line of zero.

The impulse values will cycle through (+)ve, zero, and (–)ve values cyclically.

(Note: When $w = 2\pi$, the disturbance is zero for all k).

3 *Ramp disturbance* (i.e., demand with an increasing trend component with slope = b_1):

$r_3(k) \equiv b_1 k,\ k \geq 1,\ Lag = l_3(k) \equiv l_3 \forall k.$

$$\begin{aligned} Impulse &= \sum_{k=0}^{\infty} \left(l_3(k)+1\right) r_3(k) \equiv \sum_{k=0}^{\infty} \left(l_3+1\right) r_3(k) \\ &= \lim_{k\to\infty} \left(l_3+1\right) k(k+1)/2 \\ &\to \infty \ as \ k \to \infty \end{aligned} \tag{3.10}$$

The impulse value is (+)ve and increases without bound as impulse time increases.

4 *Combined disturbances:* From the definition of the impulse above, we can see that the impulse is additive in nature and hence we have Result 1 below:

$$Result\ 1: Impulse\ of\ \left(r_3^1(k)+r_3^2(k)\right) = Impulse\ of\ \left(r_3^1(k)\right) + Impulse\ of\ \left(r_3^2(k)\right) \tag{3.11}$$

Proof:

$$\begin{aligned} Impulse\ of\ \left(r_3^1(k)+r_3^2(k)\right) &= \sum_0^k \left(l_3(k)+1\right)\left(r_3^1(k)+r_3^2(k)\right) \\ &= \sum_0^k \left(l_3(k)+1\right) r_3^1(k) + \sum_0^k \left(l_3(k)+1\right) r_3^2(k) \\ &= Impulse\ of\ \left(r_3^1(k)\right) + Impulse\ of\ \left(r_3^2(k)\right) \end{aligned} \tag{3.11}$$

And hence: impulse of a sum of disturbances = sum of the impulses of individual disturbances.

The impulse values of a combined disturbance consisting of several components would be simply the sum of the impulse values of the individual components. And it can hence have (+)ve, (–)ve, and zero values at different points of time and would be based on the signs and magnitudes (b_0, b_1, A) of the individual components.

5 *A complete supply disruption* from the first period onwards for '*m*' periods, followed by a partial supply disruption with increased in lag of L periods (normally existing lag = $l_3^0(k)$):

$$\begin{aligned} Impulse &= \sum_{k=0}^{m} \left(l_3^0(k)+1\right)\left(-\bar{F}_3^{in}(k)+D_3^{out}(k)\right) \\ &+ \sum_{k=m+1}^{\infty} \left(l_3^0(k)+1+L\right)\left(-F_3^{in}(k)+D_3^{out}(k)\right) \end{aligned} \tag{3.12}$$

where $\overline{F}_3^{in}(k)$ is the planned inflow in period k. Hence in the case of a complete disruption of inflow, the inflow deviation is the negative of the planned inflow. The term $\overline{F}_3^{in}(k)$ is the (partial and negative) inflow deviation following the resumption of inflow under increased lag.

6 *Cases of zero impulse:* To render the impulse measure useful in further studies it would be advantageous to ensure that it has a non-zero value in all cases. In cases where the value can become zero, we can make it non-zero by proper choice of the impulse time (this will always be possible since the impulse value is the value of the impulse function at $k = T_I$).

Alternatively, we can pick the maximum value of the impulse function over the interval $[0, T_I]$ as the impulse value.

For example, in the case of a sinusoidal disturbance occurring perpetually about a centre line of zero under time-invariant lag as given in equation (3.9) above, the impulse function will be sinusoidal and hence zero at $k = 2\pi(n - 1/2) / w$. In this case, we can pick the maximum value of the impulse function.

For sinusoidal disturbances, it would be logical to consider a full cycle of the disturbance in calculating the impulse value. However, if the lag is time-invariant over the cycle, this value will be zero. In such cases, we can choose the maximum value of the impulse function as the value of the impulse for use in subsequent computations. This represents the maximum value of the impulse function as it varies with time. In this case it would be $(l_3 + 1)A / \mathrm{Sin}(w / 2)$. The same would also hold for the case wherein the disturbance acts only over a *single cycle* of the sinusoid.

7 *A Dirac delta disturbance component:* One can also sometimes have a sudden large increase in demand over a very short period of time, or a sudden surge which again drops down to zero subsequently within a very short span of time. This is modelled by a Dirac delta function of the type:

$B\delta(t - a)$ or $B\delta(k - a)$ which gives a sudden jump in demand of magnitude B units at time t or k = a, where the Dirac delta function is defined as:

$$\delta(t) = \left[\begin{matrix} 1, t = 0 \\ 0, t \neq 0 \end{matrix}\right) \tag{3.13}$$

The impulse of this term would be given simply by: $(l_3(a) + 1)B$ which is easily obtained from equation (3.3) using the definition of the Dirac delta function.

8 *Practical situations:* In practical situations, we can note that a general or composite demand disturbance would usually be split into its components as under (Gouriroux and Monfort 1990):

a A mean value or centre line of a step disturbance, denoted by b_0.

b A linear or quadratic trend, denoted by $b_1t + b_2t^2$ or $b_1k + b_2k^2$, where t is the continuous time variable, and k is the discrete time variable.

c A seasonal component denoted by a sinusoidal term: *ACoswt* or *Acoswk*, where $w = \frac{2\pi}{T}$, with T being the time-period of seasonality. For example, an annual variation with time being measured in months would have $T = 12$, months, while a monthly cycle with time being measured in days would have $T = 30$ days, whereas a monthly cycle with time measured in weeks would have $T = 4$ weeks.

Of course, it is entirely possible to have a mix of Sinwt and Coswt terms also as in a Fourier series fit to data.

d A sudden large jump in demand lasting for an extremely short period, at time $t / k = a$ denoted by a Dirac delta term: $B\delta(t - a)$ or $B\delta(k - a)$.

e A residual random variation term denoted by: $\varepsilon(t)$ or $\varepsilon(k)$, and modelled by a white noise process with zero mean and variance σ^2. Thus $\varepsilon(t) = WN(0, \sigma^2)$ or $\varepsilon(k) = WN(0, \sigma^2)$, as mentioned earlier.

Thus, a general disturbance is denoted as:

$$r_3(k) \equiv b_0 + b_1 k + b_2 k^2 + ACoswk + B\delta(k - a) + \varepsilon(k) \tag{3.14}$$

A detailed computational procedure for obtaining the coefficients $\{b_0, b_1, b_2, A, B\}$ and σ^2 can be found in Gouriroux and Monfort (1990).

The first four components comprise the *mean value of the disturbance*, while the last term is the residual random variation component which is characterised by a white noise process as mentioned above.

The impulse of such a composite disturbance (excluding the last residual random variation component) can be easily computed as the sum of the impulses of the individual components.

Further, in most practical cases, the Dirac delta disturbance term may not usually be observed, as also the quadratic trend term. And hence in most cases of practical interest, the disturbance could be expected of the form:

$$r_3(k) \equiv b_0 + b_1 k + ACoswk + \varepsilon(k),$$

whose impulse value is easily computed as shown above.

We next provide numerical illustrations of impulse computations for controlled systems (since all systems of practical interest are controlled systems).

4 Numerical examples: computation of severity of disturbance in controlled systems

4.1 Computation of severity

We take a numerical example in which the demand is as given below.

The coefficients in equation (3.14) are taken as: $(b_0, b_1, b_2, A, B) = (1, 1, 0, 1, 5)$ with the surge demand occurring at $k = 3$. This represents a demand with mean value given by a unit step increase ($b_0 = 1$), with a linear trend component and no quadratic component ($b_1 = 0.5$, $b_2 = 0$), a sinusoidal component of unit amplitude (A = 1), a sudden surge of value 5 ($B = 5$) occurring at $k = 3$. The time-period of the cyclic term is taken as $T = 4$ periods. (Note: the value of the residual variation term, i.e., σ is not required in the computation of the impulse of the disturbance; its use is limited to the computation of the limiting inventory variance, which we do not take up herein).

The lag is taken as 1 period up to time $k = 3$, and 2 periods thereafter. Also, since the impulse value will be infinite, we truncate the computation at $k = T_E = 8$ periods.

The impulse is now computed as:

$$Impulse = \sum_{k=0}^{\infty/T_I} \{(l_3(k)+1)r_3(k)\} = \sum_{k=0}^{8} \{(l_3(k)+1)r_3(k)\}$$

$$= \sum_{k=0}^{3} \{(l_3(k)+1)r_3(k)\} + \sum_{k=4}^{8} \{(l_3(k)+1)r_3(k)\}$$

(since the lag changes from period 4 onwards)

$$= \sum_{k=0}^{3} (1+1)\{1+0.5k+Cos(2\pi k/4)+5\delta(k-3)\}$$

$$+ \sum_{k=4}^{8} (2+1)\{1+0.5k+Cos(2\pi k/4)\}$$

The computed values are shown as a function of time from $k = 0$ up to $k = 8$ in Table 1.

Table 1 Impulse computation

Time, k	0	1	2	3	4	5	6	7	8
Impulse (k)	4	10	12	27	39	49.5	58.5	76.5	94.5

In the computation above, the sudden jump at $k = 3$ is due to the demand surge in the third period.

The larger increases in impulse values with time beyond $k = 3$ is due to the increased lag of two periods beyond the third period.

4.2 Effect of severity of disturbance on system deviation and performance

We next take another simple example wherein we show the effect of the impulse on the system response. Herein, we mathematically derive the system mean response, thereby obviating the need for statistical validation.

We take the disturbance to be a simple step demand disturbance of magnitude b_0 demand units, acting in the first period. The standard ICs are taken to hold with all deviations at zero at the start, i.e., at $k = 0$, i.e., we have $r_3(k + 1) \equiv b_0$ valid in $k \geq 0$.

Since the performance behaviour is implicitly affected by the replenishment control scheme followed in the system, we take the case of the simplest and most frequently encountered replenishment control scheme in practice, which is the proportional control scheme. Herein, the replenishment control flow is proportional to the inventory shortfall, i.e., the replenishment quantity ordered is proportional to the current latest observed inventory shortfall, the objective being to restore the inventory levels to their design/nominal levels as quickly as possible. Hence the replenishment control flows are given by:

$$q_3(k+1) \equiv K_3 x_3(k - l_3 - 1) \tag{4.1}$$

where K_3 is the proportionality factor linking the replenishment flow deviation to the inventory deviation (proportional control), and l_3 is the lag in the replenishment inflow to the warehouse from the preceding upstream unit.

The dynamic equations describing the warehouse mean inventory levels are hence given by:

$$x_3(k+l_3+2)-x_3(k+l_3+1)-K_3x_3(k)\equiv -r_3(k+l_3+2), \text{ valid in } k\geq 0 \tag{4.2a}$$

or alternatively in operator form by:

$$\left[E^{l_3+2}-E^{l_3+1}-K_3\right]x_3(k)\equiv -E^{l_3+2}r_3(k) \tag{4.2b}$$

where the forward-Shift Operator, E is defined by: $E^r x(k) \equiv x(k + r)$ for any non-negative integer r.

The system behaviour and inventory profile over time, i.e., the *mean response*, can be derived from the solution of the above linear difference equation (LDE) (4.2) under the ICs mentioned earlier, and are given below. We take two values of lag, viz. zero lag, and single period lag, i.e., $l_3 = 0$ and $l_3 = 1$.

4.2.1 Zero lag

The region of stability for this system is: $-1 < K_3 < 0$ (obtained from elementary analysis). The solution of the system equation for the case of marginal stability (worst case scenario) with $K_3 = -1$ is obtained as:

$$x_3(k)\equiv b_0\left\{-1/|K_3|+\frac{2}{\sqrt{3}}Cos(k\pi/3-\pi/6)=-1+1.1547Cos(k\pi/3-\pi/6)\right\} \tag{4.3a}$$
$$\text{valid in } k\geq 0$$

Alternatively, the solution can be written as:

$$x_3(k)/b_0\equiv -1/|K_3|+\frac{2}{\sqrt{3}}Cos(k\pi/3-\pi/6)=-1+1.1547Cos(k\pi/3-\pi/6) \tag{4.3b}$$
$$\text{valid in } k\geq 0$$

The response shows perpetual un-damped oscillations of constant amplitude of $1.1547b_0$ units about a centre line of $-b_0$ units.

The impulse of the disturbance is obtained as:

$$Impulse(k)\equiv b_0 k.$$

The impulse value above becomes infinitely large as time increases, and hence to keep it finite we can truncate the value at the evaluation time, T_E, and hence take the impulse value as b_0T_E.

Thus, and from the above, we can see the direct effect of the severity of the disturbance on the system behaviour. The inventory deviations are directly proportional to b_0 and hence to the severity of the disturbance. And hence we can expect the responsiveness and resilience behaviour, and other operational measures also to be directly affected by the severity of disturbance. And hence in evaluating the system performance, the severity of the disturbance would have to be kept in mind, and appropriate corrections for factoring it out would be necessary.

The system solution for an intermediate value of $K_3 = -1/2$ is obtained as:

$$x_3(k)/b_0 = -2 + \left(1/\sqrt{2}\right)^k 2Cos(k\pi/4) \text{ valid in } k \geq 0 \tag{4.4}$$

which shows damped oscillations about a centre line of $-2b_0$, with damping rate of $O((1/\sqrt{2})^k)$.

Hence in both cases the inventory deviations and hence under-performance is proportional to b_0 and hence to the severity of disturbance.

4.2.2 Single period lag

The region of stability is obtained as: $-0.619 < K_3 < 0$. The best damping is obtained for $K_3 = 4/27$, ($\lambda_1 = \lambda_2 = 2/3$, $\lambda_3 = -1/3$ being the roots of the LHS operator).

The mean response for the following cases:

a best damping ($K_3 = -4/27$)

b the case of marginal stability ($K_3 = -0.619$) (minimum permissible value of K for stability)

c an intermediary value of $K_3 = -1/2$, are given respectively by equations (4.5), (4.6), and (4.7) below:

$$x_3(k)/b_0 \equiv -6.75 - (0.917)(-1/3)^k + (7.667 + 0.5k)(2/3)^k \text{ valid for } k \geq 0 \tag{4.5}$$

$$\begin{aligned} &x_3(k)/b_0 \equiv -1.615 - (0.321)(-0.619)^k + 2.75Cos(0.629k + \phi), \\ &\tan\phi = -1.0083, \text{ valid for } k \geq 0 \end{aligned} \tag{4.6a}$$

which can be closely approximated by equation (4.6b) below

$$x_3(k)/b_0 \equiv -1.615 - (0.321)(-0.619)^k + 2.75Cos\left((k+1)\pi/4\right) \text{ valid for } k \geq 0 \tag{4.6b}$$

$$\begin{aligned} &x_3(k)/b_0 = -2 + 0.2(-1/2)^k + \left(\sqrt{3}/2\right)^k (2.89)Cos(k\pi/6 + \phi), \\ &\tan\phi = 0.608\ \phi = 0.5463 \text{ radians, valid for } k \geq 0 \end{aligned} \tag{4.7}$$

The first term in all the three equations above is $-b_0 / |K_3|$ and gives the offset value in equations (4.5) and (4.7), and the centre line (average inventory level) about which oscillations occur in the second equation (4.6).

The response characteristics can be seen to be as under:

- For cases a and c: damped oscillations with negative offset.
- For case b: un-damped oscillations in perpetuity of amplitude $2.75b_0$ units about a centre line of $-1.165b_0$ units.

And hence in all three cases, the system deviation is proportional to b_0, and hence to the severity of disturbance. Hence, we can again see the direct effect of the severity of disturbance on the system deviation and performance.

Thus, in summary, in all the cases presented above and for both values of lag, the *system deviation is directly proportional to the severity of disturbance*, thereby showing in this case, the explicit effect of severity on system behaviour.

4.3 Effect of severity of disturbance on responsiveness

4.3.1 Illustration 1

We now illustrate below the effect of severity of disturbance on the responsiveness of the system using the same numerical computation above.

In the earlier literature, one of the measures of responsiveness is that of the Cumulative Back Orders till restoration to normalcy defined as under:

$$\textit{Cumulative Back Orders} = -\sum_{k=0}^{\infty} \min\{x_3(k), 0\} \tag{4.8}$$

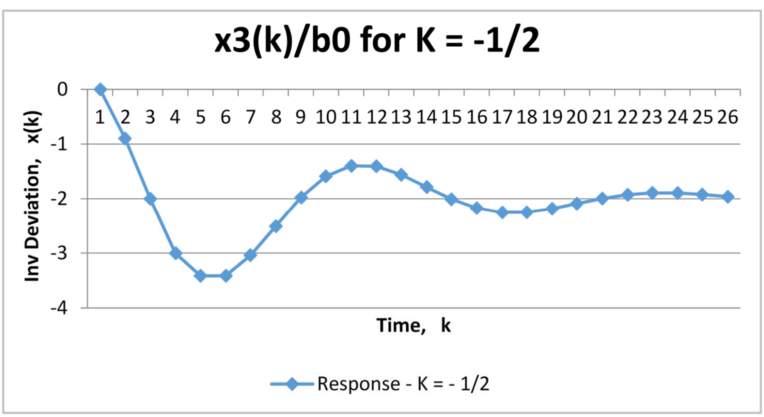
In cases where it is unbounded, we can take it as:

$$\textit{Cumulative Back Orders} = -\sum_{k=0}^{T_E} \min\{x_3(k), 0\} \tag{4.9}$$

For illustration, we take the case of single period lag and the intermediary value of $K_3 = -1/2$, for which the system response is given by equation (4.7) above. Since the cumulative back orders is unbounded, we compute the cumulative back orders till $T_E = 25$ as the measure of responsiveness.

The response is given in Figure 1 for $b_0 = 1$ unit step disturbance) from which we can observe that the cumulative back orders would again be proportional to the severity of disturbance. For $T_E = 25$, the cumulative back orders value is obtained as $52.5465b_0$ while the severity value is obtained as $50b_0$, thereby showing the direct and explicit dependence of the responsiveness measure on the severity of disturbance.

Figure 1 Illustration 1 (see online version for colours)



4.3.2 Illustration 2

We now take the demand disturbance with an additional sinusoidal component (seasonal variation), in addition to a step component. The mean disturbance is hence given by:

$$r_3(k+1) \equiv b_0 + ACoswk, \text{ valid in } k \geq 0, \text{ where } w = 2\pi / T = 2\pi / 4 \tag{4.10}$$

with T, the time-period of the seasonal cycle being 4 periods. This represents a sudden increase in demand by b_0 units along with a sinusoidal component of amplitude A units.

The disturbance occurs from the first period onwards. The system lag is taken as a single period, i.e., $l_3 = 1$.

The system equation for the *mean* inventory level in operator form is hence given by:

$$\left[E^{l_3+2} - E^{l_3+1} - K_3\right] x_3(k) \equiv -E^{l_3+2}\left\{b_0 + ACoswk\right\} \tag{4.11}$$

The annihilator of the RHS terms (Kelley and Peterson, 2001) is given by the operator:

$$(E-1)\left(E - e^{j\pi/2}\right)\left(E - e^{-j\pi/2}\right),$$

and hence the homogeneous form of the LDE is

$$(E-1)\left(E - e^{j\pi/2}\right)\left(E - e^{-j\pi/2}\right)\left[E^{l_3+2} - E^{l_3+1} - K_3\right] x_3(k) \equiv 0 \text{ valid in } k \geq 0 \tag{4.12}$$

And hence the general solution is given by:

$$x_3(k) \equiv D + \left(C_0 + C_1 k\right)(2/3)^k + C_2(-1/3)^k + A_1 Cos(k\pi/2) + A_2 Sin(k\pi/2) \tag{4.13a}$$

And plugging the solution into the original non-homogeneous equation (4.11) yields

$$\begin{aligned} x_3(k) \equiv & -(27/4)b_0 + \left(C_0 + C_1 k\right)(2/3)^k + C_2(-1/3)^k \\ & + (0.6567)Cos(k\pi/2 + \phi) \end{aligned} \tag{4.13b}$$

$$\tan\phi = 0.871, \phi = 0.7166 \text{ radians.}$$

The ICs of the system are:

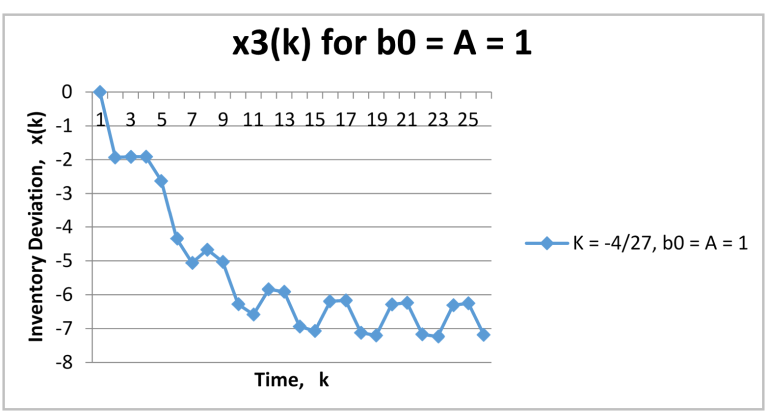
$$\left\{x_3(0) = 0, x_3(1) = -b_0 - A, x_3(2) = -2b_0\right\},$$

plugging in which yields the full solution as:

$$\begin{aligned} x_3(k) \equiv & -6.75b_0 + (0.6567)ACos(k\pi/2 + 0.7166) \\ & + \left(6.67b_0 - 1.5272A\right)(2/3)^k + \left(2b_0 + 1.29A\right)k(2/3)^k \\ & + \left(0.08b_0 + 1.033A\right)(-1/3)^k \end{aligned} \tag{4.13c}$$

The response is shown in Figure 2 for $b_0 = A = 1$ (i.e., unit step with a sinusoid of unit amplitude).

Figure 2 Illustration 2 (see online version for colours)



The last three terms die out rapidly, and for $k \geq 7$ the cumulative back orders can be obtained as:

$$Cum\ Back\ Orders(k) \cong -6.75b_0k + (32.06)b_0 - 3.25A \tag{4.14}$$

while the impulse is obtained as:

$$Impulse(k) \equiv 2b_0k - 2A\delta\left(k - a_n\right), a_n = 4n + 2, 4n + 3, \text{ for } n = 1, 2, 3, ... \tag{4.15}$$

i.e., the –2A term is present only for $k = 4n + 2$, $4n + 3$, $n = 0,1,2,3,\ldots$

And hence we can see that the responsiveness measure is a strong function of the impulse value again; and for larger values of k, will be approximately *proportional* to the impulse or severity of disturbance.

Also, we can see above that both impulse and well as the Cum. back orders are unbounded.

Hence, we truncate the computation of both at $T_E = 10$ which yields

$$Cum.\ Back\ Orders(10) = 35.5b_0 - 3.25A,\ while\ Impulse(10) = 20b_0 \tag{4.16}$$

For a larger value of $T_E = 20$, we obtain

$$Cum.\ Back\ Orders(20) = 103b_0 - 3.25A,\ while\ Impulse(20) = 40b_0 \tag{4.17}$$

And thus, the computation above illustrates the major impact of the severity of disturbance on the responsiveness performance of the chain.

4.4 Effect of severity of disturbance on recovery performance (Illustration 3)

We now present Illustration 3 wherein we illustrate the effect of severity on the recovery performance and recovery time. We take the same system as above with a lag of a single period and examine the recovery performance under two disturbances as below

Disturbance 1 A unit step acting for infinite time, whose severity is given by

$$Impulse\,1(k) \equiv 2b_0k \text{ for } k \geq 1$$

Disturbance 2 A unit step acting for a finite time, i.e., up to $k = L$, and zero thereafter, whose severity is given by:

$$Impulse2(k) \equiv \left[\begin{matrix} 2b_0k, k \leq L \\ 2b_0L, k > L \end{matrix}\right).$$

The disturbance hence is of finite severity.

We now look at the response of the system under these two disturbances with $L = 8$ periods, i.e., the disturbance 2 acts up to period 8, and then becomes zero thereafter.

Using the same methods given above the solutions for the two cases are obtained as:

- for Disturbance 1:

$$x_3(k)/b_0 \equiv -6.75 + (6.667 + 2k)(2/3)^k + 0.0833(-1/3)^k \text{ valid in } k \geq 1 \tag{4.18}$$

- for DISTURBANCE 2 (for $L = 8$):

Note: for Disturbance 2, the solution for $1 \le k \le L = 8$ is given by the same equation (4.18) above.

For $k \ge L + 1 = 9$, the system equation is given by:

$$[E^3 - E^2 + 4/27]x_3(k) \equiv 0, \text{ valid in } k \ge L+1 = 9,$$

with ICs:

$$\begin{Bmatrix} x_3(L+1) = x_3(9) = -6.108, x_3(L+2) = x_3(10) = -6.288, \\ x_3(L+3) = x_3(11) = -6.419 \end{Bmatrix} \tag{4.19}$$

and the solution by:

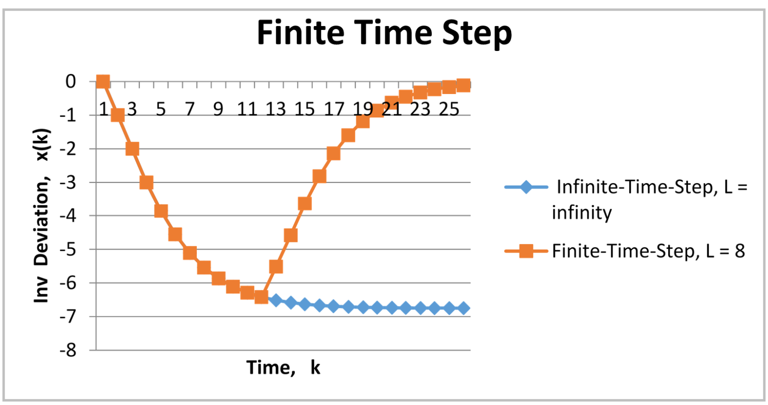
$$x_3(k) \equiv (52.08 - 6.673k)(2/3)^{k-8} + 2.249(-1/3)^{k-8}, \text{ valid in } k \ge L+1 = 9 \tag{4.20}$$

The response for Disturbance 1 settles to its negative offset value of –6.75 by about 25 periods, and thus shows complete loss of responsiveness and recovery. The system does not ever recover from the disruption but continues indefinitely with a severely depleted inventory level. And recovery time is obviously infinite.

Whereas the response for Disturbance 2 shows recovery beyond $k = L + 1 = 9$. The system recovers from the disruption with a recovery time of about 17 periods after the disturbance subsides, and hence has a finite recovery time.

The responses of the system are shown in Figure 3 for the two disturbances.

Figure 3 Illustration 3 (see online version for colours)



Thus, we can see the completely different recovery behaviour of the system under the two disturbances of different severities, thereby illustrating the major effect of severity on recovery performance.

4.5 A simulation example

We now present a simulation example for a step disturbance of magnitude 3 units, as under.

The mean disturbance is given by:

$$r_3(k) \equiv 3 \; \forall k >> 1.$$

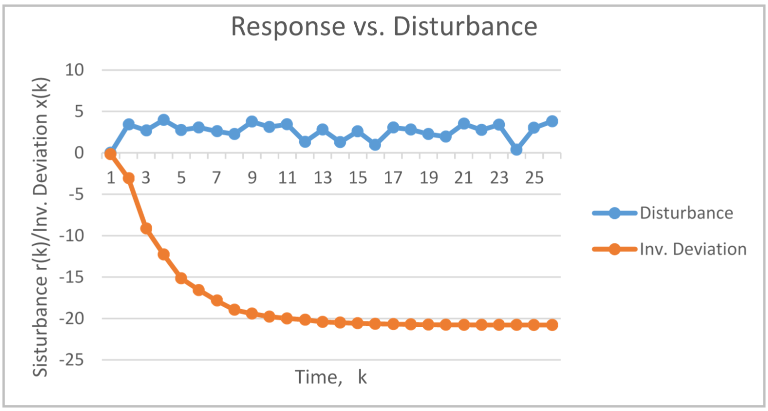
This represents a sudden increase in demand by 3 units. The system lag is taken as a single period, i.e., $l_3 = 1$. The random variance term in the disturbance is taken as unity (i.e., unity variance, $\sigma^2 = 1$).

The disturbance term hence is

$$r_3(k) \equiv 3 + \varepsilon(k) \text{ where } \varepsilon(k) \sim WN\left(0,\ \sigma^2\right) = WN\left(0,\ 1^2\right).$$

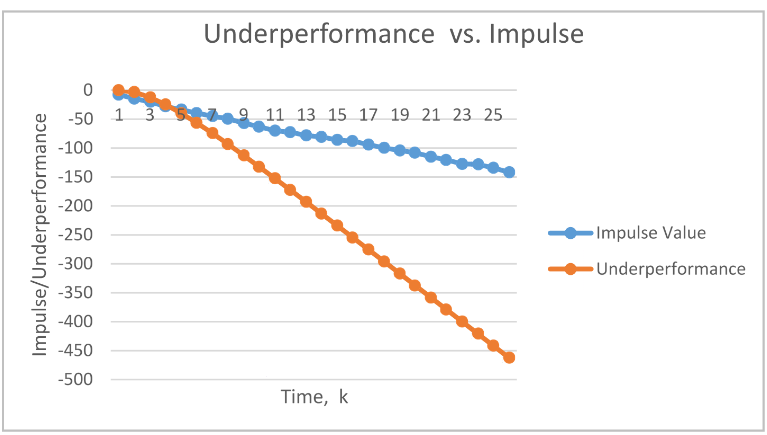
The simulated disturbance and the system response for the best damping case are shown in Figure 4.

Figure 4 Simulation example: response vs. disturbance (see online version for colours)



While the impulse function and the underperformances (cumulative deviation) are shown in Figure 5.

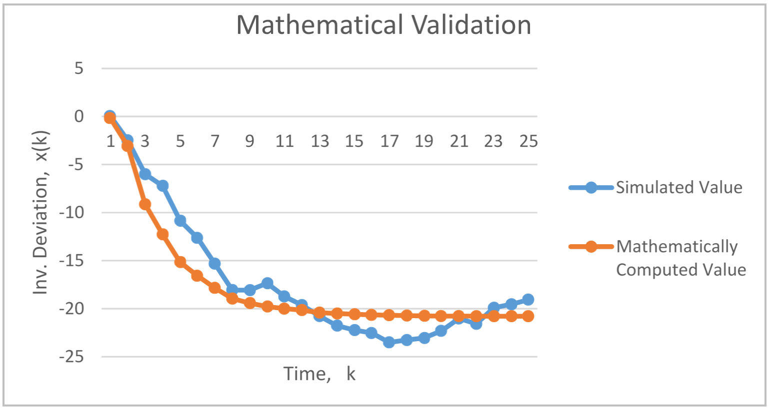
Figure 5 Simulation example: underperformance vs. impulse (see online version for colours)



And from Figure 5, we can readily observe the nearly proportionate increase in underperformance with increase in impulse values, with the proportionality constant being approximately 3.26.

The theoretical validation of the simulation results is shown in Appendix. The simulated values are plotted alongside the mathematically computed values in Figure 6, from which we can readily see the close correspondence between the two, thereby validating the simulation run.

Figure 6 Mathematical validations (see online version for colours)



Hence, the simulation results further corroborate the preceding discussions, and Figure 5 graphically illustrates the nearly proportional increase in underperformance with increase in impulse value.

The results above would have significant managerial and practical implications, which are now discussed below.

5 Managerial implications and practical significance

SC performance evaluation is now an area of growing interest and focus in SC management. And from the above illustrations it is evident that SC under-performance, and especially its responsiveness and recovery performance are impacted in a very major way by the severity of the disturbance to the chain. In some of the illustrations above, the performance measures have even been shown to be approximately proportional to the severity of disturbance. In the light of the above observations, we feel there could be a need for a *major shift in our approach to evaluating the performance of a SC* as well as its responsiveness and recovery. While hitherto our approach has been to measure its under-performance alone, and in absolute terms, it would now appear that we would need to change this approach and procedure and would need to measure SC under-performance *in relation to the severity of the disturbance* that it is subjected to.

Hence it would appear logical to keep in mind or factor out the severity of the disturbance when evaluating the responsiveness, recovery, and operational performance of a SC. And hence in the evaluation of the performance of SCs, their under-performance alone in absolute terms, and the absolute values of their performance measures may *not* give the correct or complete picture. Rather, it would be logical to suggest that the measures/metrics be used after accounting for or factoring out the severity of disturbance. Such a procedure would be able to provide a *more objective* measure of the performance of the SC system, independent of the severity of disturbance. And hence our paper proposes not only a new measure the severity of a disturbance to a SC, but also makes a case for a more objective procedure to be followed for evaluation of SC performance, by factoring out the severity of the disturbance to the chain.

The results of our study would hence be of substantial managerial significance.

One simple procedure to measure SC performance more objectively could be to *normalise* the performance measure by the severity of disturbance, to yield values *per unit severity*.

And hence one could recommend and envisage the use of such SC performance measures and metrics in future, which could factor out the severity of the disturbance to the chain, thereby yielding more objective performance measures. The results of this paper thus pave the way for future work on this topic. These and other such similar procedures could be taken up in further studies.

6 Conclusions

This paper has proposed a substantially new approach to defining and quantifying the severity of a disturbance or disruption to a SC. Using the dynamic modelling framework, the paper has proposed a precise quantitative definition and a quantitative measure of the severity of a disturbance to a SC, by way of the impulse of the disturbance. The objective of the definition and measure is to capture and quantify the disruption potential of the disturbance. The computation of the severity measure has been illustrated for various types of disturbances. Numerical illustrations have also been presented which highlight the major impact of the severity of disturbance on the responsiveness, recovery, and operational performance measures of the chain.

In the light of the above, it is observed that the absolute values of operational performance measures of a SC may not yield the complete picture, and it is suggested that they be used after factoring out the effect of the severity of disturbance. Such a procedure would then present a truer picture, and a more objective measure of the performance of the SC system, independent of the severity of disturbance.

Further work could focus on such procedures and measures, as also on computational/empirical studies.

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Appendix

Mathematical validation of the simulation example

The simulation example

Note: The numbering of the equations is given as a continuation of those in Section 4 for ease of reference.

The results of the simulation study are now compared with the mathematically predicted values to validate the simulation results. For this purpose, the mathematical solution is given below.

The system LDE is given by:

$$\left[E^{l_3+2} - E^{l_3+1} - K_3\right] x_3(k) \equiv -E^{l_3+2} r_3(k) \tag{4.2b}$$

(The best damping is obtained for $K_3 = -4/27$, ($\lambda_1 = \lambda_2 = 2/3$, $\lambda_3 = 1/3$) being the roots of the LHS operator). The deterministic part of the solution is given by:

$$x_3(k)^{\det} \equiv 3\left(-6.75 - (0.917)(-1/3)^k + (7.667 + 0.5k)(2/3)^k\right) \text{ valid for } k \geq 0 \tag{4.21}$$

While the stochastic part is evaluated as under.

The stochastic difference equation (SDE) for the stochastic part of the solution is given by:

$$\left[E^3 - E^2 + \frac{4}{27}\right] x_3(k)^{stoc} \equiv -\varepsilon(k+3) \quad \text{valid in for } k \geq 0 \tag{4.22}$$

Now, since unity is not a root of the LHS operator, the stochastic part of the solution admits an infinite moving average representation in terms of the white noise random disturbance terms, as under:

$$x_3(k)^{stoc} \equiv \sum_{l=0}^{\infty} \beta_l \varepsilon(k-l) \equiv \sum_{l=0}^{k} \beta_l \varepsilon(k-l) \tag{4.23}$$

And hence, we have the SDE for the stochastic part of the solution as:

$$\left[E^3 - E^2 + 4/27\right]\left\{\sum_{l=0}^{k} \beta_l \varepsilon(k-l)\right\} \equiv -\varepsilon(k+3) \quad \text{valid for } k \geq 0 \tag{4.24}$$

And equating the coefficients of $\varepsilon(k)$ on both sides of equation (4.24) yields as under:

Term in SDE	*Coefft. of $\varepsilon(k+3)$*	*Coefft. of $\varepsilon(k+2)$*	*Coefft. of $\varepsilon(k+1)$*	*Coefft. of $\varepsilon(k)$*	……	*Coefft. of $\varepsilon(k)$*
E^3	β_0	β_1	β_2	β_3	……	β_{k+3}
E^2	0	$-\beta_0$	$-\beta_1$	$-\beta_2$	……	β_{k+2}
E^1	0	0	0	0	……	0
4/27	0	0	0	$(4/27\,\beta_0)$	……	$\left(\frac{4}{27}\right)\beta_k$
RHS	–1	0	0	0	0	0

From which we obtain:

$$\beta_0 = -1, \beta_1 - \beta_0 = 0, \beta_2 - \beta_1 = 0, \text{ and } \beta_{k+3} - \beta_{k+2} + (4/27)\beta_k \equiv 0 \quad \forall k \geq 0 \tag{4.25}$$

With ICs given by:

$$\beta_0 = \beta_1 = \beta_2 = -1,$$

which yields the solution for the β coefficients as:

$$\beta_k \equiv (-1/9)(-1/3)^k + \{-(8/9) - (2/3)k\}(2/3)^k, \text{ valid for } k \geq 0 \tag{4.26}$$

And hence, the stochastic part of the solution is given by:

$$x_3(k)^{stoc} \equiv \sum_{l=0}^{k} \{(-1/9)(-1/3)^l + (-8/9 - (-2/3)l)(2/3)^l\}\varepsilon(k-l) \tag{4.26}$$

And hence, the full solution is given by:

$$x_3(k) \equiv x_3(k)^{\det} + x_3(k)^{stoc} \tag{4.27a}$$

Thus,

$$\begin{aligned} x_3(k) \equiv\ & 3\left(-6.75 - (0.91)(-1/3)^k + (7.667 + 0.5k)(2/3)^k\right) \\ & + \sum_{l=0}^{k} \{(-1/9)(-1/3)^l + (-8/9 - (-2/3)l)(2/3)^l\}\varepsilon(k-l) \\ & \text{valid for } k \geq 0 \end{aligned} \tag{4.27b}$$

where the random disturbance terms $\varepsilon(k) \sim WN(0, 1^2)$.

The predicted solution is plotted alongside the simulation results in Figure 6, from which we can see the close correspondence between the two, thereby validating the simulation study.